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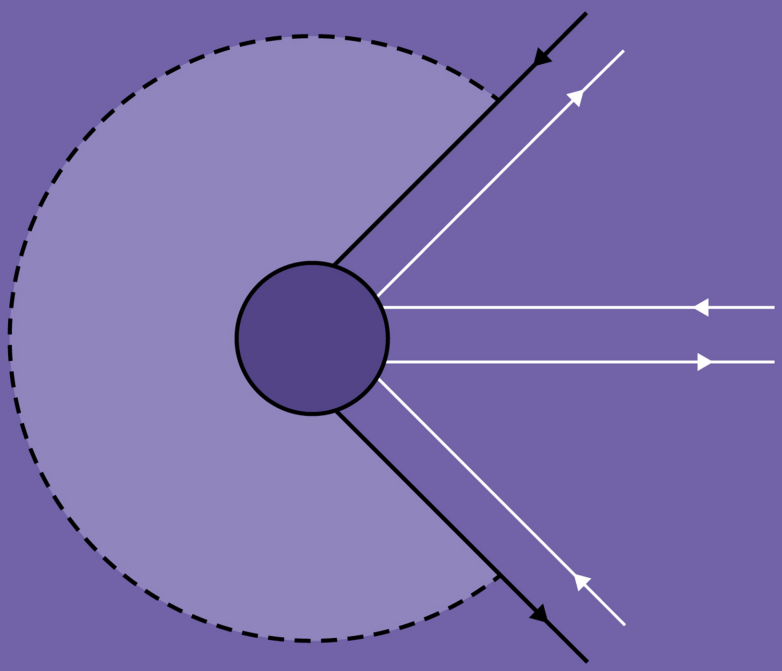
Badis Ydri

VOLUME TWO



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A Modern Course in Quantum Field Theory, Volume 2

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To my father for his continuous support throughout his life …

Saad Ydri
1943–2015

Also to my …
Nour

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Preface

This two-volume book was accepted for publication by IOPP (Institute of Physics Publishing) on 20 February 2017, submitted on 14 December 2018 and will appear in its final form during the spring of 2019. It contains a comprehensive introduction to the fundamental topic of quantum field theory starting from free fields and their quantization, renormalizable interactions, critical phenomena, the standard model of elementary particle physics, lattice field theory, the functional renormalization group equation, non-commutative field theory, topological field configurations, exact solutions of quantum field theory, supersymmetry and finally the AdS/CFT correspondence. The emphasis throughout is put on the physical principle of symmetry (especially the local principle of gauge symmetry) and on the mathematical machinery of the renormalization group equation à la Wilson. This book is the fifth book published by the author[1] and it completes therefore his in-depth detailed and constructive study of all fundamental areas of theoretical physics which took several years to complete. The author would like to thank his IOPP editor John Navas for all his help in publishing three of his books.

[1] Together with two open and free books on fundamental physics in Arabic.

Author biography

Badis Ydri

Badis Ydri—currently a professor of theoretical particle physics, teaching at the Institute of Physics, Badji Mokhtar Annaba University, Algeria—received in 2001 his PhD from Syracuse University, New York, USA and in 2011 his Habilitation from Annaba University, Annaba, Algeria.

His doctoral work, titled 'Fuzzy Physics', was supervised by Professor A P Balachandran. Professor Ydri is a research associate at the Dublin Institute for Advanced Studies, Dublin, Ireland, and a regular ICTP associate at the Abdus Salam Center for Theoretical Physics, Trieste, Italy. His postdoctoral experience comprises a Marie Curie fellowship at Humboldt University Berlin, Germany, and a Hamilton fellowship at the Dublin Institute for Advanced Studies, Ireland.

His current research directions include: the gauge/gravity duality; the renormalization group method in matrix and noncommutative field theories; noncommutative and matrix field theory; emergent geometry, emergent gravity and emergent cosmology from matrix models.

Other interests include string theory, causal dynamical triangulation, Hořava–Lifshitz gravity, and supersymmetric gauge theory in four dimensions.

He has recently published three books. His hobbies include reading philosophic works and the history of science.

Introduction

The luminous matter in the Universe is constituted of elementary fermion particles of spin 1/2 (leptons and quarks) which interact via elementary boson particles of spin 1 (gauge vector bosons) mediating the three fundamental interactions of nature: the electromagnetic interaction, the strong nuclear force and the weak nuclear interaction. The fourth fundamental force of nature (the gravitational force) is mediated instead by a tensor particle of spin 2.

These particles are all massless and these forces obey a fundamental symmetry principle called the gauge principle which can only be broken spontaneously via the Higgs particle (the breaking of the electroweak force into the observed electromagnetic force and weak interactions) which is an (the only) elementary particle of spin 0 in nature. This process of spontaneous symmetry breaking is what gives all elementary particles their measured masses and all the forces their observed strengths.

Quantum field theory is a relativistic quantum theory which describes precisely this luminous matter and its interactions. In fact, it is widely believed that quantum field theory should also describe dark matter and perhaps even dark energy (in terms of vacuum energy). This quantum field theory is perturbatively renormalizable. However, quantum field theory enjoys also non-perturbative formulation either directly (through lattice field theory, the renormalization group equation and conformal field theory) or indirectly by admitting exact solutions (especially in two dimensions but also in four dimensions via the supersymmetric gauge principle).

Furthermore, the 'modern' or 'new' quantum field theory includes also gravity via the AdS/CFT correspondence which is the most celebrated paradigm of gauge/gravity holographic duality. Hence, modern quantum field theory which governs all elementary particles and their interactions as well as gravity can be summarized in three major sub-theories:

1. The standard model of elementary particles: This provides a unified scheme of the electromagnetic force, the weak interaction, and the strong nuclear force, and is due historically to the work of Weinberg, Abdu Salam and Glashow among many other physicists. The standard model is the most successful (experimentally) quantum field theory to date and perhaps the most successful theory ever (especially its quantum electrodynamics (QED) component). It accounts for a large body of phenomenological effects and observations seen in nature in terms of only a finite (but still relatively large =19) number of parameters such as the gauge coupling constants, the Higgs vacuum expectation value, the CKM angles and the theta angle governing CP violation. The standard model is however, mostly perturbative and it includes in a fundamental way the phenomena of spontaneous symmetry breaking and is based entirely on the meta-theory of the renormalization group equation.

   The standard model consists of two parts. The first part is the electroweak force which unifies quantum electrodynamics and quantum flavordynamics

which describes the weak force. The second part of the standard model consists of quantum chromodynamics (QCD) which describes the strong force. QCD admits a non-perturbative definition given typically in terms of a lattice formulation, and lattice QCD is arguably the most sophisticated discipline in computational physics.

2. Supersymmetric gauge theory in four dimensions: This allows us a non-perturbative formulation (one in which we do not need a small parameter of expansion) of the gauge principle which can be solved exactly (like the harmonic oscillator in quantum mechanics) in many instances by means of supersymmetry and holomorphy among other things (Witten–Seiberg–Nekrasov theory). This is of paramount importance to strongly coupled systems such as quantum chromodynamics since the strong force is a highly non-perturbative interaction. However, supersymmetric gauge theory also gives a profound understanding of the phenomena of spontaneous gauge symmetry breaking and the associated phenomena of renormalization.
3. AdS/CFT duality: As stated above, the gravitational force is not mediated via a vector gauge boson but via a tensor particle of spin 2 called the graviton. The AdS/CFT duality is the theory which allows us to bring gravity and black holes into the realm of unitary quantum field theory. Although this theory emerged historically from string theory it is intrinsically a quantum field theory. It relies heavily on conformal field theory, supersymmetry and renormalization. It states simply that supergravity theory (string theory in general) in an anti-de Sitter (AdS) spacetime which is five dimensional is given precisely by a superconformal gauge field theory (CFT) living on the boundary of AdS which is an ordinary four dimensional Minkowski space-time (a concrete realization of the holographic principle). The AdS/CFT correspondence generalizes to the so-called gauge/gravity duality.

In this book we will mainly focus on the first axis (gauge interactions and the standard model of elementary particle physics). However, we will also prepare the ground for the second axis (chapters 14–16 on exact solutions of quantum field theory, monopoles and instantons and supersymmetry) and for the third axis (in chapter 17 we give a systematic overview of the AdS/CFT correspondence and then show how Einstein's gravity emerges from quantum entanglement).

The main emphasis throughout this book will be on the physical principle of symmetry (especially the role of symmetry groups in the quantum theory, their representation theory and conservation laws) but also on the mathematical machinery of the renormalization group equation (chapters 6–9, 12 and 13).

The renormalization group equation will allow us to study, beside the usual problems of quantum field theory relevant to particle physics (found in chapters 6–8), two more interesting physical problems: critical exponents of second order phase transitions in statistical physics (chapter 9) and renormalizability of non-commutative field theory (in chapter 13). Chapter 12 contains a systematic presentation of the functional renormalization group equation.

We will start the book in the usual way with canonical quantization of free fields (scalar field of spin 0 and spinor field of spin 1/2) in chapters 2 and 3. Then we will consider in chapter 4 perturbation theory of phi-four theory where the S-matrix structure of quantum field theory is exhibited explicitly. This is our first fundamental interaction in this book.

Then canonical quantization of the free abelian vector field of spin 1 is considered in chapter 5 where pure Yang–Mills gauge interactions with $SU(N)$ groups are also introduced. In chapter 6 perturbation theory of quantum electrodynamics (which describes the gauge interaction of a spinor field with a vector field) and its renormalization is considered in great detail. For example, we derive explicitly from the renormalization properties of the theory measurable physical effects such as the electron anomalous magnetic moment. Furthermore, the links to particle physics, i.e. the relations between quantum field theory correlation functions and particle physics cross sections and decay rates, are established explicitly in this chapter which shows more clearly the S-matrix structure of quantum field theory.

The path integral formalism is introduced in chapters 7 (for scalar fields) and 8 (for spinor and vector fields). In chapter 7 perturbative renormalizability of phi-four theory is considered at the two-loop order using the effective action formalism, whereas in chapter 8 the Faddeev–Popov quantization of the abelian and non-abelian vector fields is considered. Perturbative renormalizability of $SU(N)$ gauge theory coupled to matter, transforming in some representation of the gauge group, is then discussed (asymptotic freedom, anomalies, BRST and background field methods, etc).

In chapter 10 we discuss phenomenology of particle physics, then provide an explicit and detailed construction of the standard model Lagrangian and explain the phenomena of spontaneous symmetry breaking via the Higgs mechanism. In chapter 11 we give an explicit construction of scalar, spinor and vector lattice actions, then discuss the main Monte Carlo algorithms used and some sample numerical simulations.

In more detail, this book is then organized into chapters as follows:

1. **Relativistic quantum mechanics:** This chapter contains standard preparatory material. We will present an overview of special relativity [1], relativistic Klein–Gordon and Dirac wave equations and the convention in this book for Dirac spinors [2], and a self-contained discussion of representation theory of the rotation and Lorentz groups [3].
2. **Canonical quantization of free fields:** After a brief excursion in classical mechanics [4] we present in this chapter the canonical quantization of free scalar and Dirac fields with a detailed calculation of the corresponding propagators [2, 5]. Then we give a thorough discussion of symmetries starting with discrete symmetries [2], the Poincaré group and its representation theory [3, 5], symmetries in the quantum theory, internal symmetries and the role of Noether's theorem in conservation principles [5, 6].
3. **The phi-four theory:** A detailed discussion of the S-matrix, the Gell-Mann–Low formula, the LSZ reduction formulas, Wick's theorem, Green's functions, Feynman diagrams and the corresponding Feynman rules of

quantum Φ^4-theory is presented following [5]. This is our first non-trivial example of an interacting field theory and its canonical quantization.

4. **The electromagnetic field and Yang–Mills gauge interactions:** In this chapter we discuss in great detail the canonical quantization of the electromagnetic gauge field with emphasis on $U(1)$ gauge invariance and the Gupta–Bleuler method. Then a pedagogical introduction to Yang–Mills gauge interactions with $SU(2)$ and $SU(N)$ gauge groups (and even for general gauge groups) is presented. These gauge fields describe in Nature spin 1 particles (the so-called vector bosons) which encompass the carriers of the electromagnetic force (the photon γ), the nuclear strong color force (the gluons g) and the nuclear weak radioactive force (the W and Z^0 vector bosons).

   Good pedagogical references for the canonical quantization of the electromagnetic field are [5, 6].
5. **Quantum electrodynamics:** The goal in this chapter is to develop canonical perturbation theory beyond the free field approximation of QED which is an interacting (local gauge) theory of the Dirac field (electrons and positrons) and the gauge vector field (photons). The formalism of canonical quantization of QED is found in [5], whereas radiative corrections and renormalization are found in [2].
6. **Path integral quantization of scalar fields:** In this chapter we will present the path integral method which is a central tool in quantum field theory and then give a detailed account of the effective action in the case of a scalar field theory. A brief discussion of spontaneous symmetry breaking is also given. These are very standard topics and we have benefited here from the books [2, 7, 8] and the lecture notes [9].
7. **Path integral quantization of Dirac and vector fields:** We develop the powerful and elegant path integral method for spinor fields (Grassmann variables) and gauge fields (gauge fixing, Faddeev–Popov method, ghosts). Then we give two important applications based on the path integral formalism. Firstly, we present a detailed derivation of the one-loop beta function of QCD with $SU(N)$ gauge theory and matter fields in the fundamental representation and discuss the resulting phenomena of asymptotic freedom. Secondly, we present the one-loop (and in fact exact) axial or chiral anomaly in QED and the Fujikawa path integral method. We also discuss briefly the background field method and symmetries within the path integral method (Schwinger–Dyson equations and Ward identities).
8. **The Callan–Symanzik renormalization group equation:** All second-order phase transitions in Nature are described by the Callan–Symanzik renormalization group equations of Euclidean scalar field theory. In this chapter, after a detailed discussion of renormalizability of quantum field theories, in particular the scalar ϕ^4 theory, we present an explicit construction of the Callan–Symanzik renormalization group equations. Then, a detailed calculation of the critical exponents of second-order phase transitions starting from the renormalization properties of scalar ϕ^4 field theory at the two-loop order is carried out explicitly. We follow closely the book [10].

9. **Standard model:** The standard model of elementary particle physics describes all known particles and their interactions which are observed in Nature. It is based on the following grand theoretical principles:
   - Relativistic invariance.
   - It is a local gauge theory based on the gauge group $SU(3) \times SU(2)_L \times U(1)_Y$.
   - The gauge group is spontaneously broken down to $SU(3) \times U(1)_{\text{em}}$. This generates mass in a gauge-invariant way.
   - It consists of a lepton sector, a quark sector, a Higgs term and a gauge sector. The matter sector (leptons, quarks and Higgs) are coupled minimally to the gauge sector (which ensures renormalizability). The mechanism by which the symmetry is spontaneously broken is the Higgs mechanism. The Higgs field is coupled to the quarks and leptons via gauge invariant renormalizable Yukawa couplings.
   - It is a chiral gauge theory, i.e. left-handed quarks and leptons couple to the gauge field differently (in the fundamental representation) than right-handed quarks and leptons (singlet representation).
   - Renormalizability: The standard model is a renormalizable theory (interaction terms between the gauge fields and the matter fields are given by minimal coupling). The requirement of gauge invariance guarantees renormalizability and unitarity.
   - The standard model is not invariant under parity P (nor under CP where C is charge conjugation). But it is invariant under CPT where T is time reversal. This holds in the lepton sector.
   - Anomaly cancellation: This is the second quantum consistency check (after renormalizability) which states that any local symmetry like gauge symmetry cannot be allowed to be anomalous. This is satisfied in the standard model since the number of lepton families is equal to the lepton of quark families.

   Extensions of the standard model include grand unified theories GUT's (such as $SU(5)$ or $SO(10)$ or any other group which contains the standard model gauge group as a subgroup), supersymmetry (minimal supersymmetric standard model), non-commutative geometry (Connes' standard model) and stringy extensions. Unification of the three forces (color strong, electromagnetic and weak) described by the standard model with gravity is however only achieved in string theory.

   In this chapter, and after a brief excursion into the phenomenology of particle physics (isospin symmetry, quark model, neutrino oscillations, etc), we give a detailed construction of the standard model Lagrangian starting with the Glashow, Weinberg, Salam electroweak theory, then we discuss the Higgs mechanism and spontaneous symmetry breaking, Majorana fermions, neutrino mass and the seesaw mechanism, and then finally we provide an extension to the quark sector and quantum chromodynamics as well as a summary of anomaly cancellation. We will follow the general presentations of [3, 9, 11, 12].

10. **Introduction to lattice field theory:** In this chapter a quick excursion into the world of lattice field theory is taken. Scalar, fermion and gauge fields are constructed on the lattice explicitly. Then the two most used Monte Carlo algorithms in numerical simulations on the lattice (the Metropolis and the hybrid Monte Carlo algorithms) are explained within the context of very simple lattice models, namely the scalar phi-four in two dimensions and quenched electrodynamics. The classic textbooks on the subject of lattice field theory are [13–17].
11. **The Wilson and functional renormalization group equations:** The renormalization group equation is a central tool of perturbative and non-perturbative quantum field theory which is vital for a proper understanding of the renormalizability of the theory and its phase diagram. The Wilson approach [18] to the renormalization group equation is, in our opinion, the most profound description of the true nature and final goals of quantum field theory. In this chapter, and after a careful review of the original Wilson approach, we describe in great detail the functional renormalization group equation which is an exact non-perturbative formulation of the Wilson renormalization group equation. The original literature on the functional renormalization group equation includes Polchinski [19] (Polchinski's equation for the effective action) and Wetterich [20] (Wetterich's equation for the average action). See also [21].
12. **Non-commutative scalar field theory and its renormalizability:** In this chapter, and after an efficient introduction to non-commutative scalar field theory, we will apply the Wilson–Polchinski renormalization group equation, discussed in the previous chapter, to the problem of renormalizing non-commutative phi-four theory in 2 and 4 dimensions with and without the harmonic oscillator term. Non-commutative field theory is discussed in great detail in our book [22], whereas we follow closely the original programme of Grosse and Wulkenhaar [23–25] in the very difficult problem of renormalization of non-commutative phi-four theory on Moyal–Weyl spaces.
13. **Some exact solutions of quantum field theory:** The non-perturbative physics of a quantum field theory (as we have seen) can only be probed by means of Monte Carlo methods on lattices (which can become quite intricate technically and numerically) and/or by means of the exact renormalization group equation (which is always quite intricate analytically and mathematically). But sometimes exact solutions of the quantum field theory model presents themselves (lower dimensions and/or a high degree of symmetries) which allow us to access the sought-after non-perturbative physics of the theory directly. In this chapter we present as examples six models in dimension two which all enjoy exact solutions, allowing us an unprecedented look at the true heart, i.e. the non-perturbative reality, of a quantum field theory.
14. **The monopoles and instantons:** Monopoles are non-trivial topological gauge field configurations which appear in spontaneously broken gauge theory via

the Higgs mechanism. These are particle-like solitonic configurations characterized by stability and finite energy among other properties. Their stability is of a topological origin characterized by the so-called winding numbers or magnetic charges. For this reason monopoles are one of the best examples in which physics and topology become intertwined. The existence of the monopole requires the embedding of electromagnetism, i.e. the group $U(1)$, as a subgroup in a larger non-abelian group G with compact cover which then becomes broken spontaneously via the usual Higgs mechanism.

The original literature on the subject consists of 't Hooft [26] and Polyakov [27]. Some of the pedagogical (from my perspective) lectures I can mention here: Lenz [28], 't Hooft [29], Coleman [30] and Tong [31]. A comprehensive book is Shnir [32] and a comprehensive review is given by Weinberg and Yi [33].

Instantons are another fundamental topological gauge configuration, perhaps more fundamental than monopoles, which are given by events localized in spacetime and hence the other name given to them is pseudo-particles (in contrast with particles such as monopoles which are events localized in space). Instantons are also the gauge field configurations which dominate the path integral in the semi-classical limit with the trivial instanton identified precisely with the perturbative vacuum $A = 0$.

We will discuss here in great detail the theta term, the role of vacuum degeneracy, the quantization of the topological charge and the role of topology in instanton physics. More precisely, the instanton is defined as a solution of the self-duality equation with zero/finite energy which happens to saturate the Bogomolnyi bound. The BPST instanton solution is then derived explicitly. The original literature on the BPST instanton is the paper by Belavin, Polyakov, Schwartz and Tyupkin [34]. We then discuss in some detail the moduli space, the collective coordinates, the zero modes, the ADHM construction, the one-loop quantization in the background of instantons as well as the connection of instantons to quantum tunneling. We have benefited here greatly from the pedagogical presentations found in [31, 35, 36].

15. **Introducing supersymmetry:** In this chapter we introduce supersymmetry following mostly [37]. In particular, we will emphasize the formal quantum field theory aspects of the formalism of global $N = 1$ supersymmetry with a detailed calculation of the corresponding F- and d-terms following also [38]. A brief description of $N = 2$ is also given. The classic text on supersymmetry of Wess and Bagger [39] remains, in our view, one of the best books on quantum field theory. We have also benefited from [40, 41].
16. **The AdS/CFT correspondence:** The goal in this chapter is to provide a pedagogical presentation of the celebrated AdS/CFT correspondence adhering mostly to the language of quantum field theory (QFT). This is certainly possible, and perhaps even natural, if we recall that in this correspondence we are positing that quantum gravity in an anti-de Sitter

spacetime AdS_{d+1} is nothing else but a conformal field theory (CFT_d) at the boundary of AdS spacetime. Some of the reviews of the AdS/CFT correspondence which emphasize the QFT aspects and language include Kaplan [42], Zaffaroni [43] and Ramallo [44].

This chapter contains, therefore, a thorough introduction to conformal symmetries, anti-de Sitter spacetimes, conformal field theories and the AdS/CFT correspondence. The primary goal however in this chapter is the holographic entanglement entropy. In other words, how spacetime geometry as encoded in Einstein's equations in the bulk of AdS spacetime can emerge from the quantum entanglement entropy of the CFT living on the boundary of AdS.

A sample of the original literature for the holographic entanglement entropy is [45–48]. However, a very good, concise and pedagogical review of the formalism relating spacetime geometry to quantum entanglement due to Van Raamsdonk and collaborators is found in [47] and [49].

This book also includes five appendices (two on classical physics, one on representation theory of Lie groups and Lie algebras, one on homotopy theory and one contains extra exercises given as examination problems throughout the years).

This book (especially the first volume) grew from a course of lectures delivered (five times) since 2010 at Annaba University (Algeria) to theoretical physics students at the Master level (first and second years).

All illustrations found in this book were created by Dr Khaled Ramda, Z Salem and L Bouraiou. The Monte Carlo results included in the chapter 'Introduction to lattice field theory' (chapter 11) are from our original numerical simulations.

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Chapter 9

Standard model

The standard model of elementary particle physics describes all known particles and their interactions which are observed in Nature. It is based on the following grand theoretical principles:

- Relativistic invariance.
- It is a local gauge theory based on the gauge group $SU(3) \times SU(2)_L \times U(1)_Y$.
- The gauge group is spontaneously broken down to $SU(3) \times U(1)_{\text{em}}$. This generates mass in a gauge-invariant way.
- It consists of a lepton sector, a quark sector, a Higgs term and a gauge sector. The matter sector (leptons, quarks and Higgs) are coupled minimally to the gauge sector (which ensures renormalizability). The mechanism by which the symmetry is spontaneously broken is the Higgs mechanism. The Higgs field is coupled to the quarks and leptons via gauge invariant renormalizable Yukawa couplings.
- It is a chiral gauge theory, i.e. left-handed quarks and leptons couple to the gauge field differently (in the fundamental representation) than right-handed quarks and leptons (singlet representation).
- Renormalizability: The standard model is a renormalizable theory (interaction terms between the gauge fields and the matter fields are given by minimal coupling). The requirement of gauge invariance guarantees renormalizability and unitarity.
- The standard model is not invariant under parity P (nor under CP where C is charge conjugation). But it is invariant under CPT where T is time reversal. This holds in the lepton sector.
- Anomaly cancellation: This is the second quantum consistency check (after renormalizability) which states that any local symmetry like gauge symmetry cannot be allowed to be anomalous. This is satisfied in the standard model since the number of lepton families is equal to the lepton of quark families.

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Extensions of the standard model include grand unified theories (GUTs) (such as $SU(5)$ or $SO(10)$ or any other group which contains the standard model gauge group as a subgroup), supersymmetry (minimal supersymmetric standard model), non-commutative geometry (Connes' standard model) and stringy extensions. Unification of the three forces (color strong, electromagnetic and weak) described by the standard model with gravity is, however, only achieved in string theory.

In this chapter, and after a brief excursion in the phenomenology of particle physics (isospin symmetry, quark model, neutrino oscillations, etc), we give a detailed construction of the standard model Lagrangian starting with the Glashow, Weinberg, Salam electroweak theory, then we discuss the Higgs mechanism and spontaneous symmetry breaking, Majorana fermions, neutrino mass and the seesaw mechanism, and then finally we provide an extension to the quark sector and quantum chromodynamics as well as a summary of anomaly cancellation. We will follow the general presentations of [1–4].

9.1 Elements of phenomenology

9.1.1 Fundamental interactions

The strong nuclear (color) force

The physical world is constituted of elementary particles. There are four types of forces governing all known interactions between these elementary particles: gravitational, strong, electromagnetic and weak.

The gravitational force is very weak at the energy scales of particle physics and therefore it is irrelevant to the standard model of elementary particle physics (or so goes the traditional wisdom) so we will not discuss it any further.

The strong nuclear force which is the first crucial ingredient in the construction of the standard model of elementary particles is a short-ranged force with an interaction radius equal 10^{-13} cm.

It is also a force of great relevance to star formation, astrophysics and cosmology of the early Universe. For example, the heat produced in the Sun is due to the strong nuclear force when deuterium nuclei and protons are combined into helium nuclei.

Particles which feel the strong nuclear force are called hadrons. These are bound states of quarks subjected to color confinement, which is one of the most fundamental characteristic signatures of the strong nuclear force. Those hadrons with integer spins are called mesons (bound states of a quark and an anti-quark), whereas those hadrons with half-integer spins are called baryons (bound states of three quarks or three anti-quarks). Particles which decay by means of the strong force are called resonances.

The strong nuclear force is described by the theory of quantum chromodynamics (QCD) which is a local gauge theory based on the gauge group $SU(3)_c$. This is to be contrasted with the electromagnetic force described by the theory of quantum electrodynamics (QED) which is also a local gauge theory based on the gauge group $U(1)_{\text{em}}$.

A local gauge symmetry means a group of local transformations, i.e. they depend on the spacetime locations, which leave the theory (action, local Lagrangian, the quantum vacuum, the spectrum, etc) invariant.

The lower index c in $SU(3)_c$ denotes the strong nuclear charges carried by the quarks, which are called colors. There are precisely three different charges or colors denoted for example by R (red), G (green) and B (blue) and their conjugates, i.e. anti-charges or anti-colors.

The hadrons are, however, colorless and we say that they are singlet states under the color gauge group $SU(3)_c$. This condition is precisely why hadrons can only come as either baryons (three quarks) or mesons (quark and anti-quark). Indeed, singlet representations of $SU(3)_c$ can only appear in the tensor products $3 \times \bar{3} = \mathbf{1} \oplus 8$ (the singlet representation $\mathbf{1}$ here gives mesons) and $3 \times 3 \times 3 = \mathbf{1} \oplus 8 \oplus 8 \oplus 10$ (the singlet representation $\mathbf{1}$ here gives baryons) but singlet representations of $SU(3)_c$ can also appear in higher combinations (tetraquarks $qq\bar{q}\bar{q}$, pentaquarks $qqqq\bar{q}$, etc) and also in glueballs which are bound states of gluons given by the tensor product $8 \times 8 = \mathbf{1} \oplus 8 \oplus 8 \oplus 10 \oplus \bar{10} \oplus 27$.

We stress the fact that these colors have nothing to do with visual colors but they simply denote extra degrees of freedom which characterize the quarks besides their usual quantum numbers.

In a strict sense, these extra degrees of freedom are the charges acted upon by the strong nuclear force in the same way that the electric charge + (and its conjugate −) is acted upon by the electromagnetic force. This is why the electromagnetic gauge group is $U(1)_{\text{em}}$ because there is a single type of electromagnetic charge as opposed to the three types of charge of the the strong nuclear force coming from the gauge group $SU(3)_c$.

Indeed, the color charges are the sources of the strong nuclear gauge field (also called gluon fields) in the same way that the electric charge is the source of the electromagnetic field (also called photon field).

However, the photon field is chargeless, whereas the gluon field is not colorless since it carries a color and an anti-color charge and as a consequence its absorption and emission by a quark will change the color charge on the quark. This is one of the reasons why color gauge symmetry is a strongly interacting force and hence highly non-perturbative, whereas electric gauge symmetry is largely weak and perturbative. Another reason is that the photon field is not self-interacting in the abelian $U(1)_{\text{em}}$ gauge theory, whereas the gluon fields are self-interacting (cubic vertex) in the non-abelian $SU(3)_c$ gauge theory.

There are, therefore, $3 \times 3 = 9$ gluon fields *a priori* but the color trace combination $R\bar{R} + B\bar{B} + G\bar{G}$ cannot change the color charge on the quark and therefore it must be subtracted out and we are thus left with only $3 \times 3 - 1 = 8$ gluon fields. The eight gluons like the photon are neutral and massless vector particles.

Gauge symmetry (both strong nuclear and electromagnetic) are exact symmetries of nature (which translates to the masslessness of the gluons and photon) which is also, as it turns out, a requirement mathematically equivalent to the conditions of unitarity and renormalizability.

Hadrons are, however, composite objects and the fundamental degrees of freedom which appear in the QCD Lagrangian are thus the quarks and the gluons and not hadrons. As a consequence, the force between color-neutral (singlet) hadrons is a residual nuclear force which arises from the fundamental color force in the same way that Van der Waals forces between neutral atoms arise from electromagnetism between the constituents electrons.

The photons, as we have said, do not carry electric charge and therefore they create no new electromagnetic field around them (the photon gauge field is not self-interacting). The electromagnetic field is therefore the strongest around the charge which created it and then it becomes weaker as we go further away from the charge.

In contrast, the gluon gauge field is self-interacting and gluons and quarks exhibit another remarkable phenomenon (beside color confinement) called asymptotic freedom.

Gluons carry color charges and therefore they create around them a new color field corresponding to new gluons and these will create new gluons and so on and so forth. Therefore, the color field created by a color charge will be enhanced by the new color fields of the generated virtual gluons and the field will therefore tend to increase as we move away from the color charge and not decrease as in the case of the electromagnetic field. The color charge inside is thus masked by vacuum polarization and becomes effectively larger as seen from larger distances. This is the so-called vacuum polarization effect.

This can also be characterized by the running of the coupling constant $\alpha_s = g_s^2/4\pi$ of QCD with the energy scale q^2 (that is why it is running) which is given by the famous perturbative formula

$$\alpha_s(q) = \frac{12\pi}{\left(11N - 2n_f\right)\ln q^2/\Lambda_{\rm QCD}^2}. \tag{9.1}$$

In this equation N is the number of colors, i.e. $N = 3$, n_f is the number of quark flavors and $\Lambda_{\rm QCD}$ is the QCD scale at which the almost free quarks at high energies (where we can use effectively perturbation theory) become confined in bound states of the known hadrons.

Indeed, for $q^2 \gg \Lambda_{\rm QCD}^2$ the effective coupling constant $\alpha_s(q)$ becomes small, indicating that quarks and gluons are almost free and perturbation theory is a valid prescription.

Whereas when $q^2 \sim \Lambda_{\rm QCD}^2$ we see that the effective coupling constant becomes infinite indicating that bound states of hadrons have formed, quarks and any color state are confined, and perturbation theory is no longer a valid approximation.

The QCD scale is not a free parameter of the theory (like the gauge coupling constant $\alpha_s = g_s/4\pi$) but is determined from experiment to be in the range between 100 and 200 MeV.

The renormalization group evolution of the running gauge coupling constant $\alpha_s(q)$ given by the above formula does, therefore, state that the strength of the strong nuclear force (the color interaction) as measured by $\alpha_s(q)$ becomes negligible at high

energies $q^2 \gg \Lambda^2_{\rm QCD}$ and, therefore, quarks and gluons become approximately free in this regime. This is the so-called asymptotic freedom.

The electromagnetic force

The electromagnetic force is the second most universal force in nature after gravity. However, on the smaller-than-cosmic scales it is arguably the most ubiquitous force around us. A prime example is the fact that all chemical and biological processes which lie at the origin of life on Earth are electromagnetic in essence.

The quantum theory of electromagnetic interactions is QED, which is the most precise physical theory ever constructed. It is a perturbative local gauge theory based on the gauge group $U(1)_{\rm em}$ where the expansion parameter is the electromagnetic fine structure constant given in terms of the electric charge e (which is precisely the electromagnetic gauge coupling constant, i.e. $g_{\rm em} = e$) by the formula

$$\alpha_{\rm em} = \frac{e^2}{4\pi\hbar c}. \tag{9.2}$$

All elementary particles (leptons and quarks) take part in electromagnetic interactions. But leptons (in particular the electron which is the original member of this great family) can only interact via the electromagnetic force and do not feel the strong color force at all. However, the leptons can also interact via the weak radioactive force but, as we will see, the weak and the electromagnetic forces share the same origin in the electroweak theory.

The exact local gauge invariance based on the abelian gauge group $U(1)_{\rm em}$ leads to a massless gauge vector field which corresponds to the photon γ. And since the photon is neutral and does not self-interact, due to the abelian structure of its gauge group $U(1)_{\rm em}$, the properties of the QED vacuum will only contain the effect of the leptons.

In contrast, the QCD vacuum relevant for the case of the strong color force contains the effect of the quarks and the gluons since the gluons are colored fields (not neutral under $SU(3)_c$) and self-interact through a cubic vertex because of the non-abelian structure of the gauge group $SU(3)_c$.

Indeed, the QED vacuum behaves very much like a dielectric medium since it is filled with virtual electron–positron pairs, and thus, if we place a positive charge in the center, then the negative virtual electrons will be attracted to the central positive charge, whereas the positive virtual positrons will be repelled. A vacuum polarization then results which partially screens the positive charge at the center. In other words, the virtual electrons partially cancel the field of the central positive charge which then appears to be reduced. The true charge inside is only manifest at smaller distances, i.e. the charge increases for small distances corresponding to higher energies.

In the case of QCD there is a competing effect working against the above behavior due to the presence of virtual gluons which polarizes the QCD vacuum in the opposite direction, i.e. the charge decreases for small distances corresponding to higher energies and the quarks and gluons appear free. As we have said, this is

partially because the eight gluons are color charged under their non-abelian gauge group, whereas the photon is electrically neutral under its abelian gauge group. This fundamental physical difference of QCD as compared to QED is called asymptotic freedom.

The charge, and therefore the gauge coupling constant, in both theories depend on the distance, i.e. on the energy scale q^2, and the running of the electromagnetic gauge coupling constant (in the sense of the renormalization group equation) is found to be given by the equation

$$\alpha_{\text{em}}(q) = \frac{3\pi\alpha_{\text{em}}(m_e c)}{3\pi - \alpha_{\text{em}}(m_e c)\ln q^2/\left(4m_e^2 c^2\right)}. \tag{9.3}$$

The theory beyond the QED scale, which is effectively given by $\Lambda_{\text{QED}} = m_e c$, hits the so-called Landau pole and the Feynman diagrams perturbative series breaks down. QED beyond this pole is ill defined and as such QED is not UV complete. In contrast, QCD is an exact theory defined at all length scales and it is believed to be truly fundamental all thanks precisely to the phenomena of asymptotic freedom discussed above.

Weak interactions

This is another short-range (of the order of 10^{-16} cm) nuclear force which is profoundly different to the strong nuclear force. A prime example of weak interactions is β-decays and radioactivity in general. Most stable particles decay via weak interactions although some (such as π^0, η^0 and Σ^0) decay via electromagnetic interactions. All processes involving the neutrino ν (such as deuterium nucleus production in the Sun) are weak processes. All processes involving parity P violation are weak processes.

In the rest frame of a decaying particle under the influence of weak interactions the decay rate Γ is found to be proportional to $G_F^2 m^5$ where m is the mass of the particle and G_F is the dimensional Fermi constant which measures the intensity of weak interactions. Its value is given by

$$G_F = 1.16639(1) \times 10^5 (\hbar^3 c^3)\ \text{GeV}^2. \tag{9.4}$$

The lifetime of the particle is given by the inverse of the decay rate, viz $\tau = 1/\Gamma$. Typically, lifetimes of particles decaying under the weak interactions are in the range 10^3–10^{-10} s which is large compared with the lifetimes 10^{-16} s (electromagnetic) and 10^{-23}–10^{-24} s (strong). This is due precisely to the weakness of weak interactions (and hence the name) compared to electromagnetic and strong interactions.

All weak interactions found in Nature correspond to a spontaneously broken gauge group as opposed to the electromagnetic and color interactions which correspond to exact gauge groups given, respectively, by $U(1)_{\text{em}}$ and $SU(3)_c$.

Indeed, weak interactions are described by the Glashow, Salam, and Weinberg (GSW) electroweak theory in which weak and electromagnetic interactions are unified in a single gauge group given by $SU(2)_{\text{EW}} \otimes U(1)_{\text{EW}}$ which then suffers a spontaneous symmetry breaking down to the electromagnetic gauge group

$U(1)_{\rm em}$ due to the vacuum condensation of the Higgs field. The pattern of gauge symmetry breaking is then given by

$$SU(2)_{\rm EW} \otimes U(1)_{\rm EW} \longrightarrow U(1)_{\rm em}. \tag{9.5}$$

If the electroweak gauge group were not broken, then exact local symmetry requires the existence of four massless gauge vector fields. Two of these vector fields are neutral (so the photon can be identified with one of them) and two are charged.

However, from experiments we know that weak interactions are short-ranged with effective radius of the order of $R_W \sim 10^{-16}$ cm and, therefore, by the Heisenberg uncertainty principle the carriers of these interactions must be vector particles with masses of the order of $M_W \sim \hbar/R_W c$ (Yukawa force).

Thus, in order for the massless gauge vector fields of the exact electroweak gauge group $SU(2)_{\rm EW} \otimes U(1)_{\rm EW}$ to acquire mass, the gauge group must be broken spontaneously. This is normally achieved by adding to the weak Lagrangian a scalar field (the Higgs) term coupled minimally to the electroweak gauge fields. The Higgs field consists of a massless scalar (under Lorentz group, i.e. it has spin 0) doublet (under $SU(2)_{\rm EW}$) consisting of components which are neutral and charged (under $U(1)_{\rm EW}$). This Higgs field will typically acquire a non-trivial vacuum expectation value (a condensate) when choosing, in a dynamical way, a particular vacuum among the infinity of possible vacua of the scalar potential and, when it does, the symmetry $SU(2)_{\rm EW} \otimes U(1)_{\rm EW}$ gets broken with three vector fields acquiring masses (Z^0 and $W^{\pm}$) while the fourth one remains massless (the photon) corresponding, therefore, to the remaining unbroken subgroup $U(1)_{\rm em}$. This is the so-called Higgs mechanism which was suggested first by Schwinger [5] and then used in an efficient way by Higgs [6].

Thus, spontaneous symmetry breaking is indeed profoundly different from explicit symmetry breaking since it is not done by any *ad hoc* procedure. It is, in fact, a dynamical breaking caused by the vacuum condensate of the Higgs boson, i.e. when the Higgs scalar field chooses a particular direction in field space, and thus although the local weak Lagrangian is always invariant under the gauge group the vacuum is not.

The first confirmation of the GSW electroweak theory came in 1973 with the discovery of neutral vector currents, i.e. processes mediated by the neutral gauge vector particle Z^0. These massive gauge vector particles (Z^0 and $W^{\pm}$) were discovered in 1983 with masses

$$m_W = 80.41 \pm 0.10\ \text{GeV}/c^2, \quad mZ = 91.187 \pm 0.007\ \text{GeV}/c^2. \tag{9.6}$$

Other important parameters of the GSW electroweak theory are the gauge coupling constants g (of the gauge group $SU(2)_{\rm EW}$) and g' (of the gauge group $U(1)_{\rm EW}$) which are given in terms of the gauge coupling constant e (of the gauge group $U(1)_{\rm em}$) and the so-called Weinberg angle θ_W (which is an experimental parameter) by the relations

$$\tan\theta_W = \frac{g'}{g}, \quad g = e\sin\theta_W. \tag{9.7}$$

The intensity of weak interactions in the GSW electroweak theory is given in terms of g by the weak fine structure constant α_e which is defined by

$$\alpha_e = \frac{g^2}{4\pi}, \quad \frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}(\hbar c)^3}. \tag{9.8}$$

9.1.2 Elementary particles and their quantum numbers

In summary, elementary particles are of three kinds:

- **Quarks/hadrons**: These interact via all three types of fundamental interactions (color strong, electromagnetic and weak). Hadrons are colorless composite objects formed from quarks which are elementary fermionic particles with a half-integer spin. Hadrons subdivide into two major families:
  - **Baryons**: These are colorless bound states of three quarks with a half-integer spin, i.e. they are fermions. Examples include the proton p and the neutron n.
  - **Mesons**: These are colorless bound states of a quark and an anti-quark with an integer spin, i.e. they are bosons. Examples include the pions $\pi^{\pm}$ and π^0.
- **Leptons**: These interact only via the electromagnetic and weak interactions and do not participate in the nuclear strong interaction. They are elementary fermionic particles with a half-integer spin. These are the electron e^-, the muon μ^-, the tau τ^-, their neutrinos ν_e, ν_μ, ν_τ, and their anti-particles.
- **Gauge vector bosons:** These are the field quanta, i.e. the carriers, of the various forces which are massless spin 1 objects. These are the photon (electromagnetic), gluons (color nuclear), $W^{\pm}$, Z^0 (weak).

The stable particles include the electron, the positron, the proton, the anti-proton, the three neutrinos and the three anti-neutrinos and the photon.

The rest are all unstable particles which come either as metastable particles (which decay under weak or electromagnetic interactions and which may be counted with stable particles) or more importantly as resonances (which decay under strong interactions). Resonances are very short-lived particles (of the order of 10^{-23} s) which does not allow them to be registered in track detectors and which are characterized by a mass dispersion spectrum (not a definite mass value) the maximum of which defines the resonance mass.

The states of these particles are characterized by quantum numbers corresponding to external symmetries (such as energy–momentum and angular momentum) or internal symmetries (such as isospin and hypercharge). Most of the quantum numbers correspond to continuous symmetries but some of them correspond to discrete symmetries (C, P and T). Some of the quantum numbers are exact (such as the electric and the color charges) associated with underlying exact symmetries while others are inexact quantum numbers, yet they are still considered 'good quantum numbers', meaning that they are conserved quantum numbers reflecting approximate symmetries.

For example, spin 1/2 particles (baryons and leptons) are characterized by the lepton numbers L_e, L_μ, L_τ, $L = L_e + L_\mu + L_\tau$ and by the baryon number B. These are good quantum numbers in the sense that they represent accidental global $U(1)$ symmetries which might be broken by non-perturbative quantum effects.

Baryons are associated with $B = 1$, anti-baryons are associated with $B = -1$ and non-baryons are associated with $B = 0$.

Similarly, electrons are associated with the electron lepton number $L_e = L = +1$, whereas the positron is associated with the electron lepton number $L_e = L = -1$. Similarly, muons and anti-muons are associated the muon lepton numbers $L_\mu = L = +1$ and $L_\mu = L = -1$, respectively, while taus and anti-taus are associated the tau lepton numbers $L_\tau = L = +1$ and $L_\tau = L = -1$ respectively. All non-leptons are associated with the lepton numbers $L_e = L_\mu = L_\tau = L = 0$.

The lepton and baryon numbers L and B are only good quantum numbers since, although no reactions with violation of either B or L (or even the individual lepton numbers) have been observed in Nature, there does not exist a true exact local symmetry underlying their conservation.

Indeed, in the standard model both L and B are accidental global symmetries which can only be violated by non-perturbative electroweak effects (the so-called sphaleron which is very similar to the instanton and arises from the ABJ chiral anomaly) but the combination B–L remains exactly conserved even at the quantum level.

For example, lepton number conservation forbids the decay

$$\mu^- \longrightarrow e^- + \gamma.$$

While baryon number conservation forbids the decay

$$p \longrightarrow e^+ + \pi^0.$$

Hence, lepton and baryon numbers are good quantum numbers of Nature as described by the standard model. In fact, the conservation of the baryon number is what underlies the stability of matter. Indeed, the stability of the proton is due to the conservation of the baryon number in the same way that the stability of the electron is due to the conservation of the electric charge. But the electron is absolutely stable (its lifetime is actually infinite) since the conservation of the electric charge is based on an exact local symmetry principle associated with the photon while the stability of the proton is not absolute in this sense. The Super-Kamiokande limits on some decay channels of the proton are [7]:

$$\tau(p \longrightarrow e^+ + \pi^0) > 1.4 \times 10^{34} \text{ yr}, \quad \tau(p \longrightarrow \bar{\nu} + K^+) > 5.9 \times 10^{33} \text{ yr}.$$

Thus, the lifetime of the proton of 10^{35} years is much larger than the age of the Universe of 10^9 years but still it is not infinite like that of the electron. This underlies the dominance of matter over antimatter in the Universe.

In summary, matter (dominated by baryons which are much heavier than leptons) is not absolutely stable since the proton is not absolutely stable and this manifests itself as the asymmetry between matter and antimatter. The baryon number B, as well as the lepton number L in the standard model (as opposed to the electric

charge), may be violated by sphalerons. These violations are, however, very negligible but can become unsuppressed at high temperature (like in the early Universe). It is expected that any minute violation of the baryon number can lead to a matter–antimatter asymmetry which is essential to the physics of the early Universe.

This is achieved in the standard model if small neutrino masses are accommodated via the seesaw mechanism by adding right-handed neutrinos (leading also to neutrino oscillation). The lepton number becomes violated in this case (through the decays of the right-handed neutrinos into leptons) but B–L remains an exact quantum number. The sphalerons convert this lepton asymmetry into baryon asymmetry and thus we obtain the dominance of matter over antimatter. This process is called leptogenesis. Hence, baryon number violation leads to, or is equivalent to, matter–antimatter asymmetry.

Another good quantum number is the strangeness, s, which is conserved only in the strong and electromagnetic interactions but violated in weak interactions. Strange particles such as the Λ and K^0 are produced in the scattering of nucleons with pions, for example

$$\pi^- + p \longrightarrow \Lambda + K^0.$$

This is a strong reaction and, therefore, its cross-section is large (which means a large probability for the creation of these strange particles). Yet, the lifetimes of Λ and K^0 are very small, of the order of 10^{-10} s, characteristic of weak interactions. For example, through the decays

$$\Lambda \longrightarrow p + \pi, \quad K^0 \longrightarrow \pi^+ + \pi.$$

This strange behavior (production through strong but decay only through weak) was explained by Gell-Mann and Nishijima by the assumption that the strangeness quantum number s is conserved in strong interactions while it is violated in weak interactions. And thus the above strange particles cannot decay into lighter hadrons via the strong interaction and as a consequence they enjoy a longer lifetime because their decays can only proceed through the weak interaction with the rule

$$|\Delta s| = 0.$$

The strangeness quantum number is obviously additive. Other additive good quantum numbers which are conserved in the strong and electromagnetic interactions and violated in the weak interaction are the charm quantum number c and the beauty quantum number b (the top quark t is extremely heavy) with rules given by

$$|\Delta c| = |\Delta b| = 1.$$

Another very important good quantum number is the isospin I_3 which is conserved in the strong interaction. This is approximately true with the two lightest quarks u and d corresponding to an $SU(2)$ isospin symmetry but also works quite reasonably if we include the strange quark s corresponding to an $SU(3)$ isospin symmetry.

9.1.3 Yukawa theory and isospin symmetry

From the point of view of the strong interaction, the proton and the neutron are two different states of the same particle (the nucleon) since they are degenerate, i.e. they have almost the same mass ($m_n - m_p = 1.293$ MeV to be compared with the scale of nuclear physics, which is of the order of 100 MeV) and also since the mass of nuclei depends primarily on the total number of nucleons A and not separately on the numbers of protons Z and the number of neutrons $A - Z$.

This can be expressed more effectively with the (experimentally verified) principle of the nuclear force charge independence, viz

$$V_{pn} = V_{pp} = V_{nn} = V_N. \tag{9.9}$$

This result underlies the hypothesis advanced by Yukawa in 1935 that the strong interactions between nucleons are carried out by exchanging virtual particles (the pions). These carriers must be massive reflecting the short-range nature of the strong interaction in the same way that interactions between electrons are carried out by exchanging massless photons reflecting the long-range nature of the electromagnetic interaction.

Thus, by emitting and absorbing pions the proton and neutron can turn into each other. The coupling $p \leftrightarrow n$ requires the existence of the charged pions π^+ and π^-, whereas the couplings $p \leftrightarrow p$ and $n \longrightarrow n$ require the existence of the neutral pion π^0. We have then the fundamental virtual process

$$N \longrightarrow N' + \pi. \tag{9.10}$$

This can be generated by a local Lagrangian density called Yukawa coupling given by

$$\mathcal{L} = \bar{N}(x)N(x)\pi(x). \tag{9.11}$$

As an example we consider the process

$$p + n \longrightarrow p + n. \tag{9.12}$$

This proceeds via the exchange of a virtual π^+ as follows. The proton turns into neutron by emitting a virtual π^+ which is then absorbed by the initial neutron which then turns into proton.

The mass of the pion can be estimated using the Heisenberg uncertainty relation $\Delta E \Delta t \geqslant \hbar/2$ to obtain the formula

$$m_\pi \sim \frac{\hbar}{Rc}. \tag{9.13}$$

By substituting the known range of the strong force given by $R \sim 10^{-13}$ cm we obtain the estimate $m_\pi = 280 m_e$. The pions were consequently discovered in cosmic rays in 1947.

This result can also be confirmed by computing the interaction Yukawa potential V_N of two nucleons separated by a distance r. We find (see exercise 2)

$$V_N = -\frac{1}{4\pi}\frac{\exp(-m_\pi r)}{r}. \tag{9.14}$$

In summary, the proton and the neutron, from the point of view of the strong force, are only two different states of the nucleon. These two states are characterized by the so-called isospin quantum number I_3. So the proton has isospin $I_3 = 1/2$ and the neutron has isospin $I_3 = -1/2$. The nucleon isospin wavefunction will then take the form

$$\psi_N = a_p\psi_p + a_n\psi_n, \quad \psi_p = \begin{pmatrix}1\\0\end{pmatrix}, \quad \psi_n = \begin{pmatrix}0\\1\end{pmatrix}. \tag{9.15}$$

We say then that the nucleon provides the fundamental representation $I = 1/2$ of the isospin global symmetry group $SU(2)$. It is then not difficult to check for the nucleon the Gell-Mann–Nishijima formula relating the electric charge Q_3, the isospin I_3 and the baryon number B given by

$$Q = I_3 + \frac{B}{2}. \tag{9.16}$$

The isospin generators S_i^{is} (generators of $SU(2)$) must then satisfy the usual angular momentum algebra given by the commutation relations

$$\left[S_i^{\text{is}}, S_j^{\text{is}}\right] = i\epsilon_{ijk}S_k^{\text{is}}. \tag{9.17}$$

For example, in the fundamental representation $I = 1/2$ the isospin generators S_i^{is} are given by Pauli matrices, i.e. $S_i^{\text{is}} = \sigma_i/2$. These generators S_i^{is} commute with the isospin square operator $(\vec{S}^{\text{is}})^2$ given by

$$(\vec{S}^{\text{is}})^2 = \sum_{i=1}^{3}\left(S_i^{\text{is}}\right)^2. \tag{9.18}$$

A general irreducible representation I of the isospin symmetry $SU(2)$ will then be defined by the $2I + 1$ eigenstates of the two operators $(\vec{S}^{\text{is}})^2$ and S_3^{is} given, respectively, by the eigenvalues $I(I + 1)$ and $I_3 = I, I - 1, \ldots, -I$. On all these eigenstates the above Gell-Mann–Nishijima formula always holds.

However, if the strangeness quantum number is included then we can form other isospin doublets (or isodoublets) such as the K-mesons and the Ξ-baryons given, respectively, by

$$\begin{pmatrix}K^+\\K^0\end{pmatrix} \quad (s = +1), \quad \begin{pmatrix}\bar{K}^0\\K^-\end{pmatrix} \quad (s = -1), \quad \begin{pmatrix}\Xi^0\\\Xi^-\end{pmatrix} \quad (s = -2). \tag{9.19}$$

But, if strangeness is included then the Gell-Mann–Nishijima formula is changed as

$$Q = I_3 + \frac{Y}{2}, \quad Y = B + s. \tag{9.20}$$

The new quantum number Y is called hypercharge and is also separately conserved in the strong force like the isospin I_3. Although the conservation of I_3 and Y is only

approximate in the strong force, their above combination is exactly conserved since the electric charge Q is conserved in all interactions. Viewing the Gell-Mann–Nishijima formula from the perspective of the weak interactions where the electric charge is also exactly conserved we observe that I_3 corresponds to the weak isospin group $SU(2)_{\rm EW}$, whereas Y corresponds to the weak hypercharge group $U(1)_{\rm EW}$, which are both broken down to the electromagnetic group $U(1)_{\rm em}$ generated by the electric charge Q.

In the Yukawa hypothesis the strong interactions between nucleons are (approximately) mediated by pions exchange. As a consequence, the nucleon–pion interaction $N \longrightarrow N' + \pi$ must be invariant under the $SU(2)$ isospin group. It follows, therefore, that pions must have isospin $I = 1$ or $I = 0$. But, since there are three degenerate pions, we conclude that the isospin of the pion is $I = 1$ where the eigenstate $I_3 = +1$ is π^+, the eigenstate $I_3 = 0$ is π^0 and the eigenstate $I_3 = -1$ is π^-. In other words, the pions provide the $I = 1$ vector representation of the $SU(2)$ isospin group. This irreducible representation is given explicitly by the matrices

$$I_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad I_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{9.21}$$

9.1.4 Eightfold way and quark model

If strangeness s is added to the isospin I_3 then the global $SU(2)$ isospin symmetry is enlarged to a global flavor symmetry group $SU(3)$ which contains the original isospin transformations as a subgroup. As we will see, this larger symmetry group corresponds to the three lightest quarks u, d and s although the mass of the strange quark s is significantly larger than the masses of the up and down quarks u and d (recall that $m_s = 150$ MeV, $m_d = 8$ MeV and $m_u = 5$ MeV).

Thus, the $SU(3)$ flavor symmetry is even less exact than the approximate $SU(2)$ isospin symmetry. Yet, many hadrons are observed to form unitary irreducible representations (also called unitary multiplets) of the $SU(3)$ flavor symmetry consisting in many cases of eight elements, i.e. 8-plets or octets, and hence this model is called the 'eightfold way' (which is due originally to Neeman and Gell-Mann in 1961). The particles entering a given multiplet can be viewed as different states of the same particle which are degenerated in the electric charge, spin, parity and baryon number.

Later it was appreciated that the eightfold way is a very natural consequence of the quark model, i.e. of the fact that hadrons are bound states of three quarks or of a quark and anti-quark.

The flavor group $SU(3)$ actually contains three independent $SU(2)$'s (one of them is the isospin group) which can be exhibited as follows.

In the strong and electromagnetic interactions both strangeness and isospin are conserved quantum numbers, i.e. $\Delta s = \Delta I_3 = 0$. On the other hand, in semi-leptonic weak interactions (the final state is formed out of leptons and hadrons) we have $\Delta s = 0$ and $|\Delta I_3| = 1$ or $|\Delta s| = 1$ and $|\Delta I_3 = 1/2|$, whereas for non-leptonic weak

interactions (the final state is formed out of hadrons) we have $|\Delta s| = 1$ and $|\Delta I_3 = 1/2|$. Thus, the changes in the strangeness and the isospin quantum numbers are correlated in the following way

$$\begin{aligned} \Delta s = 0, \quad |\Delta I_3| &= 1, 0 \\ |\Delta s| = 1, \quad |\Delta I_3| &= \frac{1}{2}. \end{aligned} \tag{9.22}$$

We plot states and multiplets in the plane (I_3, s) as shown on figure 9.1. Obviously, the isospin raising and lowering generators $S_{\pm}^{\text{is}}$ will change the isospin value as $\Delta I_3 = \pm 1$ on the I_3-axis, i.e. they also correspond to $\Delta s = 0$. The isospin raising and lowering operators give then the states on the I_3-axis defined by the vectors $S_{\pm}^{\text{is}}$, i.e. with the points $(I_3, s) = (\pm 1, 0)$. This corresponds to the $SU(2)$ isospin subgroup inside $SU(3)$.

We construct a different $SU(2)$ subgroup of $SU(3)$ by introducing the so-called U-spin with generators S_i^U also satisfying the angular momentum algebra (9.17). In some sense the U-spin is a combination of the isospin and the strangeness. The U-spin raising and lowering generators $S_{\pm}^U$ will change the isospin values as $\Delta I_3 = \mp 1/2$ while they change the strangeness as $\Delta s = \pm 1$. The U-spin raising and lowering operators yield then the states in the (I_3, s) on the axis defined by the vectors $S_{\pm}^U$ which are given by the points $(I_3, s) = (\mp 1/2, \pm 1)$.

A third $SU(2)$ subgroup of $SU(3)$ is constructed by introducing the so-called V-spin with generators S_i^V satisfying also the angular momentum algebra (9.17). The V-spin raising and lowering generators $S_{\pm}^V$ will change the isospin values as $\Delta I_3 = \pm 1/2$ while they change the strangeness as $\Delta s = \pm 1$. The V-spin raising and lowering operators yield then the states in the (I_3, s) plane on the axis defined by the vectors $S_{\pm}^V$ which are given by the points $(I_3, s) = (\pm 1/2, s = \pm 1)$.

The $SU(3)$ generators consist, therefore, of the six raising and lowering generators ($S_{\pm}^{\text{is}}$, $S_{\pm}^U$ and $S_{\pm}^V$) together with the isospin generator S_3^{is} (corresponding to the isospin quantum number I_3) and the hypercharge generator Y (corresponding to the strangeness quantum number s since $Y = s + B$).

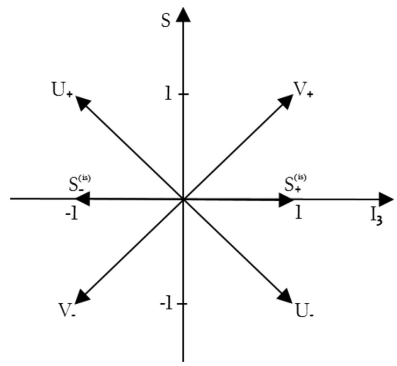

Figure 9.1. The isospin, the U-spin and the V-spin in the (I_3, s) plane.

In general, the Lie algebra of $SU(N)$ consists of $N^2 - 1$ generators where the dimension of the Cartan subalgebra (a maximally commuting subalgbera) defines the rank of the group given by $N - 1$. For $SU(3)$ the Cartan subalgebra is $\{S_3^{\text{is}},\ Y\}$ (the rank is therefore equal to 2) where the eigenvalues label the states of the unitary multiplets, i.e. irreducible representations of $SU(3)$. This is the mathematical reason why we have drawn the states in the plane $(I_3,\ s)$ in the first place.

Some of the unitary multiplets under $SU(3)$ include:

- The eight fermion baryons with spin $J = 1/2$, parity $P = +1$ and baryon number $B = +1$: $(p,\ n)$ $(Y = +1)$, Λ^0 $(Y = 0)$, $(\Sigma^+,\ \Sigma^0,\ \Sigma^-)$ $(Y = -1)$, $(\Xi^0,\ \Xi^-)$ $(Y = -2)$.
- The eight pseudo-scalar mesons with spin $J = 0$, parity $P = -1$ and baryon number $B = 0$: $(\pi^+,\ \pi^0,\ \pi^-)$ $(Y = 0)$, $(K^+,\ K^0)$ $(Y = +1)$, $(\bar{K}^-,\ \bar{K}^0)$ $(Y = -1)$, η $(Y = 0)$.
- The eight vector mesons with spin $J = 1$, parity $P = -1$ and baryon number $B = 0$: $(\rho^+,\ \rho^0,\ \rho^-)$ $(Y = 0)$, $(K^{*+},\ K^{*0})$ $(Y = +1)$, $(\bar{K}^{*-},\ \bar{K}^{*0})$ $(Y = -1)$, ω^0 $(Y = 0)$.
- The nine fermion baryons with spin $J = 3/2$, parity $P = +1$ and baryon number $B = +1$: $(\Delta^{++},\ \Delta^+,\ \Delta^0,\ \Delta^-)$ $(Y = +1)$, $(\Sigma^{*+},\ \Sigma^{*0},\ \Sigma^{*-})$ $(Y = 0)$, $(\Xi^{*0},\ \Xi^{*-})$ $(Y = -1)$.
- The pseudo-scalar meson singlet with spin $J = 0$, parity $P = -1$ and baryon number $B = 0$: η' $(Y = 0)$.
- The vector meson singlet with spin $J = 1$, parity $P = -1$ and baryon number $B = 0$: φ^0 $(Y = 0)$.

The eightfold way structure of hadrons can be explained by the quark model (proposed by Gell-Mann and Zweig in 1964). Thus, all hadrons are constructed from three elementary spin one-half particles (these are the u, d and s quarks) which together provide the fundamental representation $\mathbf{3}$ of $SU(3)$, whereas the corresponding anti-quarks provide the anti-fundamental representation $\bar{\mathbf{3}}$ of $SU(3)$. We have then the fundamental triplets

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad \bar{q} = (\bar{u},\ \bar{d},\ \bar{s}). \tag{9.23}$$

The isospin quantum numbers are $I_3(u) = 1/2$, $I_3(d) = -1/2$ and $I_3(s) = 0$, the strangeness quantum numbers are $s(u) = s(d) = 0$ and $s(s) = -1$, whereas the baryon numbers are $B(u) = B(d) = B(s) = 1/3$. Then, from the Gell-Mann–Nishijima formula we obtain the electric charges $Q(u) = 2/3$, $Q(d) = Q(s) = -1/3$.

The mesons, since they are bound states of a quark and an anti-quark, are obtained from the tensor product $\mathbf{3} \otimes \bar{\mathbf{3}}$ which decomposes under the action of $SU(3)$ as

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}. \tag{9.24}$$

The $\mathbf{1}$ are precisely the pseudo-scalar meson singlet η' and the vector meson singlet φ^0, whereas the $\mathbf{8}$ are the meson octets $J^P = 0^-$ and $J^P = 1^-$ which only differ by the values of their spin, i.e. one octet is pseudo-scalar and the other is vector. See figure 9.2.

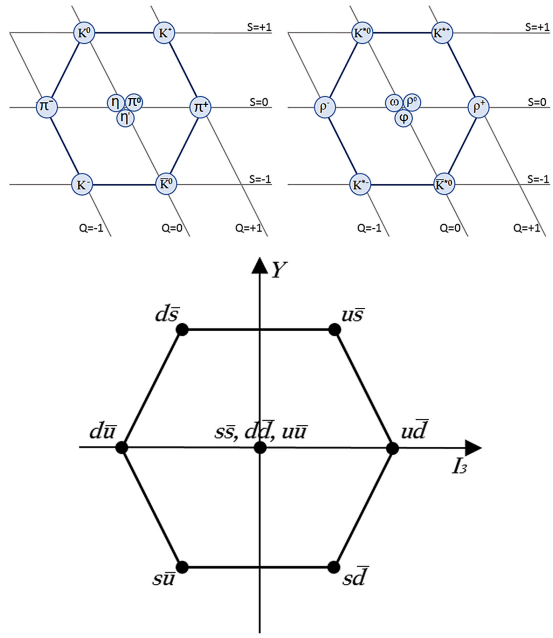

Figure 9.2. The pseudo-scalar and vector meson octets $\mathbf{8}^0$ and $\mathbf{8}^1$ and their quark contents from $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$.

On the other hand, the baryons are bound states of three quarks which are obtained from the tensor product $\mathbf{3} \times \mathbf{3} \otimes \mathbf{3}$ which decomposes under the action of $SU(3)$ as

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}. \tag{9.25}$$

The decouplet is symmetric in flavor, the singlet is antisymmetric in flavor, whereas the octets have mixed symmetry. The full wave-function of any baryon (since it is a fermion) must be antisymmetric under the exchange of any two quarks. In fact, it must be antisymmetric in color and symmetric in flavor, spin and space together. The ground state (low-laying states) baryons correspond to $l = 0$, i.e. the space wavefunction is symmetric and hence the spin wavefunction must also be symmetric.

As an example, we note that the $\frac{3}{2}^+$ baryon formed from three up-quarks (and hence it has a symmetric flavor wavefunction by construction) has a vanishing angular momentum (and hence it has a symmetric space wavefunction). But it also has parallel spins, i.e. the spin wavefunction is completely symmetric. This state cannot, therefore, have an antisymmetric total wavefunction unless there was an extra hidden quantum number which is the so-called color charge.

The symmetric wavefunction in flavor and spin corresponding to the low-laying baryons can be obtained by considering the spin-flavor symmetry group $SU(6)$ (since each quark can be in three flavor states and two spin states). The tensor product $\mathbf{6} \times \mathbf{6} \otimes \mathbf{6}$ decomposes under the action of $SU(6)$ as

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56}_S \oplus \mathbf{70}_M \oplus \mathbf{70}_M \oplus \mathbf{20}_A. \tag{9.26}$$

The low-lying baryons are found in the symmetric multiplet $\mathbf{56}_S$ which decomposes under the $SU(3)$ flavor group as

$$\mathbf{56}_S = \mathbf{10}^{\frac{3}{2}} \oplus \mathbf{8}^{\frac{1}{2}}. \tag{9.27}$$

The $\mathbf{8}^{\frac{1}{2}}$ corresponds to the baryon octet $J^P = \frac{1}{2}^-$ whereas the nine baryons corresponding to $J^P = \frac{3}{2}^+$ can only be placed in the decouplet $\mathbf{10}^{\frac{3}{2}}$. Hence, a particle with spin $J = 3/2$, parity $P = +1$, electric charge $Q = -1$ and strangeness $s = -3$ is missing from the $\frac{3}{2}^+$ spectrum given above. See figure 9.3.

Furthermore, it is observed that the difference of masses between adjacent isomultiplets in the $\frac{3}{2}^+$ spectrum is almost constant given by 146 MeV. This allows us to also predict the mass of the tenth missing particle to be given by $m_\Omega = 1676$ MeV.

This missing particle, called later Ω^-, cannot decay via the strong interaction because of the conservation of strangeness and can only decay weakly yielding a long lifetime of the order of 10^{-10} s. This particle was discovered in 1964 at the Brookhaven accelerator which was a major confirmation of $SU(3)$ flavor symmetry.

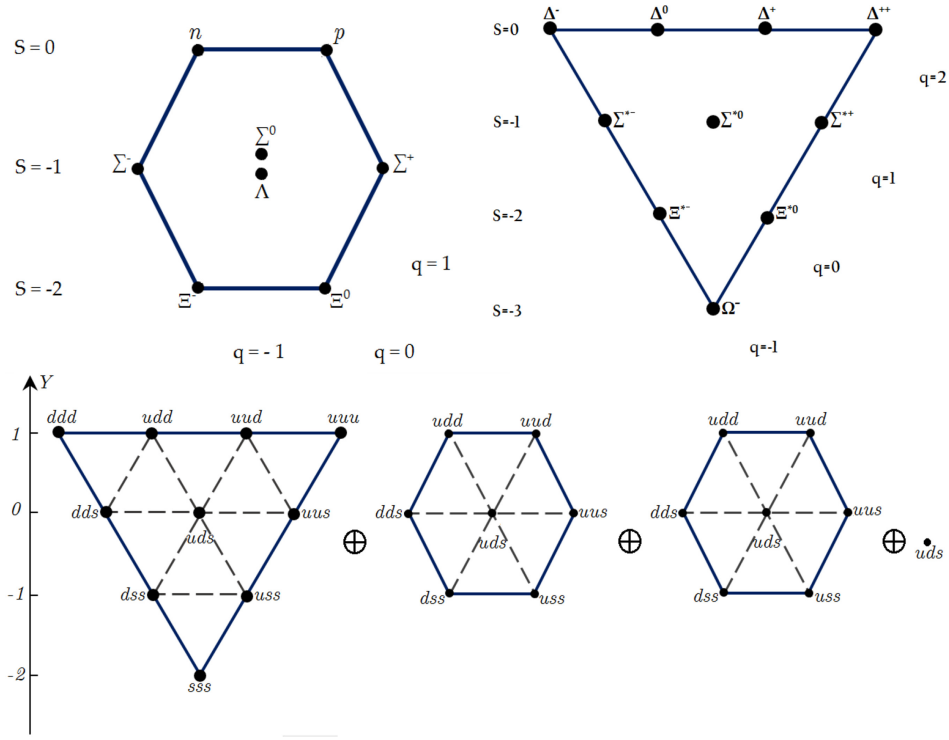

Figure 9.3. The baryon octet $\mathbf{8}^{\frac{1}{2}}$ and the baryon decouplet $\mathbf{10}^{\frac{3}{2}}$ and their quark contents from $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$.

9.1.5 Neutrino oscillations

The three neutrinos ν_e, ν_μ and ν_τ are in fact eigenstates of the weak interaction but they are not mass or physical eigenstates due to the small non-zero mass of the right-handed neutrinos. This leads to mixing, individual lepton numbers violation and neutrino oscillations. The mechanism of neutrino mixing will be described in great detail later but now we will concentrate on the phenomena of neutrino oscillations, which is a very natural consequence of mixing.

We will suppose for simplicity that we have two flavors of neutrinos: ν_e and ν_μ. The mixing of neutrinos means, in particular, that the physical or mass neutrino state is actually given by a linear combination of the form

$$|\nu\rangle = \alpha|\nu_e\rangle + \beta|\nu_\mu\rangle. \tag{9.28}$$

The states $|\nu_e\rangle$ and $|\nu_\mu\rangle$ are the weak eigenstates satisfying $\langle\nu_e|\nu_\mu\rangle = 0$. The mass eigenstates will be denoted by $|\nu_1\rangle$ and $|\nu_2\rangle$ with mass eigenvalues m_1 and m_2, respectively, satisfying also $\langle\nu_1|\nu_2\rangle = 0$. The mixing is given by the linear combination (where θ is the mixing angle)

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle, \quad |\nu_\mu\rangle = -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle. \tag{9.29}$$

The mass eigenstates (being the eigenstates of the Hamiltonian) are obtained from solving Schrödinger's equation starting from some initial states $\nu_1(0)$ and $\nu_2(0)$, viz

$$\begin{aligned} \nu_1 \equiv \nu_1(t) &= \exp(-iE_1t/\hbar)\nu_1(0), \\ \nu_2 \equiv \nu_2(t) &= \exp(-iE_2t/\hbar)\nu_2(0). \end{aligned} \tag{9.30}$$

At time $t = 0$ what is being produced via some weak interaction is actually $\nu_e(0)$ and/or $\nu_\mu(0)$ with some 3-momentum p. Let us assume for simplicity that ν_μ is being produced in the pion decay

$$\pi^+ \longrightarrow \mu^+ + \nu_\mu. \tag{9.31}$$

The initial states $\nu_1(0)$ and $\nu_2(0)$ are then determined in terms of $\nu_e(0)$ and $\nu_\mu(0)$ by the relations (9.29). They also have 3-momentum p and since the neutrino masses are very small their energies are given by

$$\frac{E_i}{c} \sim p + \frac{m_i^2c^2}{p} + \cdot s \tag{9.32}$$

We can show that at time t the weak neutrino eigenstates take the form (with $\Delta m^2 = m_2^2 - m_1^2$ and $\delta = t(cp + m_1^2c^3/2p)/\hbar$)

$$\begin{aligned} \nu_\mu(t) &= \exp(-i\delta)\left(\nu_1(0)\cos\theta + \exp\left(-it\frac{\Delta m^2c^3}{2\hbar p}\right)\nu_2(0)\sin\theta\right) \\ \nu_e(t) &= \exp(-i\delta)\left(-\nu_1(0)\sin\theta + \exp\left(-it\frac{\Delta m^2c^3}{2\hbar p}\right)\nu_2(0)\cos\theta\right). \end{aligned} \tag{9.33}$$

The probability amplitude that the muon neutrino produced at $t = 0$ becomes at time t an electron neutrino after propagating a distance $L = ct = Et/p$ is given by $\langle \nu_e(t)|\nu_\mu\rangle$, whereas the probability amplitude that the muon neutrino produced at $t = 0$ remains at time t a muon neutrino after propagating a distance $L = ct = Et/p$ is given by $\langle \nu_\mu(t)|\nu_\mu(0)\rangle$. We compute immediately the probabilities

$$P(\nu_\mu \longrightarrow \nu_e,\, L) = |\langle \nu_e(t)|\nu_\mu(0)\rangle|^2 = \sin^2 2\theta \sin^2 \frac{\Delta m^2 c^3 L}{4\hbar E}. \tag{9.34}$$

$$P(\nu_\mu \longrightarrow \nu_\mu,\, L) = |\langle \nu_\mu(t)|\nu_\mu(0)\rangle|^2 = 1 - \sin^2 2\theta \sin^2 \frac{\Delta m^2 c^3 L}{4\hbar E}. \tag{9.35}$$

We stress here the fact that the neutrinos are not in fact decaying but they are simply oscillating back and forth between flavors as they propagate in space.

An example is given by solar neutrinos. The number of solar electron neutrinos, which are produced in the center of the Sun, detected at the Earth's surface is observed to be half of that expected. This is explained by the fact that these solar electron neutrinos are oscillating to muon neutrinos with probability

$$\begin{aligned} P(\nu_e \longrightarrow \nu_\mu,\, L) &= \sin^2 2\theta \sin^2 \frac{\Delta m^2 c^3 L}{4\hbar E} \\ &= 2 \sin^2 \theta \cos^2 \theta \left(1 - \cos \frac{\Delta m^2 c^3 L}{4\hbar E}\right) \\ &\simeq 2 \sin^2 \theta \cos^2 \theta \\ &\simeq \frac{1}{2}: \quad \text{observation}. \end{aligned} \tag{9.36}$$

This fixes the mixing angle θ to be around $\pi/4$ corresponding to $\sin^2 2\theta \simeq 1$. A more precise measurement gives $\sin^2 2\theta \simeq 0.82$.

Another example is given by cosmic rays neutrinos. The basic decay is given by pions decaying to muons and the subsequent decay of muons to electrons given by the process

$$\pi^+ \longrightarrow \mu^+ + \nu_\mu \longrightarrow e^+ + \nu_e + \bar{\nu}_\mu + \nu_\mu. \tag{9.37}$$

Thus, it is evident that at the Earth's surface we expect to observe twice as many cosmic muon neutrinos as cosmic electron neutrinos. But the observation reveals that both fluxes are virtually the same, indicating that muon neutrinos have oscillated not to electron neutrinos but to tau neutrinos which involve another mixing angle.

9.2 The standard model

9.2.1 The Glashow–Weinberg–Salam electroweak theory and lepton sector

The GWS electroweak theory, which is the first component of the standard model, is a spontaneously broken local gauge theory [8–10].

It is based on the gauge group $SU(2)_L \times U(1)_Y$ where $SU(2)_L$ is the group of electroweak isospin and $U(1)_Y$ is the group of electroweak hypercharge.

The Lagrangian of the GWS electroweak theory is further constrained by the two fundamental requirements of (1) relativistic Lorentz covariance and (2) renormalizability.

This theory contains vector fields (carriers of the interactions) which transform in the adjoint representation of the gauge group as well as scalar (Higgs) and spinor (matter) fields. The renormalizability requirement fixes the mass dimensions of scalars, spinors and vector fields to be 1, 3/2 and 1, respectively, with a corresponding Lagrangian which does not contain any parameters of negative mass dimension.

The massless gauge vector fields associated with the non-abelian gauge group $SU(2)_L$ are denoted by W^1, W^2 and W^3 with gauge coupling constant g, whereas the massless gauge vector field associated with the abelian gauge group $U(1)_Y$, is denoted by B with gauge coupling constant g'. The pure Lagrangian density involving these gauge fields is given by the usual formula

$$\mathcal{L}_{\text{pure gauge}} = -\frac{1}{4} G^r_{\mu\nu} G^{r\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}. \tag{9.38}$$

The $G^r_{\mu\nu}$ and $B_{\mu\nu}$ are the field strengths associated with the non-abelian gauge field W^r_μ and the abelian gauge field B_μ respectively, given by

$$G^r_{\mu\nu} = \partial_\mu W^r_\nu - \partial_\nu W^r_\mu + g\epsilon_{rst} W^s_\mu W^t_\nu, \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{9.39}$$

The electroweak isospin generator S^W_3 (which generates together with S^W_1 and S^W_2 the $SU(2)_L$ gauge transformations) and the electroweak hypercharge generator Y^W (which generates $U(1)_Y$ gauge transformations) are related by the Gell-Mann–Nishijima formula given by

$$Q = S^W_3 + \frac{Y^W}{2}. \tag{9.40}$$

The electromagnetic charge generator Q generates the unbroken subgroup $U(1)_{\text{em}}$ of electromagnetic gauge transformations as we will see. From the knowledge of the electric charge and the third component of the isospin we can always calculate the value of the hypercharge.

Next, we fix the matter content of the electroweak part of the standard model. We start with leptons which is the more involved part.

It is well established that weak interactions violate parity P and thus left-handed and right-handed leptons should enter the theory differently. We observe in nature three families of left-handed leptons which transform in the fundamental representation under the isospin group $SU(2)_L$, i.e. they are doublets with isospin $S^W = 1/2$, viz

$$\psi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \quad S^W = \frac{1}{2}, \quad Y^W = -1. \tag{9.41}$$

The left-handed and right-handed parts of a spinor ψ are defined, as usual, by the relations

$$\psi_L = \frac{1-\gamma_5}{2}\psi, \quad \psi_R = \frac{1+\gamma_5}{2}\psi, \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{9.42}$$

On the other hand, the right-handed leptons (as a very good approximation we assume that neutrinos are truly massless and that there are no right-handed neutrinos) transform as singlets under $SU(2)_L$, viz

$$\psi_R = e_R, \mu_R, \tau_R, \quad S^W = 0, \quad Y^W = -2. \tag{9.43}$$

The facts that left-handed leptons transform in the fundamental representation and right-handed leptons transform as singlet, guarantees that the theory will be P-violating. The left-handed and right-handed leptons transform, respectively, under the $SU(2)_L \times U(1)_Y$ electroweak gauge transformations as

$$\begin{aligned} \psi_L &\longrightarrow \psi_L' = \exp\left(i\left(\vec{\alpha}\cdot\vec{S}^W + \beta\frac{Y^W}{2}\right)\right)\psi_L, \\ \psi_R &\longrightarrow \psi_R' = \exp\left(i\beta\frac{Y^W}{2}\right)\psi_R, \end{aligned} \tag{9.44}$$

where $\vec{\alpha}$ and β are local functions on spacetime, viz $\vec{\alpha} \equiv \vec{\alpha}(x)$ and $\beta \equiv \beta(x)$.

In order to write down couplings between the gauge fields W_μ^r and B_μ and the leptonic matter fields $\psi_{L,\,R}$ we adopt the canonical prescription of minimal coupling which will also guarantee for us renormalizability of the resulting theory. Thus, we introduce a covariant derivative as usual by the relation

$$\nabla_\mu = \partial_\mu - ig\vec{W}_\mu\cdot\vec{S}^W - ig'B_\mu\frac{Y^W}{2}. \tag{9.45}$$

The left-handed leptons are in the fundamental representation of $SU(2)_L$ which means that the generators S_r^W are the Pauli matrices, i.e. $S_r^W = \tau_r/2$ while under $U(1)_Y$ they have a hypercharge $Y^W = -1$. The action of the covariant derivative on the left-handed leptons is, therefore, given by

$$\nabla_\mu\psi_L = \left(\partial_\mu - igW_\mu^r\frac{\tau_r}{2} + i\frac{g'}{2}B_\mu\right)\psi_L. \tag{9.46}$$

Similarly, the right-handed leptons are singlet under $SU(2)_L$ which means that the generators S_r^W are simply 0 while under $U(1)_Y$ they have a hypercharge $Y^W = -2$. The action of the covariant derivative on the right-handed leptons is, therefore, given by

$$\nabla_\mu\psi_R = (\partial_\mu + ig'B_\mu)\psi_R. \tag{9.47}$$

A gauge invariant (under $SU(2)_L \times U(1)_Y$) kinetic Lagrangian density for the leptonic matter fields can then be written directly as

$$\mathcal{L}_{\text{kinetic lepton}} = \bar{\psi}_L i\gamma^\mu\nabla_\mu\psi_L + \bar{\psi}_R i\gamma^\mu\nabla_\mu\psi_R. \tag{9.48}$$

The leptons are observed to be massive in nature including the neutrinos. However, the canonical lepton mass term

$$m\bar{\psi}_L\psi_R + m\bar{\psi}_R\psi_L \tag{9.49}$$

is not gauge invariant under $SU(2)_L \times U(1)_Y$ since the left-handed and right-handed leptons transform differently as we have seen. Hence, this mass term is not allowed in the Lagrangian.

The lepton masses in the standard model are generated via the Higgs mechanism [6] in which the symmetry $SU(2)_L \times U(1)_Y$ is spontaneously broken down to the electromagnetic symmetry $U(1)_{\text{em}}$ which is really the symmetry observed in nature. In this way we also obtain masses for the weak carriers $W^{\pm}$ and Z^0 while the photon γ remains massless.

The Higgs mechanism underlying spontaneous symmetry breaking requires the introduction of a scalar field ϕ which transforms in the fundamental representation of $SU(2)_L$, i.e. as a doublet with isospin $S^W = 1/2$, with a weak hypercharge $Y^W = +1$. We have then

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad S^W = \frac{1}{2}, \quad Y^W = +1. \tag{9.50}$$

Therefore, the covariant derivative acting on the Higgs field is given by

$$\nabla_\mu \phi = \left(\partial_\mu - igW_\mu^{\ r}\frac{\tau_r}{2} - i\frac{g'}{2}B_\mu\right)\phi. \tag{9.51}$$

The Lagrangian of the scalar fields is given by an $SU(2)_L \times U(1)_Y$-invariant scalar ϕ^4-theory, viz

$$\mathcal{L}_{\text{Higgs}} = (\nabla_\mu\phi)^\dagger\nabla_\mu\phi - V(\phi),\ V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \tag{9.52}$$

The coupling of the scalar fields and lepton fields is fixed by the requirements of gauge invariance and renormalizability and, therefore, it can only be the gauge invariant Yukawa coupling given by

$$\mathcal{L}_{\text{Yukawa}} = -\sum_{l=e,\mu,\tau} f_l\bar{\psi}_{lL}\phi\psi_{lR} + \text{h.c.} \tag{9.53}$$

The f_l are the Yukawa parameters.

The total Lagrangian for the GWS electroweak theory in the lepton sector is given by the sum of the Lagrangians (9.38), (9.48), (9.52) and (9.53).

9.2.2 Higgs mechanism and spontaneous symmetry breaking

The GWS electroweak theory is a gauge theory which requires gauge fixing. For example, we can perform a local $SU(2)_L$ gauge transformation which sets the upper component of the scalar field to 0 while it sets the lower component equal to a real number which can be performed explicitly as follows. The scalar field ϕ can be parametrized as

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} i\phi_1 + \phi_2 \\ -i\phi_3 + H \end{pmatrix} = \phi_0 + \phi', \quad \phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{9.54}$$

The real constant v is chosen such that

$$\langle \phi' \rangle = 0. \tag{9.55}$$

This means that, at the leading order in perturbation theory, ϕ_0 is the minimum of the potential $V(\phi)$ and hence

$$v^2 = -\frac{\mu^2}{2\lambda}. \tag{9.56}$$

In other words, we must have $\mu^2 < 0$ and $\lambda > 0$ (second-order phase transition). The doublet ϕ_0 is, therefore, the vacuum expectation value of the scalar field ϕ, viz

$$\langle \phi \rangle = \phi_0. \tag{9.57}$$

The doublet ϕ_0 has quantum numbers $S_3^W = -1/2$ and $Y^W = 1$ and thus it transforms non-trivially under $SU(2)_L$ and $U(1)_Y$ respectively. However, we observe that this expectation value is invariant under a residual $U(1)_{\text{em}}$ generated precisely by the electric charge Q defined in equation (9.40). Indeed, we have

$$Q\phi_0 = 0 \rightarrow \exp(i\alpha Q)\phi_0 = \phi_0. \tag{9.58}$$

In other words, even after the scalar field ϕ acquires a vacuum expectation value (a vev) v the theory is still invariant under the gauge transformations generated by Q, although it is not separately invariant under $SU(2)_L$ and $U(1)_Y$ gauge transformations, and hence it will contain a massless gauge boson (the photon) corresponding to the linear combination of the generators S_3^W and Y^W given by Q. From this it is very clear that the pattern of breaking is indeed given by

$$SU(2)_L \times U(1)_Y \longrightarrow U(1)_{\text{em}}. \tag{9.59}$$

The above parametrization (9.54) can also be rewritten (in the lowest order in the powers of ϕ_r) as

$$\phi = \exp(i\tau_r\phi_r/v)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H \end{pmatrix}. \tag{9.60}$$

We can then perform an $SU(2)_L$ gauge transformation given explicitly by $U = \exp(-i\tau_r\phi_r/v)$ to bring this configuration into the form

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H \end{pmatrix}. \tag{9.61}$$

The field H is precisely the Higgs field.

By substituting equation (9.61) into the kinetic term of the scalar fields given by the first term in equation (9.52) we can exhibit the mass terms of the three gauge

bosons associated with the three gauge symmetries which are spontaneously broken by the vacuum condensation ϕ_0. We have

$$\begin{aligned}(\nabla^\mu\phi)^\dagger(\nabla_\mu\phi)|_{\phi=\phi_0} &= \frac{g^2v^2}{8}\left(W^{\mu 1}W^1_\mu + W^{\mu 2}W^2_\mu\right) \\ &\quad + \frac{v^2}{8}(gW^{\mu 3} - g'B^\mu)\left(gW^{\ 3}_\mu - g'B_\mu\right) \\ &= \frac{g^2v^2}{8}\left(W^{\mu 1}W^1_\mu + W^{\mu 2}W^2_\mu\right) + \frac{(g^2+g'^2)v^2}{8}Z^\mu Z_\mu.\end{aligned} \tag{9.62}$$

The neutral vector boson field Z_μ is clearly proportional to the combination $gW^{\ 3}_\mu - g'B_\mu$. More precisely, the neutral vector boson field Z_μ and the photon field A_μ are defined in terms of W^3_μ and B_μ by the relation

$$\begin{aligned}\begin{pmatrix}Z_\mu\\ A_\mu\end{pmatrix} &= \frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix}g & g'\\ g' & g\end{pmatrix}\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix} \\ &= \begin{pmatrix}\cos\theta_W & -\sin\theta_W\\ \sin\theta_W & \cos\theta_W\end{pmatrix}\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix}.\end{aligned} \tag{9.63}$$

The Weinberg angle θ_W is defined by

$$\tan\theta_W = \frac{g'}{g}. \tag{9.64}$$

From the mass term (9.62) we can read the mass values of the vector bosons $W^\pm_\mu = (W^1_\mu \mp iW^2_\mu)/\sqrt{2}$ and Z_μ to be given by

$$m_W = \frac{gv}{2}, \quad m_Z = \frac{v\sqrt{g^2+g'^2}}{2}. \tag{9.65}$$

Thus, the massless Nambu–Goldstone modes $\phi_{1,2,3}$ associated with the spontaneously broken gauge symmetries are eaten, i.e. they can be eliminated by gauge symmetry as shown by equation (9.61), by the corresponding three vector bosons ($W^\pm_\mu$ and Z_μ) which become, as a consequence, massive with three extra degrees of freedom, i.e. their extra longitudinal components. The total number of degrees of freedom thus remains fixed. The remaining unbroken gauge symmetry (associated with Q) yields a massless vector boson (the photon) which is then purely transversal. The fourth scalar field is precisely the Higgs field H which can be shown, by substitution into the potential $V(\phi)$, to have a mass equal to

$$m^2_H = 2\lambda v^2. \tag{9.66}$$

Also, by substitution into the Yukawa coupling (9.53) we get the mass for leptons

$$m_l = \frac{f_l v}{\sqrt{2}}. \tag{9.67}$$

Thus, the neutrinos do not acquire mass in this Higgs mechanism.

The interaction between the Higgs and the vector bosons is also obtained by substituting the configuration (9.61) into the kinetic term of the scalar fields given by the first term in equation (9.52). We get

$$\frac{g^2}{4}\left[W_\mu^* W^\mu + \frac{1}{2\cos^2\theta_W} Z_\mu Z^\mu\right][H^2 + 2vH]. \tag{9.68}$$

Thus, the coupling constants between the Higgs field and the vector bosons W and Z are given by gm_W and gm_Z, respectively, which are much larger than the coupling constant between the Higgs field and the leptons.

Finally, the interaction between the Higgs and the leptons is obtained by substituting the configuration (9.61) into the leptonic kinetic term given by the Lagrangian density (9.48). We get, after some calculation, the electromagnetic interaction, the charged current weak interaction, and the neutral current weak interaction given by the terms

$$-e\bar{l}\gamma^\mu l A_\mu, \quad |e| = \frac{gg'}{\sqrt{g^2 + g'^2}} \tag{9.69}$$

$$\frac{g}{2\sqrt{2}}\left[\bar{\nu}_l\gamma^\mu(1-\gamma_5)lW_\mu^* + \bar{l}\gamma^\mu(1-\gamma_5)\nu_l W_\mu\right] \tag{9.70}$$

$$\frac{g}{4\cos\theta_W}\left[\bar{\nu}_l\gamma^\mu(1-\gamma_5)\nu_l + \bar{l}\gamma^\mu(4\sin^2\theta_W - 1 + \gamma_5)l\right]Z_\mu. \tag{9.71}$$

9.2.3 Majorana fermions

The neutrino is truly a special particle in Nature as described by the standard model. For example, it is widely believed that the neutrino is not a Dirac spinor like the electron but rather it is a Majorana spinor which means, in particular, that it is its own anti-particle.

A Dirac spinor is, in some sense, the canonical spinor in introductory quantum field theory although it does not really provide an irreducible representation of the Lorentz group. Indeed, a 4-component Dirac spinor ψ is a reducible representation of the Lorentz group given by the direct sum of the two irreducible representations (1/2, 0) and (0, 1/2) of the Lorentz group corresponding to the 2-component Weyl spinors χ and $\bar{\eta}$ respectively. We can then write explicitly

$$\psi = \begin{pmatrix} \chi \\ \bar{\eta} \end{pmatrix}. \tag{9.72}$$

The Lorentz invariant objects constructed from these 2-component Weyl spinors χ and $\bar{\eta}$ are shown easily to be given by

$$(i\sigma^2\chi)^T\chi = \chi^\alpha\chi_\alpha, \quad (-i\sigma^2\bar{\eta})^T\bar{\eta} = \bar{\eta}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}. \tag{9.73}$$

The χ^α are the components of the spinor $i\sigma^2\chi$ given by $\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta$ (with the convention $\epsilon^{12} = +1$) where χ_α are the components of the spinor χ. Similarly, $\bar{\eta}_{\dot{\alpha}}$ are the components of the spinor $-i\sigma^2\bar{\eta}$ given by $\bar{\eta}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\eta}^{\dot{\beta}}$ (with the convention $\epsilon_{\dot{1}\dot{2}} = -1$) where $\bar{\eta}^{\dot{\alpha}}$ are the components of the spinor $\bar{\eta}$.

The four different spinors χ_α, χ^α, $\bar{\chi}^{\dot{\alpha}}$ and $\bar{\chi}_{\dot{\alpha}}$ transform under the Lorentz transformation $\psi \longrightarrow \psi' = S(\Lambda)\psi$ where

$$\begin{aligned} S(\Lambda) &= \exp\left(\frac{i}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\right) = \begin{pmatrix} M & 0 \\ 0 & M^{-1\dagger} \end{pmatrix}, \\ M &= \exp\left(-\frac{1}{2}\lambda^{\mu\nu}\sigma_{\mu\nu}\right) \end{aligned} \tag{9.74}$$

as

$$\chi_\alpha \longrightarrow \chi'_\alpha = M_\alpha^{\ \beta}\chi_\beta \tag{9.75}$$

$$\chi^\alpha \longrightarrow \chi'^\alpha = (M^{-1T})^\alpha_{\ \beta}\chi^\beta \tag{9.76}$$

$$\bar{\chi}^{\dot{\alpha}} \longrightarrow \bar{\chi}'^{\dot{\alpha}} = (M^{-1\dagger})^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\chi}^{\dot{\beta}} \tag{9.77}$$

$$\bar{\chi}_{\dot{\alpha}} \longrightarrow \bar{\chi}'_{\dot{\alpha}} = (M^*)^{\dot{\beta}}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}}. \tag{9.78}$$

Thus, dotted spinors are nothing other than the complex conjugate of undotted spinors, viz

$$\bar{\chi}_{\dot{\alpha}} = \chi^*_\alpha, \quad \bar{\chi}^{\dot{\alpha}} = (\chi^\alpha)^*. \tag{9.79}$$

The Weyl spinors χ_α, χ^α, $\bar{\eta}^{\dot{\alpha}}$ and $\bar{\eta}_{\dot{\alpha}}$ are obviously related to the left-handed and right-handed spinors by the relations

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\psi = \begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix}, \quad \psi_R = \frac{1}{2}(1 + \gamma_5)\psi = \begin{pmatrix} 0 \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix} \tag{9.80}$$

$$\bar{\psi}_L = \frac{1}{2}\bar{\psi}(1 + \gamma_5) = (0, \bar{\chi}_{\dot{\alpha}}), \quad \bar{\psi}_R = \frac{1}{2}\bar{\psi}(1 - \gamma_5) = (\eta^\alpha, 0). \tag{9.81}$$

A Lorentz invariant Dirac mass term can then be constructed as follows

$$\mathcal{L}_{\text{Dirac}} = -m\bar{\psi}\psi = -m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) = -m(\bar{\chi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} + \eta^\alpha\chi_\alpha). \tag{9.82}$$

Now we are in a position to define Majorana spinors.

A Majorana spinor is a Dirac spinor satisfying a reality condition known as the Majorana condition in which the spinor ψ_M is equal to its charge conjugated spinor $(\psi_M)^c = C\bar{\psi}_M^T$ where C is the charge conjugation operator. The Majorana condition reads then

$$(\psi_M)^c = \psi_M. \tag{9.83}$$

A Majorana spinor provides, therefore, an irreducible real representation of the Lorentz group. Indeed, a Majorana spinor can be constructed from a single Weyl spinor as follows

$$\psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{\psi}_M = (\chi^\alpha, \bar{\chi}_{\dot{\alpha}}). \tag{9.84}$$

As a consequence, the left-handed and right-handed spinors corresponding to a given Majorana spinor are not independent since they are related by charge conjugation. Indeed, we can show that

$$\psi_M = \psi_L + (\psi^c)_R, \tag{9.85}$$

where

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\psi = \begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix}, \quad \bar{\psi}_L = \frac{1}{2}\bar{\psi}(1 + \gamma_5) = (0, \bar{\chi}_{\dot{\alpha}}), \tag{9.86}$$

and

$$(\psi^c)_R = (\psi_L)^c = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \quad \overline{(\psi)^c_R} = (\chi^\alpha, 0). \tag{9.87}$$

A Lorentz invariant Majorana mass term can then be constructed as follows

$$\begin{aligned} \mathcal{L}_{\text{Majorana}} &= -m\bar{\psi}_M\psi_M \\ &= -m\left(\bar{\psi}_L(\psi^c)_R + \overline{(\psi^c)}_R\psi_L\right) = -m\left(\bar{\chi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} + \chi^\alpha\chi_\alpha\right) \\ &= -m\left(\bar{\psi}_L(\psi_L)^c + \overline{(\psi_L)^c}\psi_L\right). \end{aligned} \tag{9.88}$$

Another Majorana mass term is obtained by the exchange $L \leftrightarrow R$. We get

$$\begin{aligned} \mathcal{L}_{\text{Majorana}} &= -m\left(\bar{\psi}_R(\psi^c)_L + \overline{(\psi^c)}_L\psi_R\right) \\ &= -m\left(\bar{\psi}_R(\psi_R)^c + \overline{(\psi_R)^c}\psi_R\right). \end{aligned} \tag{9.89}$$

9.2.4 Neutrino mass and seesaw mechanism

We introduce right-handed neutral fields which are assumed to be singlets under the gauge group $SU(2)_L \times U(1)_L$, and thus they have isospin and hypercharge quantum numbers equal zero, which means in particular that they do not interact with vector bosons. They will be denoted by

$$N_{lR}, \quad l = e, \tau, \mu. \tag{9.90}$$

The Yukawa coupling (9.53) is then extended as (with $f_l = h_{ll}$)

$$\mathcal{L}_{\text{Yukawa}} = -\sum_{l=e,\mu,\tau} h_{ll}\bar{\psi}_{lL}\phi\psi_{lR} - \sum_{l,l'=e,\mu,\tau} h'_{ll'}\bar{\psi}_{lL}\phi^c N_{l'R} + \text{h.c.} \tag{9.91}$$

Since the hypercharges of $\bar{\psi}_L$ and ψ_R are $Y^W = +1$ and $Y^W = -2$, respectively, while the hypercharge of ϕ is $Y^W = -2$ only the charge conjugate scalar field ϕ^c with

hypercharge $Y^W = +2$ can appear in the second term to maintain gauge invariance. The charge conjugate field ϕ^c is given explicitly by

$$\phi^c = \frac{1}{\sqrt{2}}\begin{pmatrix} v + H \\ 0 \end{pmatrix}. \tag{9.92}$$

First, we assume that N_{lR} is a Dirac spinor, i.e. N_{lR} is the right-handed neutrino ν_{lR}. After substitution with the configurations (9.61) and (9.92) into the Lagrangian we obtain the neutrino mass term

$$\mathcal{L}_{\text{mass}} = -\sum_{l,l'=e,\mu,\tau} [M_{ll'}\bar{\nu}_{lR}\nu_{l'L} + M_{l'l}\bar{\nu}_{l'L}\nu_{lR}]. \tag{9.93}$$

The mass matrix M is not diagonal but it is symmetric if we assume that the Yukawa parameters $h'_{ll'}$ satisfy $h'_{ll'} = h'_{l'l}$. It is given in terms of the vacuum expectation value v of the Higgs field (scale of the electroweak theory) by

$$M = \frac{v}{\sqrt{2}}\begin{pmatrix} h'_{ee} & h'_{e\mu} & h'_{e\tau} \\ h'_{\mu e} & h'_{\mu\mu} & h'_{\mu\tau} \\ h'_{\tau e} & h'_{\tau\mu} & h'_{\tau\tau} \end{pmatrix}. \tag{9.94}$$

The matrix M can be diagonalized as $M = U^{-1}DU$ where $D = \text{diag}(m_1, m_2, m_3)$ with the immediate consequence that the states ν_e, ν_μ, ν_τ are not physical states and hence these neutrinos do not have definite mass values, i.e. they do not constitute a basis of mass eigenstates. The physical states are, in fact, mixed states given precisely by the linear combinations ν_1, ν_2 and ν_3 defined by

$$\nu_j = \sum_{l=e,\mu,\tau} U_{jl}\nu_l. \tag{9.95}$$

We consider now the case when N_{lR} is not necessarily the right-handed neutrino which will lead to a Majorana neutrino. By substituting with the configurations (9.61) and (9.92) into the Lagrangian we obtain the neutrino Dirac mass term

$$-\sum_{l,l'=e,\mu,\tau} M_{ll'}\bar{\nu}_{lL}N_{l'R} + \text{h.c.} \tag{9.96}$$

However, in this case the following Majorana mass term is also allowed (with symmetric mass matrices F and D)

$$-\frac{1}{2}\sum_{l,l'=e,\mu,\tau} F_{ll'}\overline{(N_{lR})^c}N_{l'R} - -\frac{1}{2}\sum_{l,l'=e,\mu,\tau} D_{ll'}\overline{(\nu_{lL})^c}\nu_{l'L} + \text{h.c.} \tag{9.97}$$

This mass term is Lorentz invariant in an obvious way. The F Majorana mass term is also gauge invariant under $SU(2)_L \times U(1)_Y$ (since N_{lR} is a singlet under this gauge group) while it breaks the global accidental B–L symmetry of the standard model (due to lepton number violation). The mass matrix F is completely arbitrary since it is independent of any scale.

On the other hand, the mass matrix D must be set equal to 0 since the D Majorana mass term is clearly not gauge invariant under $SU(2)_L \times U(1)_Y$.

We obtain, by adding the above Dirac and Majorana mass terms together, the complete neutrino mass term

$$-\frac{1}{2}(\bar{\nu}_L, \overline{(N^c)_L})\begin{pmatrix} 0 & M \\ M & F \end{pmatrix}\begin{pmatrix} (\nu^c)_R \\ N_R \end{pmatrix} + \text{h.c.} \tag{9.98}$$

For a single generation the calculation is trivial since F and M are just numbers. The diagonalization of the matrix

$$\mathcal{M} = \begin{pmatrix} 0 & M \\ M & F \end{pmatrix} \tag{9.99}$$

is achieved by means of the orthogonal matrix

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \tan 2\theta = \frac{2M}{F}. \tag{9.100}$$

We have then $\mathcal{M} = O^{-1}\mathcal{M}'O$ where $\mathcal{M}' = \text{diag}(-m_1, m_2)$. It is easy to show that

$$m_{1,2} = \frac{1}{2}\sqrt{F^2 + 4M^2} \mp \frac{F}{2}. \tag{9.101}$$

In other words, $m_{1,2} \geqslant 0$. Thus, we must redefine the diagonal mass matrix as $\mathcal{M}' = \mathcal{M}_+ D$ where $\mathcal{M}_+ = \text{diag}(m_1, m_2)$ and $D = \text{diag}(-1, 1)$. The physical neutrino states are defined by

$$\begin{pmatrix} \xi_{1L} \\ \xi_{2L} \end{pmatrix} = O\begin{pmatrix} \nu_L \\ (N^c)_L \end{pmatrix}, \quad \begin{pmatrix} \xi_{1R} \\ \xi_{2R} \end{pmatrix} = DO\begin{pmatrix} (\nu^c)_R \\ N_R \end{pmatrix}. \tag{9.102}$$

By construction these states ξ_1 and ξ_2 have mass values m_1 and m_2, respectively, i.e. the mass term takes the form

$$\mathcal{L}_{\text{mass}} = -m_1\bar{\xi}_{1L}\xi_{1R} - m_2\bar{\xi}_{2L}\xi_{2R} + \text{h.c.} \tag{9.103}$$

Explicitly these physical neutrino states are given by

$$\begin{aligned} \xi_1 &= \xi_{1L} + \xi_{1R} = (\nu_L - (\nu^c)_R)\cos\theta - ((N^c)_L - N_R)\sin\theta \\ \xi_2 &= \xi_{2L} + \xi_{2R} = (\nu_L + (\nu^c)_R)\sin\theta + ((N^c)_L + N_R)\cos\theta. \end{aligned} \tag{9.104}$$

These states are Majorana spinors since they satisfy

$$(\xi_1)^c = -\xi_1, \quad (\xi_2)^c = \xi_2. \tag{9.105}$$

In the case that $M \ll F$ (since M is due to the coupling with the Higgs field and therefore it is of the order of the charged lepton, whereas F is arbitrary) we obtain the mass values

$$m_1 \sim \frac{M^2}{F} \sim \frac{m_l^2}{F} \equiv m_{\nu_l}, \quad m_2 = F \equiv m_{N_l}. \tag{9.106}$$

It is not difficult to convince ourselves further that in the limit $M \ll F$ the right-handed field N_R is dominated mostly by the heavy neutrino ν_2, whereas the left-handed neutrino ν_L consists mostly of the light neutrino ν_1.

This is what is called the seesaw mechanism, in which the neutrino acquires a very small mass since the Majorana mass matrix F is very large, of the order of the grand unification theories scale, while the Dirac mass matrix M is naturally much smaller, of the order of the electroweak scale, i.e. of the order of the vacuum expectation value v.

9.2.5 Quantum chromodynamics and quark sector

Now we add quarks to the GWS electroweak theory and then extend the theory to the full standard model by including the strong color interactions of quarks.

The quarks transform non-trivially under the electroweak gauge group $SU(2)_L \times U(1)_Y$ as doublets in the fundamental representation for the left-handed quarks and in the singlet representation for the right-handed quarks. For the left-handed quarks, we have the doublets

$$Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \quad S^W = \frac{1}{2}, \quad Y^W = \frac{1}{3}, \tag{9.107}$$

whereas for the right-handed quarks we have the singlets

$$\begin{aligned} U_R &= u_R,\, c_R,\, t_R, \quad S^W = 0, \quad Y^W = \frac{4}{3} \\ D_R &= d_R,\, s_R,\, b_R, \quad S^W = 0, \quad Y^W = -\frac{2}{3}. \end{aligned} \tag{9.108}$$

In analogy with the $SU(2)_L \times U(1)_Y$ gauge invariant kinetic lepton term (9.48) we can immediately write down the $SU(2)_L \times U(1)_Y$ gauge invariant kinetic quark term as

$$\mathcal{L}_{\text{kinetic quark}} = \sum_i \left(\bar{Q}_{iL} i\gamma^\mu \nabla_\mu Q_{iL} + \bar{U}_{iR} i\gamma^\mu \nabla_\mu U_{iR} + \bar{D}_{iR} i\gamma^\mu \nabla_\mu D_{iR} \right). \tag{9.109}$$

The action of the covariant derivative on the doublets and the singlets is defined as before by equation (9.45).

However, quarks (compared to leptons) carry an extra index corresponding to the strong color charge which transforms in the fundamental representation of the color gauge group $SU(3)$. If the $SU(3)$ gauge field and the $SU(3)$ coupling constant are denoted by A_μ and g_3, respectively, then the covariant derivative (9.45) acting on quarks should be modified as

$$\nabla_\mu Q_{iL} = \left(\partial_\mu - i g_3 A_\mu - i g \vec{W}_\mu \cdot \vec{S}^W - i g' B_\mu \frac{Y^W}{2} \right) Q_{iL} \tag{9.110}$$

$$\begin{aligned}\nabla_\mu U_{iR} &= \left(\partial_\mu - ig_3 A_\mu - ig' B_\mu \frac{Y^W}{2}\right) U_{iR}, \\ \nabla_\mu D_{iR} &= \left(\partial_\mu - ig_3 A_\mu - ig' B_\mu \frac{Y^W}{2}\right) D_{iR}.\end{aligned} \tag{9.111}$$

The field strength associated with the non-abelian gauge field $A_\mu = A_\mu^a t_a$, where t_a are the generators of $SU(3)$ satisfying $[t_a, t_b] = if_{abc}t_c$ with f_{abc} being the $SU(3)$ structure constants, is defined by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_3 f_{abc} A_\mu^b A_\nu^c. \tag{9.112}$$

The pure gauge action (9.38) should then be modified by adding the contribution of the color gauge field A_μ as follows

$$\mathcal{L}_{\text{pure gauge}} = -\frac{1}{4} G_{\mu\nu}^r G^{r\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}. \tag{9.113}$$

The theory of QCD is obtained by setting the $SU(2)$ and $U(1)$ gauge fields equal to zero, viz $B_\mu = W_\mu^r = 0$. The action of the covariant derivative on the quarks q_i becomes

$$\nabla_\mu q_i = (\partial_\mu - ig_3 A_\mu) q_i. \tag{9.114}$$

We then obtain an $SU(3)$ local gauge theory with a Lagrangian given by (including also a mass term for each quark)

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \sum_i \bar{q}_i (i\gamma^\mu \nabla_\mu - m_i) q_i. \tag{9.115}$$

If the number of quark flavors is N_f then this Lagrangian enjoys, besides $SU(3)$ local gauge invariance, a $U(N_f)$ flavor symmetry which is enhanced to an $U(N_f)_L \times U(N_f)_R$ flavor symmetry in the chiral limit $m_i \longrightarrow 0$.

In the standard model, the quarks acquire mass through the Higgs mechanism similar to the leptons. In analogy with equation (9.53), or more precisely with equation (9.91), the Lorentz invariant, gauge invariant and renormalizable coupling between the Higgs scalar field ϕ and the quark fields Q_L, U_R and D_R are given by the additional Yukawa coupling

$$\mathcal{L}'_{\text{Yukawa}} = -\sum_{ij} G_{ij} \bar{Q}_{iL} \phi D_{jR} - \sum_{ij} F_{ij} \bar{Q}_{iL} \phi^c U_{jR} + \text{h.c.} \tag{9.116}$$

The Higgs scalar doublet ϕ^c in the second term is the charge conjugate field to ϕ, viz $\phi^c = i\tau_2 \phi^*$ or equivalently $(\phi^c)_i = \epsilon_{ij} \phi^{*j}$ (with the convention $\epsilon_{12} = +1$). By substituting the configurations (9.61) and (9.92) into this Lagrangian we obtain mass terms for the quarks of the form

$$\mathcal{L}_{\text{mass}} = -\sum_i m_i \bar{q} q. \tag{9.117}$$

The masses m_i are proportional to the vacuum expectation value v of the Higgs field and to the Yukawa parameters G_{ij} or F_{ij}.

The complete Lagrangian for the standard model is given by the sum of the Lagrangians (9.113), (9.48), (9.52), (9.91), (9.109) and (9.116).

We have already seen that weak interactions of leptons have a very special impact on the lepton mass matrix. Indeed, we have observed that the down-leptons (the electron, the muon and the tau) have a diagonal mass matrix, whereas the up-leptons (the neutrinos), have a non-diagonal mass matrix which requires diagonalization. In other words, the weak eigenstates of the neutrinos are very different from their mass or physical eigenstates. Hence, the weak eigenstates of the neutrinos are mixed states (which causes lepton number violation in an obvious way) given in terms of the mass eigenstates by the linear combinations

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \mathcal{M}^{\mathrm{NM}} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{9.118}$$

The neutrino mixing matrix $\mathcal{M}^{\mathrm{NM}}$ is given in terms of the so-called neutrino mixing angles $\theta_{ij}^{\mathrm{NM}}$ which will be given below.

In a similar way, weak interactions of quarks have a very special impact on the quark mass matrix. In this case, we observe that the up-quarks (u, c, t) have a diagonal mass matrix, whereas the down-quarks (d, s, b), have a non-diagonal mass matrix which again requires diagonalization, and as a consequence, the weak eigenstates of the down-quarks are given in terms of the physical or mass eigenstates by the linear combinations

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \mathcal{M}^{\mathrm{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{9.119}$$

We should stress that the primed quarks d', s' and b' are in fact the weak eigenstates and, therefore, one should make the replacement $q \longrightarrow q'$ in equations (9.107) and (9.108).

The quark mixing matrix $\mathcal{M}^{\mathrm{CKM}}$ is a unitary matrix called the Cabibbo–Kobayashi–Maskawa (CKM) matrix [11, 12] given in terms of the so-called quark mixing angles $\theta_{12}^{\mathrm{CKM}}$, $\theta_{13}^{\mathrm{CKM}}$, $\theta_{23}^{\mathrm{CKM}}$ and the phases $\exp(\pm i\delta_{13})$ (which describes CP violation) by the equation (with $c_{ij} = \cos\theta_{ij}^{\mathrm{CKM}}$ and $s_{ij} = \sin\theta_{ij}^{\mathrm{CKM}}$)

$$\mathcal{M}^{\mathrm{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\exp(-i\delta_{13}) \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\exp(i\delta_{13}) & c_{12}c_{23} - s_{12}s_{23}s_{13}\exp(i\delta_{13}) & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\exp(i\delta_{13}) & -c_{12}s_{23} - s_{12}c_{23}s_{13}\exp(i\delta_{13}) & c_{23}c_{13} \end{pmatrix}. \tag{9.120}$$

The neutrino mixing matrix $\mathcal{M}^{\rm NM}$ is obtained from the CKM matrix $\mathcal{M}^{\rm CKM}$ by the substitution $\theta_{ij}^{\rm CKM} \longrightarrow \theta_{ij}^{\rm NM}$ where the neutrino mixing angles are given by

$$\theta_{12}^{\rm NM} = \frac{\pi}{4} - \theta_{12}^{\rm CKM}, \quad \theta_{23}^{\rm NM} = \frac{\pi}{4} - \theta_{23}^{\rm CKM}, \quad \theta_{13}^{\rm NM} = \theta_{13}^{\rm CKM}. \tag{9.121}$$

In summary:

- The neutrinos are produced in the weak eigenstates ν_e, ν_μ, ν_τ which are NOT equal to the physical eigenstates denoted usually by ν_1, ν_2 and ν_3 which have definite mass values. This is described in the standard model by a mixing mass matrix called the Pontecorvo, Maki, Nakagawa, Sakata (PMNS) mixing matrix. This is what leads to the observed oscillation between the various flavor of neutrinos ν_e, ν_μ, ν_τ.
- Neutrino oscillation does NOT mean decay but it simply means spontaneous flip of the neutrino flavor as it propagates freely in space.
- The down-quarks (d, s, b) are the mass eigenstates but the weak eigenstates are quite different denoted typically by the same symbols with primes. Thus, there is also a non-diagonal mass matrix corresponding to the down-quarks (d, s, b) called the CKM mixing matrix which plays exactly the same role for quarks as the PMNS matrix plays for neutrinos.
- Thus, there must exist, in principle, an oscillation between the weakly produced down-quark flavors (similar to the neutrino oscillation), which is not observed due to the fact that quarks are confined (as opposed to neutrinos which exist as asymptotic particles) and, therefore, they suffer hadronization, i.e. they become hadrons, very quickly after production on timescales of the order of the inverse of $\Lambda_{\rm QCD}$. Hence, the linear combinations of weak eigenstates of quarks decohere very quickly leaving only the mass eigenstates of quarks which are observed in detectors.

9.2.6 Anomaly cancellation

The standard model, as a chiral gauge theory (it couples differently to the left-handed and right-handed quarks and leptons), will necessarily involve triangle diagrams in which internal lines are fermions, external lines are gauge fields and the vertices involve the γ^5. These diagrams, as we have seen, suffer from the Adler, Bell, Jackiw anomaly and their existence signals a fundamental unresolvable conflict between renormalization and symmetry which in the case of local gauge invariance (if anomalous) will cost the consistency of the whole theory (gauge symmetry is essential for renormalizability and unitarity).

Therefore, gauge anomaly cancellation (for the three groups $U(1)$, $SU(2)$ and $SU(3)$ plus gravity) is a fundamental further grand requirement that must be imposed on the structure of the standard model and which turns out to be satisfied by the fact that the number of quark generations is equal to the number of lepton generations.

9.3 Exercises

Exercise 1:

- Use the Heisenberg principle to show that the short-range of weak interactions ($R_W \sim 10^{-16}$ cm) gives indeed a mass of the order of the mass of the W vector bosons. The color force is also short-ranged ($R_S \sim 10^{-13}$ cm) yet the vector bosons in this case (the gluons) are absolutely massless although they should have been massive by a naive application of the Heisenberg principle. Explain why? Hint: Asymptotic states, confinement and mass gap.
- In Yukawa theory the strong nuclear force between the proton and the neutron is mediated by the exchange of virtual pions. By using the Heisenberg uncertainty principle $\Delta E \Delta t \geqslant \hbar/2$ derive the formula (9.13). By using this formula calculate an estimate for the mass of the pions using the experimentally known range of the strong force.

Exercise 2:

The pion field is created by a nucleon density $\rho(x)$ in the same way that the photon field is created by an electron density. However, since the pion is a spin zero particle it must obey the Klein–Gordon equation. Show that the pion field $\phi(\vec{x})$ generated by a static nucleon configuration $\rho(\vec{x})$ is given by

$$\phi(\vec{x}) = \frac{1}{4\pi} \int d^3\vec{x}' \rho(\vec{x}') \frac{\exp(-m_\pi |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}. \tag{9.122}$$

Show that the nuclear interaction energy of the system is given by

$$H = -\frac{1}{8\pi} \int d^3\vec{x}' \rho(\vec{x}') \int d^3\vec{x} \rho(\vec{x}) \frac{\exp(-m_\pi |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}. \tag{9.123}$$

Derive then the interaction Yukawa potential V_N of two nucleons separated by a distance r.

Exercise 3:

- Determine the transformation property under the $SU(2)$ isospin symmetry of the antinucleon $\bar{N}C$ where C is charge conjugation. What is the transformation property of $\zeta = -i\sigma_2(\bar{N}C)$?
- The deuteron is a two-nucleon system which is spatially symmetric but, by the Pauli principle, its total wavefunction must be antisymmetric. Determine the isospin of the deuteron and its isospin wavefunction.
- Determine the isotriplet and the isosinglet states for a nucleon–antinucleon system $N\bar{N}$.

Exercise 4:

- Show explicitly that we can bring the Higgs scalar field ϕ by means of a local $SU(2)_L$ gauge transformation into the form

$$\phi_{\text{fixed}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H \end{pmatrix}. \tag{9.124}$$

- Calculate $\nabla_\mu \phi$ at $\phi = \phi_{\text{fixed}}$.
- Show by substitution into $V(\phi)$ that the mass of the Higgs field H is given by

$$m_H^2 = 2\lambda v^2. \tag{9.125}$$

  Show also that the other three scalar fields $\phi_{1,\,2,\,3}$ remain massless (Nambu–Goldstone modes associated with the broken gauge symmetries).
- Calculate the kinetic term $(\nabla_\mu \phi)^\dagger (\nabla^\mu \phi)$.
- Show that the transformation (9.63) diagonalizes the kinetic term. Determine the mass values of the vector bosons $W_\mu^\pm = (W_\mu^1 \mp i W_\mu^2)/\sqrt{2}$ and Z_μ.
- Determine the mass values of the leptons by substituting into the Yukawa coupling.
- Show that the interaction between the Higgs and the vector bosons is given by

$$\frac{g^2}{4}\left[W_\mu^* W^\mu + \frac{1}{2\cos^2\theta_W} Z_\mu Z^\mu \right][H^2 + 2vH]. \tag{9.126}$$

- Show that the electromagnetic interaction, the charged current interaction, and the neutral current weak interaction between the vector bosons and the leptons are given by the terms

$$-e\bar{l}\gamma^\mu l A_\mu, \quad |e| = \frac{gg'}{\sqrt{g^2 + g'^2}} \tag{9.127}$$

$$\frac{g}{2\sqrt{2}}\left[\bar{\nu}_l \gamma^\mu (1 - \gamma_5) l W_\mu^* + \bar{l}\gamma^\mu (1 - \gamma_5)\nu_l W_\mu\right] \tag{9.128}$$

$$\frac{g}{4\cos\theta_W}\left[\bar{\nu}_l \gamma^\mu (1 - \gamma_5)\nu_l + \bar{l}\gamma^\mu (4\sin^2\theta_W - 1 + \gamma_5) l\right] Z_\mu. \tag{9.129}$$

Exercise 5:

- Show that the Majorana mass matrices F and D are symmetric.
- Show that the sum of the Dirac mass term and the Majorana mass term (with $D = 0$) can be brought into the form

$$-\frac{1}{2}\left(\bar{\nu}_L,\, \overline{(N^c)_L}\right)\mathcal{M}\begin{pmatrix} (\nu^c)_R \\ N_R \end{pmatrix} + \text{h.c.} \tag{9.130}$$

  Determine $\mathcal{M}$ and calculate its eigenvalues.
- Show explicitly that the physical neutrino states, in the case of a single generation, are given by the linear combinations

$$\xi_1 = \xi_{1L} + \xi_{1R} = (\nu_L - (\nu^c)_R)\cos\theta - ((N^c)_L - N_R)\sin\theta \tag{9.131}$$

$$\xi_2 = \xi_{2L} + \xi_{2R} = (\nu_L + (\nu^c)_R)\sin\theta + ((N^c)_L + N_R)\cos\theta. \tag{9.132}$$

- Show that the states ξ_1 and ξ_2 are indeed Majorana spinors.
- Show that in the limit $M \ll F$ the right-handed field N_R is dominated mostly by the heavy neutrino ν_2, whereas the left-handed neutrino ν_L consists mostly of the light neutrino ν_1.

Exercise 6:

The kaons K^0 and $\bar{K}^0$ are strange particles (with $s = +1$ and $s = -1$ respectively) which are produced in strong reactions but decay in weak reactions (thus violating conservation of strangeness) and they decay mostly to pions π^0 and $\pi^\pm$.

- Determine the kaon eigenstates K_L and K_S of the CP operation.
- It is assumed that weak interaction conserves CP but it does not conserve C and P separately. Determine the decay modes into pions of K_S and K_L.
- Compare with the experimental result

$$\frac{\sigma(K_L \longrightarrow 2\pi)}{\sigma(K_L \longrightarrow \text{all other modes})} = (2.0 \pm 0.4) \times 10^{-3} \neq 0. \tag{9.133}$$

  What do you conclude?

Solution 6:

- The kaons are 0^- particles and thus they are pseudo-scalars under parity operation P with parity eigenvalue $\mathcal{P} = -1$. On the other hand, the kaons K^0 and $\bar{K}^0$ are anti-particles of one another and hence charge conjugation operation C sends K^0 to $\bar{K}^0$ and vice versa. It is then not difficult to show that the eigenstates of the CP operation are, in fact, given by the short-lived and long-lived kaons K_S and K_L given by the linear combinations

$$|K_S\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle), \quad CP = +1 \tag{9.134}$$

$$|K_L\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle), \quad CP = -1. \tag{9.135}$$

- The K-short and K-long are what actually decay in weak interactions. The hadronic weak decays are $K_S \longrightarrow 2\pi$ and $K_L \longrightarrow 3\pi$. Indeed, we can show that the final states have the following eigenvalues

$$CP|\pi^+\pi^-\rangle = +|\pi^+\pi^-\rangle. \tag{9.136}$$

$$CP|\pi^0\pi^0\pi^0\rangle = -|\pi^0\pi^0\pi^0\rangle, \quad CP|\pi^+\pi^-\pi^0\rangle = -|\pi^+\pi^-\pi^0\rangle. \tag{9.137}$$

- Experimentally we observe

$$\frac{\sigma(K_L \longrightarrow 2\pi)}{\sigma(K_L \longrightarrow \text{all other modes})} = (2.0 \pm 0.4) \times 10^{-3} \neq 0. \tag{9.138}$$

  The fact that this ratio is non-zero signifies that CP is violated in kaon decays, i.e. CP is in fact violated in weak interactions. By the CPT theorem we conclude then that weak interactions are not symmetric under time reversal T which is a remarkable result.

Exercise 7:

Show that the probability that the muon neutrino produced at $t = 0$ in the process $\pi^+ \longrightarrow \mu^+ + \nu_\mu$ becomes at time t an electron neutrino after propagating a distance $L = ct = Et/p$ in space is given by

$$P(\nu_\mu \longrightarrow \nu_e, L) = |\langle \nu_e(t)|\nu_\mu(0)\rangle|^2 = \sin^2 2\theta \sin^2 \frac{\Delta m^2 c^3 L}{4\hbar E}. \tag{9.139}$$

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Badis Ydri

Chapter 10

Introduction to lattice field theory

In this chapter we take a quick excursion into the world of lattice field theory. Scalar, fermion and gauge fields are constructed on the lattice explicitly. Then, the two most used Monte Carlo algorithms in numerical simulations on the lattice (the Metropolis and the hybrid Monte Carlo algorithms) are explained within the context of very simple lattice models, namely the scalar phi-four in two dimensions and quenched electrodynamics. The classic textbooks on the subject of lattice field theory are [1–5].

10.1 The lattice ϕ_2^4

We start by discussing scalar phi-four theory in two dimensions on the lattice which is one of the simplest lattice models. This model is also of great relevance to the Ising model and to the Ising universality class. References for this section include the lattice field theory textbook [2] and the original articles [6–8].

10.1.1 The model and its phase structure

The Euclidean ϕ^4 action with $O(N)$ symmetry is given by

$$S[\phi] = \int d^dx \left(\frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}m^2 \phi^i \phi^i + \frac{\lambda}{4}(\phi^i \phi^i)^2 \right). \tag{10.1}$$

We will employ lattice regularization in which $x = an$, $\int d^dx = a^d \sum_n$, $\phi^i(x) = \phi^i_n$ and $\partial_\mu \phi^i = \left(\phi^i_{n+\hat{\mu}} - \phi^i_n\right)/a$. The lattice action reads

$$S[\phi] = \sum_n \left(-2\kappa \sum_\mu \Phi^i_n \Phi^i_{n+\hat{\mu}} + \Phi^i_n \Phi^i_n + g(\Phi^i_n \Phi^i_n - 1)^2 \right). \tag{10.2}$$

The mass parameter m^2 is replaced by the so-called hopping parameter κ and the coupling constant λ is replaced by the coupling constant g where

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$$m^2a^2 = \frac{1-2g}{\kappa} - 2d, \;\; \frac{\lambda}{a^{d-4}} = \frac{g}{\kappa^2}. \tag{10.3}$$

The fields ϕ_n^i and Φ_n^i are related by

$$\phi_n^i = \sqrt{\frac{2\kappa}{a^{d-2}}}\,\Phi_n^i. \tag{10.4}$$

The partition function is given by

$$\begin{aligned} Z &= \int \prod_{n,i} d\Phi_n^i \, e^{-S[\phi]} \\ &= \int d\mu(\Phi)\, e^{2\kappa \sum_n \sum_\mu \Phi_n^i \Phi_{n+\hat{\mu}}^i}. \end{aligned} \tag{10.5}$$

The measure $d\mu(\phi)$ is given by

$$\begin{aligned} d\mu(\Phi) &= \prod_{n,i} d\Phi_n^i \, e^{-\sum_n (\Phi_n^i \Phi_n^i + g(\Phi_n^i \Phi_n^i - 1)^2)} \\ &= \prod_n \left(d^N \vec{\Phi}_n \, e^{-\vec{\Phi}_n^2 - g(\vec{\Phi}_n^2 - 1)^2} \right) \\ &\equiv \prod_n d\mu(\Phi_n). \end{aligned} \tag{10.6}$$

This is a generalized Ising model. Indeed, in the limit $g \longrightarrow \infty$, the dominant configurations are such that $\Phi_1^2 + \cdots + \Phi_N^2 = 1$, i.e. points on the sphere S^{N-1}. Hence

$$\frac{\int d\mu(\Phi_n) f(\vec{\Phi}_n)}{\int d\mu(\Phi_n)} = \frac{\int d\Omega_{N-1} f(\vec{\Phi}_n)}{\int d\Omega_{N-1}}, \;\; g \longrightarrow \infty. \tag{10.7}$$

For $N = 1$ we obtain

$$\frac{\int d\mu(\Phi_n) f(\vec{\Phi}_n)}{\int d\mu(\Phi_n)} = \frac{1}{2}(f(+1) + f(-1)), \;\; g \longrightarrow \infty. \tag{10.8}$$

Thus, the limit $g \longrightarrow \infty$ of the $O(1)$ model is precisely the Ising model in d dimensions. The limit $g \longrightarrow \infty$ of the $O(3)$ model corresponds to the Heisenberg model in d dimensions. The $O(N)$ models on the lattice are thus intimately related to spin models.

There are two phases in this model. A disordered (paramagnetic) phase characterized by $\langle \Phi_n^i \rangle = 0$ and an ordered (ferromagnetic) phase characterized by $\langle \Phi_n^i \rangle = v_i \neq 0$. The boundary occurs at some value κ_c of κ. We have then

$$\kappa > \kappa_c\text{: broken, ordered, ferromagnetic} \tag{10.9}$$

$$\kappa < \kappa_c\text{: symmetric, disordered, paramagnetic.} \tag{10.10}$$

The critical line $\kappa_c = \kappa_c(g)$ interpolates in the $\kappa - g$ plane between the two lines given by

$$\kappa_c = \frac{N}{4d},\ g \longrightarrow \infty \tag{10.11}$$

$$\kappa_c = \frac{1}{2d},\ g \longrightarrow 0. \tag{10.12}$$

For $d = 4$ the critical value at $g = 0$ is $\kappa_c = 1/8$ for all N. This critical value can be derived in a different way as follows. We know that the renormalized mass at one-loop order in the continuum ϕ^4 with $O(N)$ symmetry is given by the equation

$$\begin{aligned} a^2 m_R^2 = am^2 + \frac{(N+2)\lambda}{16\pi^2} + \frac{(N+2)\lambda}{16\pi^2} a^2 m^2 \ln a^2 m^2 \\ + \frac{(N+2)\lambda}{16\pi^2} a^2 m^2 \mathbf{C} + a^2 \times \text{finite terms.} \end{aligned} \tag{10.13}$$

The lattice space a is formally identified with the inverse cut-off $1/\Lambda$, viz

$$a = \frac{1}{\Lambda}. \tag{10.14}$$

Thus, we obtain in the continuum limit $a \longrightarrow 0$ the result

$$\begin{aligned} a^2 m^2 \longrightarrow -\frac{(N+2)\lambda}{16\pi^2} + \frac{(N+2)\lambda}{16\pi^2} a^2 m^2 \ln a^2 m^2 \\ + \frac{(N+2)\lambda}{16\pi^2} a^2 m^2 \mathbf{C} + a^2 \times \text{finite terms.} \end{aligned} \tag{10.15}$$

In other words (with $r_0 = (N+2)/8\pi^2$)

$$a^2 m^2 \longrightarrow a^2 m_c^2 = -\frac{r_0}{2}\lambda + O(\lambda^2). \tag{10.16}$$

This is the critical line for small values of the coupling constant as we will now show. Expressing this equation in terms of κ and g we obtain

$$\frac{1-2g}{\kappa} - 8 \longrightarrow -\frac{r_0}{2}\frac{g}{\kappa^2} + O(\lambda^2). \tag{10.17}$$

This can be brought to the form

$$\left[\kappa - \frac{1}{16}(1-2g)\right]^2 \longrightarrow \frac{1}{256}[1 + 16 r_0 g - 4g] + O(g^2/\kappa^2). \tag{10.18}$$

We get the result

$$\kappa \longrightarrow \kappa_c = \frac{1}{8} + \left(\frac{r_0}{2} - \frac{1}{4}\right) g + O(g^2). \tag{10.19}$$

This result is of fundamental importance. The continuum limit $a \longrightarrow 0$ corresponds precisely to the limit in which the mass approaches its critical value. This happens for every value of the coupling constant and hence the continuum limit $a \longrightarrow 0$ is the limit in which we approach the critical line. The continuum limit is, therefore, a second-order phase transition.

10.1.2 The hybrid Monte Carlo algorithm

We associate to the fields Φ_n^i canonical momenta P_n^i. Then, we consider the following Hamiltonian

$$\begin{aligned} H[\phi,\, P] = & \frac{1}{2}\sum_n P_n^i P_n^i \\ & + \sum_n \left(-2\kappa \sum_\mu \Phi_n^i \Phi_{n+\hat{\mu}}^i + \Phi_n^i \Phi_n^i + g\left(\Phi_n^i \Phi_n^i - 1\right)^2 \right). \end{aligned} \tag{10.20}$$

The Hamilton equations of motion are

$$\begin{aligned} \frac{\partial H}{\partial P_n^i} &= \dot{\Phi}_n^i = P_n^i \\ \frac{\partial H}{\partial \Phi_n^i} &= -\dot{P}_n^i = V_n^i. \end{aligned} \tag{10.21}$$

The force is given by

$$\begin{aligned} V_n^i &= \frac{\partial S}{\partial \Phi_n^i} \\ &= -2\kappa \sum_\mu \left(\Phi_{n+\hat{\mu}}^i + \Phi_{n-\hat{\mu}}^i \right) + 2\Phi_n^i + 4g\Phi_n^i \left(\Phi_n^j \Phi_n^j - 1 \right). \end{aligned} \tag{10.22}$$

The leap-frog, or Stormer–Verlet, algorithm, which maintains the symmetry under time reversal and the conservation of the phase space volume of the above Hamilton equations, is then given by the equations

$$P_n^i\left(t + \frac{\delta t}{2}\right) = (P)_n^i(t) - \frac{\delta t}{2} V_n^i(t) \tag{10.23}$$

$$\Phi_n^i(t + \delta t) = \Phi_n^i(t) + \delta t P_n^i\left(t + \frac{\delta t}{2}\right) \tag{10.24}$$

$$P_n^i(t + \delta t) = P_n^i\left(t + \frac{\delta t}{2}\right) - \frac{\delta t}{2} V_n^i(t + \delta t). \tag{10.25}$$

We recall that $t = n\delta t$, $n = 0, 1, 2, \ldots, \nu - 1, \nu$ where the point $n = 0$ corresponds to the initial configuration $\Phi_n^i(0)$, whereas $n = \nu$ corresponds to the final

configuration $\Phi_n^i(T)$ where $T = \nu\delta t$. This algorithm does not conserve the Hamiltonian due to the systematic error associated with the discretization, which goes as $O(\delta t^2)$, but as can be shown, the addition of a Metropolis accept–reject step will nevertheless lead to an exact algorithm.

The hybrid Monte Carlo algorithm in this case can be summarized as follows:

(1) Choose $P(0)$ such that $P(0)$ is distributed according to the Gaussian probability distribution $\exp\left(-\frac{1}{2}\sum_n P_n^i P_n^i\right)$. In particular we choose P_n^i such that

$$P_n^i = \sqrt{-2\ln(1 - x_1)}\cos 2\pi(1 - x_2), \tag{10.26}$$

where x_1 and x_2 are two random numbers uniformly distributed in the interval [0, 1]. This step is crucial if we want to avoid ergodic problems.

(2) Find the configuration $(\Phi(T), P(T))$ by solving the above differential equations of motion.

(3) Accept the configuration $(\Phi(T), P(T))$ with a probability

$$\min(1, \exp(-\Delta H[\Phi, P])), \tag{10.27}$$

where ΔH is the corresponding change in the Hamiltonian when we go from $(\Phi(0), P(0))$ to $(\Phi(T), P(T))$.

(4) Repeat.

10.1.3 Renormalization and continuum limit

The continuum and lattice actions for Φ^4 theory in two dimensions with $N = 1$ are given, with some slight change of notation, by

$$S[\phi] = \int d^2x\left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\mu_0{}^2\phi^2 + \frac{\lambda}{4}\phi^4\right) \tag{10.28}$$

$$S[\phi] = \sum_n\left(-2\kappa\sum_\mu \Phi_n\Phi_{n+\hat{\mu}} + \Phi_n^2 + g\left(\Phi_n^2 - 1\right)^2\right) \tag{10.29}$$

$$\mu_0{}^2 = m^2 \tag{10.30}$$

$$\mu_{0l}^2 \equiv \mu_0{}^2 a^2 = \frac{1 - 2g}{\kappa} - 4, \;\; \lambda_l \equiv \lambda a^2 = \frac{g}{\kappa^2}. \tag{10.31}$$

In the simulations we will start by fixing the lattice quartic coupling λ_l and the lattice mass parameter μ_{0l}^2 which then allows us to fix κ and g as

$$\kappa = \frac{\sqrt{8\lambda_l + \left(\mu_{0l}^2 + 4\right)^2} - \left(\mu_{0l}^2 + 4\right)}{4\lambda_l} \tag{10.32}$$

$$g = \kappa^2\lambda_l. \tag{10.33}$$

The phase diagram will be drawn originally in the μ_{0l}^2–λ_l plane. This is the lattice phase diagram. This should be extrapolated to the infinite volume limit $L = Na \longrightarrow \infty$.

The Euclidean quantum field theory phase diagram should be drawn in terms of the renormalized parameters and is obtained from the lattice phase diagram by taking the limit $a \longrightarrow 0$. In two dimensions the Φ^4 theory requires only mass renormalization while the quartic coupling constant is finite. Indeed, the bare mass μ_0^2 diverges logarithmically when we remove the cut-off, i.e. in the limit $\Lambda \longrightarrow \infty$ where $\Lambda = 1/a$ while λ is independent of a. As a consequence, the lattice parameters will go to zero in the continuum limit $a \longrightarrow 0$.

We know that mass renormalization is due to the tadpole diagram which is the only divergent Feynman diagram in the theory and takes the form of a simple reparametrization given by

$$\mu_0^2 = \mu^2 - \delta\mu^2, \tag{10.34}$$

where μ^2 is the renormalized mass parameter and $\delta\mu^2$ is the counter term which is fixed via an appropriate renormalization condition. The ultraviolet divergence $\ln \Lambda$ of μ_0^2 is contained in $\delta\mu^2$ while the renormalization condition will split the finite part of μ_0^2 between μ^2 and $\delta\mu^2$. The choice of the renormalization condition can be quite arbitrary. A convenient choice suitable for Monte Carlo measurements and which distinguishes between the two phases of the theory is given by the usual normal ordering prescription [6].

Quantization at one-loop gives explicitly the 2-point function

$$\Gamma^{(2)}(p) = p^2 + \mu_0^2 + 3\lambda \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \mu_0^2}. \tag{10.35}$$

A self-consistent Hartree treatment gives then the result

$$\begin{aligned} \Gamma^{(2)}(p) &= p^2 + \mu_0^2 + 3\lambda \int \frac{d^2k}{(2\pi)^2} \frac{1}{\Gamma^{(2)}(k)} \\ &= p^2 + \mu^2 + 3\lambda \int \frac{d^2k}{(2\pi)^2} \frac{1}{\Gamma^{(2)}(k)} - \delta\mu^2 \\ &= p^2 + \mu^2 + 3\lambda \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \mu^2} \\ &\quad - \delta\mu^2 + \text{two-loop}. \end{aligned} \tag{10.36}$$

This should certainly work in the symmetric phase where $\mu^2 > 0$. We can also write this as

$$\Gamma^{(2)}(p) = p^2 + \mu^2 + \Sigma(p), \;\; \Sigma(p) = 3\lambda A_{\mu^2} - \delta\mu^2 + \text{two-loop}. \tag{10.37}$$

A_{μ^2} is precisely the value of the tadpole diagram given by

$$A_{\mu^2} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \mu^2}. \tag{10.38}$$

The renormalization condition which is equivalent to normal ordering the interaction in the interaction picture in the symmetric phase is equivalent to the choice

$$\delta\mu^2 = 3\lambda A_{\mu^2}. \tag{10.39}$$

A dimensionless coupling constant can then be defined by

$$f = \frac{\lambda}{\mu^2}. \tag{10.40}$$

The action becomes

$$S[\phi] = \int d^2x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\mu^2 \left(1 - 3f A_{\mu^2}\right)\phi^2 + \frac{f\mu^2}{4}\phi^4 \right). \tag{10.41}$$

For sufficiently small f the exact effective potential is well approximated by the classical potential with a single minimum at $\phi_{\text{cl}} = 0$. For larger f, the coefficient of the mass term in the above action can become negative and as a consequence a transition to the broken symmetry phase is possible, although in this regime the effective potential is no longer well approximated by the classical potential. Indeed, a transition to the broken symmetry phase was shown to be present in [8], where a duality between the strong-coupling regime of the above action and a weakly coupled theory normal ordered with respect to the broken phase was explicitly constructed.

The sites on the lattice are located at $x_\mu = n_\mu a$ where $n_\mu = 0, \ldots, N-1$ with $L = Na$. The plane waves on a finite volume lattice with periodic boundary conditions are $\exp(ipx)$ with $p_\mu = m_\mu 2\pi/L$ where $m_\mu = -N/2+1, -N/2+2, \ldots, N/2$ for N even. This means that the zero of the x-space is located at the edge of the box while the zero of the p-space is located in the middle of the box. We have, therefore, the normalization conditions $\sum_x \exp(-i(p-p')x) = \delta_{p,p'}$ and $\sum_p \exp(-i(x-x')p) = \delta_{x,x'}$ where, for example, $\sum_p = \sum_m / L^2$. In the infinite volume limit defined by $L = Na \longrightarrow \infty$ with a fixed we have $\sum_p \longrightarrow \int_{-\pi/a}^{\pi/a} d^2p/(2\pi)^2$. It is not difficult to show that on the lattice the propagator $1/(p^2+\mu^2)$ becomes $a^2/\left(4\sum_\mu \sin^2 ap_\mu/2 + \mu_l^2\right)$ [3]. Thus, on a finite volume lattice with periodic boundary conditions, the Feynman diagram A_{μ^2} takes the form

$$\begin{aligned} A_{\mu^2} &= \sum_{p_1,p_2} \frac{a^2}{4\sin^2 ap_1/2 + 4\sin^2 ap_2/2 + \mu_l^2} \\ &= \frac{1}{N^2} \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \frac{1}{4\sin^2 \pi m_1/N + 4\sin^2 \pi m_2/N + \mu_l^2}. \end{aligned} \tag{10.42}$$

In the last line we have shifted the integers m_1 and m_2 by $N/2$. Hence, on a finite volume lattice with periodic boundary conditions equation (10.34), together with equation (10.39), becomes

$$F\left(\mu_l^{\,2}\right) = \mu_l^{\,2} - 3\lambda_l A_{\mu_l^2} - \mu_{0l}^{\,2} = 0. \tag{10.43}$$

Given the critical value of μ_{0l}^2 for every value of λ_l we need then to determine the corresponding critical value of $\mu_l^{\,2}$. This can be done numerically using the Newton–Raphson algorithm. The continuum limit $a \longrightarrow 0$ is then given by extrapolating the results into the origin, i.e. taking $\lambda_l = a^2\lambda \longrightarrow 0$, $\mu_l^{\,2} = a^2\mu^2 \longrightarrow 0$ in order to determine the critical value

$$f_c = \lim_{\lambda_l, \mu_l^2 \longrightarrow 0} \frac{\lambda_l}{\mu_{lc}^{\,2}}. \tag{10.44}$$

10.1.4 Numerical calculation of the critical line

We measure as observables the average value of the action, the specific heat, the magnetization, the susceptibility and the Binder cumulant defined, respectively, by

$$\langle S \rangle \tag{10.45}$$

$$C_v = \langle S^2 \rangle - \langle S \rangle^2 \tag{10.46}$$

$$M = \frac{1}{N^2}\langle m \rangle, \;\; m = \left| \sum_n \phi_n \right| \tag{10.47}$$

$$\chi = \langle m^2 \rangle - \langle m \rangle^2 \tag{10.48}$$

$$U = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}. \tag{10.49}$$

We note the use of the absolute value in the definition of the magnetization since the usual definition $M = \langle \sum_n \phi_n \rangle / N^2$ is automatically zero on the lattice because of the symmetry $\phi \longrightarrow -\phi$. The specific heat diverges at the critical point logarithmically as the lattice size is set to infinity. The susceptibility shows also a peak at the critical point, whereas the Binder cumulant exhibits a fixed point for all values of N.

This is a relatively simple numerical problem so we can run quick Monte Carlo simulations ourselves to uncover the main features of the non-perturbative physics.

Indeed, we ran simulations with $T_{\rm th} + T_{\rm mc} \times T_{\rm co}$ steps where $T_{\rm th}$ is thermalization time and $T_{\rm mc}$ is measurement time. We used $T_{\rm th} = 2^{13}$ and $T_{\rm mc} = 2^{14}$. Every two successive measurements are separated by $T_{\rm co} = 2^3$ steps to reduce auto-correlations. We use RAN2 as our random numbers generator and the jackknife method to estimate error bars. The hybrid Monte Carlo code used in these simulations can be found in the last chapter of [10].

We have considered lattices with $N = 16$, 32 and 49 and values of the quartic coupling given by $\lambda_l = 1$, 0.7, 0.5, 0.25. Some results are shown in figure 10.1. The

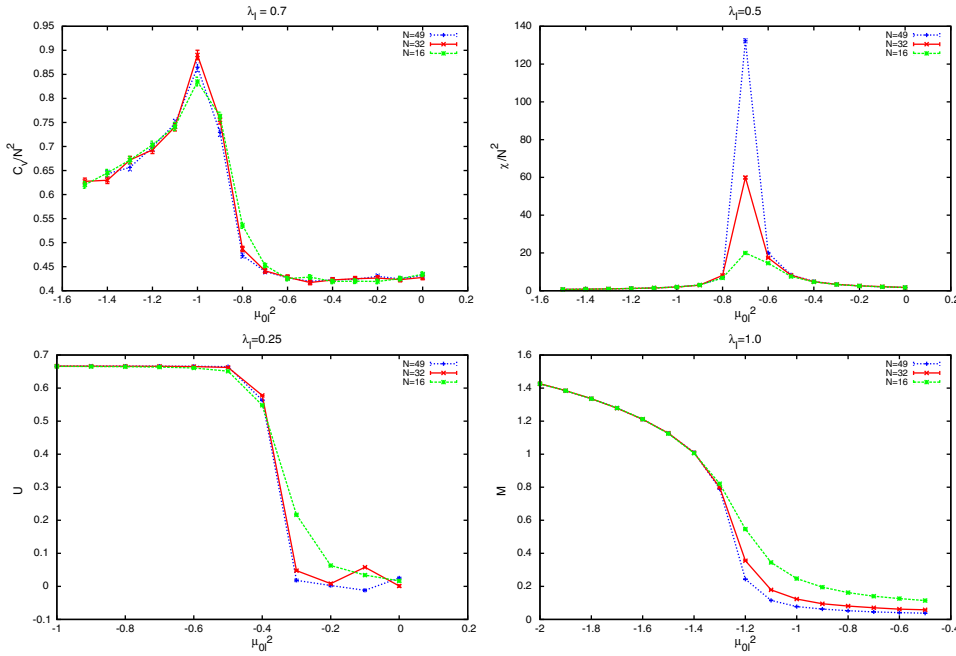

Figure 10.1. Various observables for ϕ_2^4 on the lattice using the hybrid Monte Carlo algorithm.

Table 10.1. The critical values.

λ_l	μ_{0l*}^2	μ_{l*}^2
1.0	-1.25 ± 0.05	1.00×10^{-2}
0.7	-0.95 ± 0.05	6.89×10^{-2}
0.5	-0.7 ± 0.00	5.52×10^{-2}
0.25	-0.4 ± 0.00	2.53×10^{-2}

critical value μ_{0l*}^2 for each value of λ_l is found from averaging the values at which the peaks in the specific heat and the susceptibility occur. The results are shown in the second column of table 10.1. The final step is to take the continuum limit $a \longrightarrow 0$ in order to find the critical value μ_{l*}^2 by solving the renormalization condition (10.43) using the Newton–Raphson method. This is an iterative method based on a single iteration given by $\mu_{l*}^2 = \mu_{l*}^2 - F/F'$. The corresponding results are shown in the third column of table 10.1. The critical line is shown in figure 10.2 with a linear fit going through the origin given by

$$\lambda_l = (9.88 \pm 0.22)\mu_{l*}^2. \tag{10.50}$$

This should be compared with the much more precise result $\lambda_l = 10.8\mu_{l*}^2$ published in [7]. The above result is sufficient for our purposes here.

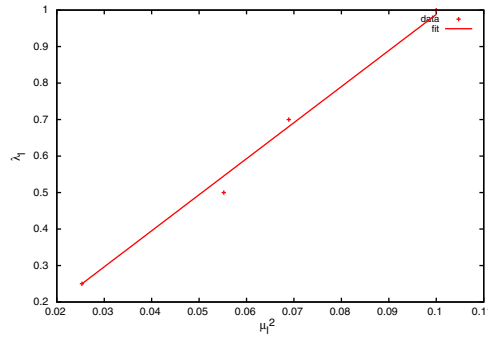

Figure 10.2. The critical line.

10.2 Fermions on the lattice

10.2.1 The fermion doubling problem on the lattice

The non-perturbative formulation of chiral gauge theories is a long standing program in quantum field theory. It seems clear that one should regularize these theories with all symmetries intact. But there is a major problem associated with conventional lattice approaches to this program which is rooted in topological features. The Nielsen–Ninomiya no-go theorem [9] states that if we want to maintain chiral symmetry then, under plausible assumptions, one cannot avoid the doubling of fermions in the usual lattice formulations.

To see this, let us recall how one encounters the doubling problem on a four-dimensional Euclidean lattice. Our discussion and notations will follow that of [4] or [3]. The Dirac equation in Minkowski spacetime is

$$\left(i\gamma^\mu\partial_\mu - M\right)\psi(x) = 0. \tag{10.51}$$

The 4×4 Dirac matrices γ^μ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ where $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The spinor $\psi(x)$ is a 4-component field. The equation of motion for ψ and $\bar\psi = \psi^+\gamma^0$ follows from the variation of the action

$$S_F = \int d^4x\bar\psi(x)\left(i\gamma^\mu\partial_\mu - M\right)\psi(x). \tag{10.52}$$

The Euclidean action S_F^{Eucl} is obtained by (i) making the replacement $x_0 \longrightarrow -ix_4$ wherever x_0 appears explicitly, (ii) substituting $\psi^E(x) = \psi(\vec{x}, x_4)$ for $\psi(x) = \psi(\vec{x}, t)$ and (iii) multiplying the obtained expression by $-i$. Since in Euclidean space the Lorentz group is replaced by the four-dimensional rotation group we introduce new γ-matrices γ_μ^E as follows $\gamma_4^E = \gamma^0$, $\gamma_i^E = -i\gamma^i$. They satisfy $\left\{\gamma_\mu^E, \gamma_\nu^E\right\} = 2\delta_{\mu\nu}$. The Euclidean action is given by

$$S_F^{\text{Eucl}} = \int d^4x\bar\psi^E(x)\left(\gamma_\mu^E\partial_\mu + M\right)\psi^E(x). \tag{10.53}$$

We will drop the labels 'Eucl' and 'E' in the following. Thus, the canonical continuum fermion action in Euclidean four-dimensional space is

$$\mathcal{L} = \bar{\psi}(\gamma_\mu \partial_\mu + M)\psi. \tag{10.54}$$

This has the symmetry $\psi \longrightarrow e^{i\theta}\psi$ and the symmetry $\psi \longrightarrow e^{i\theta\gamma_5}\psi$ when $M = 0$. The associated conserved currents are known to be given by $J_\mu = \bar{\psi}\gamma_\mu\psi$ and $J_\mu^5 = \bar{\psi}\gamma_\mu\gamma_5\psi$ where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. It is also a known result that in quantum theory one cannot maintain the conservation of both of these currents simultaneously in the presence of gauge fields.

A regularization which maintains exact chiral invariance of the above action can be achieved by replacing the Euclidean four-dimensional spacetime by a four-dimensional hypercubic lattice of N^4 sites. Every point on the lattice is specified by four integers which we denote collectively by $n = (n_1, n_2, n_3, n_4)$ where n_4 denotes Euclidean time. Clearly, each component of the 4-vector n is an integer in the range $-\frac{N}{2} \leqslant n_\mu \leqslant \frac{N}{2}$. The lattice is assumed to be periodic. Thus, we have $x_\mu = an_\mu$, where a is the lattice spacing and $L = aN$ is the linear size of the lattice. Now to each site $x = an$ we associate a spinor variable $\psi(n) = \psi(x)$ and the derivative $\partial_\mu\psi(x)$ is replaced by

$$\partial_\mu\psi(x) \longrightarrow \frac{1}{a}\hat{\partial}_\mu\psi(n) = \frac{1}{2a}[\psi(n + \hat{\mu}) - \psi(n - \hat{\mu})]. \tag{10.55}$$

The vector $\hat{\mu}$ is the unit vector in the μ-direction. With this prescription the action (10.53) becomes (with $\hat{M} = aM$ and $\hat{\psi} = a^{\frac{3}{2}}\psi$)

$$\begin{aligned} S_F &= \sum_n \sum_m \sum_\alpha \sum_\beta \bar{\hat{\psi}}_\alpha(n) K_{\alpha\beta}(n, m)\hat{\psi}_\beta(m) \\ K_{\alpha\beta}(n, m) &= \frac{1}{2}\sum_\mu (\gamma_\mu)_{\alpha\beta}\left(\delta_{m,n+\hat{\mu}} - \delta_{m,n-\hat{\mu}}\right) + \hat{M}\delta_{\alpha\beta}\delta_{m,n}. \end{aligned} \tag{10.56}$$

Next we will compute the Euclidean two-point function (propagator) which is given by the expression

$$\langle \hat{\psi}_\alpha(n)\bar{\hat{\psi}}_\beta(m)\rangle = K^{-1}_{\alpha\beta}(n, m). \tag{10.57}$$

The inverse is defined by $\sum_l \sum_\lambda K^{-1}_{\alpha\lambda}(n, l)K_{\lambda\beta}(l, m) = \delta_{\alpha\beta}\delta_{nm}$. The delta function δ_{nm} is obviously given by

$$\delta_{mn} = \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4} e^{i\hat{k}(n-m)}. \tag{10.58}$$

Thus

$$K_{\alpha\beta}(n, m) = \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4} e^{i\hat{k}(n-m)} K_{\alpha\beta}(\hat{k}), \tag{10.59}$$

where

$$K_{\alpha\beta}(\hat{k}) = i\sum_\mu (\gamma_\mu)_{\alpha\beta} \sin\hat{k}_\mu + \hat{M}\delta_{\alpha\beta}. \tag{10.60}$$

Above we have used the fact that $\hat{k}\hat{\mu} = \hat{k}_\mu$. Next we write

$$K^{-1}_{\alpha\beta}(n, m) = \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4} e^{i\hat{k}(n-m)} K^{-1}_{\alpha\beta}(\hat{k}). \tag{10.61}$$

By using $\sum_l e^{i(\hat{k}'-\hat{k})l} = (2\pi)^4\delta^4(\hat{k}' - \hat{k})$ we obtain from $\sum_l \sum_\lambda K^{-1}_{\alpha\lambda}(n, l)K_{\lambda\beta}(l, m) = \delta_{\alpha\beta}\delta_{nm}$ the condition $\sum_\lambda K^{-1}_{\alpha\lambda}(\hat{k})K_{\lambda\beta}(\hat{k}) = \delta_{\alpha\beta}$. The solution is given by

$$K^{-1}_{\alpha\beta}(\hat{k}) = \left(\frac{1}{i\sum_\mu \gamma_\mu \sin\hat{k}_\mu + \hat{M}} \right)_{\alpha\beta} = \frac{-i\sum_\mu (\gamma_\mu)_{\alpha\beta} \sin\hat{k}_\mu + \hat{M}\delta_{\alpha\beta}}{\sum_\mu (\sin\hat{k}_\mu)^2 + \hat{M}^2}. \tag{10.62}$$

Hence

$$\begin{aligned} \langle \hat{\psi}_\alpha(n)\bar{\hat{\psi}}_\beta(m)\rangle &= \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4} e^{i\hat{k}(n-m)} \frac{-i\sum_\mu (\gamma_\mu)_{\alpha\beta} \sin\hat{k}_\mu + \hat{M}\delta_{\alpha\beta}}{\sum_\mu (\sin\hat{k}_\mu)^2 + \hat{M}^2} \\ &= a^3 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{-i\sum_\mu (\gamma_\mu)_{\alpha\beta} \frac{\sin ak_\mu}{a} + M\delta_{\alpha\beta}}{\sum_\mu \left(\frac{\sin ak_\mu}{a}\right)^2 + M^2}. \end{aligned} \tag{10.63}$$

Equivalently

$$\begin{aligned} \langle \psi_\alpha(n)\bar{\psi}_\beta(m)\rangle = \lim_{a\to 0} &\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \\ &\times \frac{-i\sum_\mu (\gamma_\mu)_{\alpha\beta} \frac{\sin ak_\mu}{a} + M\delta_{\alpha\beta}}{\sum_\mu \left(\frac{\sin ak_\mu}{a}\right)^2 + M^2}. \end{aligned} \tag{10.64}$$

It is obvious that there are two different regions of the Brillouin zone which give rise to the continuum propagator. These are the small momentum region $k_\mu = 0$ and the large momentum region $k_\mu = \pm\pi/a$. To see this more explicitly let us rewrite the above propagator by separating the integral over each component k_μ in the following way

$$\begin{aligned} \langle \psi_\alpha(n)\bar{\psi}_\beta(m)\rangle = \lim_{a\to 0} &\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{\prod_{\nu\neq\mu} dk_\nu}{(2\pi)^3} e^{i\sum_{\nu\neq\mu} k_\nu (x-y)_\nu} \\ &\times \left[\int_{-\frac{\pi}{a}}^{-\frac{\pi}{2a}} \frac{dk_\mu}{2\pi} K(k_\mu) + \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{dk_\mu}{2\pi} K(k_\mu) + \int_{\frac{\pi}{2a}}^{\frac{\pi}{a}} \frac{dk_\mu}{2\pi} K(k_\mu) \right]. \end{aligned} \tag{10.65}$$

The kernel $K(k_\mu)$ is

$$K(k_\mu) = \left[i \sum_{\nu \neq \mu} \gamma_\nu \frac{\sin(ak_\nu)}{a} + i\gamma_\mu \frac{\sin(ak_\mu)}{a} + M \right]^{-1} e^{ik_\mu (x-y)_\mu}. \tag{10.66}$$

Now, by changing the integration variable in the first term to $\tilde{k}_\mu = k_\mu + \frac{\pi}{a}$ and in the last term to $\tilde{k}_\mu = k_\mu - \frac{\pi}{a}$ one obtains

$$\begin{aligned} \langle \psi_\alpha(n) \bar{\psi}_\beta(m) \rangle = \lim_{a \longrightarrow 0} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{\prod_{\nu \neq \mu} dk_\nu}{(2\pi)^3} e^{i \sum_{\nu \neq \mu} k_\nu (x-y)_\nu} \\ \times \left[\int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{dk_\mu}{2\pi} K(k_\mu) + \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{d\tilde{k}_\mu}{2\pi} \tilde{K}(\tilde{k}_\mu) \right]. \end{aligned} \tag{10.67}$$

In the above

$$\begin{aligned} \tilde{K}(\tilde{k}_\mu) = K\left(\tilde{k}_\mu \pm \frac{\pi}{a} \right) = \cos \frac{\pi}{a} (x-y)_\mu \\ \times \left[i \sum_{\nu \neq \mu} \gamma_\nu \frac{\sin(ak_\nu)}{a} - i\gamma_\mu \frac{\sin(a\tilde{k}_\mu)}{a} + M \right]^{-1} e^{i\tilde{k}_\mu (x-y)_\mu}. \end{aligned} \tag{10.68}$$

For small lattice spacing the first term of equation (10.67) is dominated by small momenta $k_\mu \longrightarrow 0$ which leads to the propagator $[i\gamma_\mu k_\mu + M + O(a^2)]^{-1}$, whereas the second term is dominated by large momenta $\tilde{k}_\mu \longrightarrow 0$ which also leads to a similar continuum propagator. It is a similar propagator and not the same because of the minus sign in $\tilde{K}(\tilde{k}_\mu)$ which can be absorbed by redefining the γ_μ and, therefore, redefining the chirality operator so that it corresponds to a different fermion species.

Thus, for each spacetime dimension we have two different regions in momentum space where the discrete theory (10.56) gives the continuum fermion propagator. Altogether we obtain in the continuum $2^4 = 16$ independent fermion species corresponding to $2^4 = 16$ different regions of integration. Of these, 15 regions correspond to high momentum excitations of the order of $\pm\pi/a$ with the correct continuum propagator. In d spacetime dimensions we will have 2^d fermion species.

This fermion doubling phenomenon is due to the fact that our regularization scheme preserves exact chiral invariance, i.e. there is no chiral anomaly. Putting it differently, the effect of these extra fermions is to cancel exactly the chiral anomaly. Clearly, in order to obtain the correct continuum limit we must eliminate these extra fermion species.

10.2.2 Wilson fermions

In this section we will modify the lattice action (10.56) by a term which vanishes in the naive continuum limit but is not invariant under chiral transformations.

We consider

$$S_F^W = S_F - \frac{r}{2}\sum_n \sum_\alpha \bar{\hat{\psi}}_\alpha(n)\hat{\partial}^2\hat{\psi}_\alpha(n), \tag{10.69}$$

where

$$S_F = \sum_n \sum_m \sum_\alpha \sum_\beta \bar{\hat{\psi}}_\alpha(n) K_{\alpha\beta}(n, m)\hat{\psi}_\beta(m), \tag{10.70}$$

and

$$K_{\alpha\beta}(n, m) = \frac{1}{2}\sum_\mu (\gamma_\mu)_{\alpha\beta}\left(\delta_{m,n+\hat{\mu}} - \delta_{m,n-\hat{\mu}}\right) + \hat{M}\delta_{\alpha\beta}\delta_{m,n}. \tag{10.71}$$

The parameter r is the Wilson parameter. The operator $\hat{\partial}^2$ is the lattice Laplacian defined as follows. The lattice derivative was defined by the expression

$$\hat{\partial}_\mu\psi(n) = \frac{1}{2}[\psi(n + \hat{\mu}) - \psi(n - \hat{\mu})]. \tag{10.72}$$

Alternatively, we can define the lattice derivative by the expressions

$$\hat{\partial}_\mu^R\psi(n) = \psi(n + \hat{\mu}) - \psi(n) \tag{10.73}$$

$$\hat{\partial}_\mu^L\psi(n) = \psi(n) - \psi(n - \hat{\mu}). \tag{10.74}$$

We use these last two expressions to construct the lattice Laplacian as follows

$$\hat{\partial}^2\psi(n) = \sum_\mu \hat{\partial}_\mu^R\hat{\partial}_\mu^L\psi(n) = \sum_\mu (\psi(n + \hat{\mu}) + \psi(n - \hat{\mu}) - 2\psi(n)) = \sum_\mu \bar{\partial}_\mu\psi(n). \tag{10.75}$$

By setting $\hat{\psi} = a^{\frac{3}{2}}\psi$ and $\hat{\partial}^2 = a^2\partial^2$ we can see that the Wilson term vanishes linearly with a in the continuum limit. The Wilson action can be put in the form

$$S_F^W = \sum_n \sum_m \sum_\alpha \sum_\beta \bar{\hat{\psi}}_\alpha(n) K_{\alpha\beta}^W(n, m)\hat{\psi}_\beta(m), \tag{10.76}$$

where

$$\begin{aligned} K^W_{\alpha\beta}(n,\, m) &= \frac{1}{2}\sum_\mu (\gamma_\mu)_{\alpha\beta}\left(\delta_{m,n+\hat{\mu}} - \delta_{m,n-\hat{\mu}}\right) + \hat{M}\delta_{\alpha\beta}\delta_{m,n} \\ &\quad - \frac{r}{2}\delta_{\alpha\beta}\sum_\mu \left(\delta_{m,n+\hat{\mu}} + \delta_{m,n-\hat{\mu}} - 2\delta_{m,n}\right) \\ &= \frac{1}{2}\sum_\mu \left[(\gamma_\mu - r)_{\alpha\beta}\delta_{m,n+\hat{\mu}} - (\gamma_\mu + r)_{\alpha\beta}\delta_{m,n-\hat{\mu}}\right] \\ &\quad + (\hat{M} + 4r)\delta_{\alpha\beta}\delta_{m,n}. \end{aligned} \tag{10.77}$$

This expression breaks chiral symmetry for $r \neq 0$ even for $\hat{M} \longrightarrow 0$. As before, we can write

$$S^W_F = \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4} e^{i\hat{k}(n-m)} K^W_{\alpha\beta}(\hat{k}). \tag{10.78}$$

In this case

$$\begin{aligned} K^W_{\alpha\beta}(\hat{k}) &= i\sum_\mu (\gamma_\mu)_{\alpha\beta}\sin\hat{k}_\mu + \hat{M}\delta_{\alpha\beta} + 2r\sum_\mu \sin^2\frac{\hat{k}_\mu}{2}\delta_{\alpha\beta} \\ &= i\sum_\mu (\gamma_\mu)_{\alpha\beta}\sin\hat{k}_\mu + \hat{M}(\hat{k})\delta_{\alpha\beta}. \end{aligned} \tag{10.79}$$

We can see that we will obtain the result

$$\langle\psi_\alpha(n)\bar{\psi}_\beta(m)\rangle = \lim_{a\longrightarrow 0}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{-i\sum_\mu (\gamma_\mu)_{\alpha\beta}\frac{\sin ak_\mu}{a} + M(k)\delta_{\alpha\beta}}{\sum_\mu \left(\frac{\sin ak_\mu}{a}\right)^2 + M^2(k)}, \tag{10.80}$$

where

$$M(k) = M + \frac{2r}{a}\sum_\mu \sin^2\frac{ak_\mu}{2}. \tag{10.81}$$

We note that when $k_\mu \longrightarrow 0$, $M(k) \longrightarrow M$ when $a \longrightarrow 0$ and when $k_\mu \longrightarrow \pm\frac{\pi}{a}$, $M(k) \longrightarrow \infty$ when $a \longrightarrow 0$. In other words, the modes near the corners of the BZ will have an infinite mass in the continuum limit and hence they decouple. This solves the fermion doubling problem but at the expense of breaking chiral symmetry.

10.2.3 Staggered fermions

Another solution of the fermion doubling problem is given by staggered fermions which goes as follows. We start from the action

$$S_F = \frac{1}{2}\sum_n \sum_{\alpha,\beta} \sum_\mu \left[\bar{\hat{\psi}}_\alpha(n)(\gamma_\mu)_{\alpha\beta}\hat{\psi}_\beta(n+\hat{\mu}) - \bar{\hat{\psi}}_\alpha(n)(\gamma_\mu)_{\alpha\beta}\hat{\psi}_\beta(n-\hat{\mu})\right] + \hat{M}\sum_n \sum_\alpha \bar{\hat{\psi}}_\alpha(n)\hat{\psi}_\alpha(n). \tag{10.82}$$

We spin diagonalize this expression by making the local change of variables[1]

$$\hat{\psi}(n) = T(n)\chi(n), \;\; \bar{\hat{\psi}}(n) = \bar{\chi}(n)T^+(n). \tag{10.83}$$

The $T(n)$ are unitary $2^{\frac{d}{2}} \times 2^{\frac{d}{2}}$ matrices which also satisfy

$$T^+(n)\gamma_\mu T(n+\hat{\mu}) = \eta_\mu(n)\mathbf{1}. \tag{10.84}$$

The action becomes

$$S_F = \frac{1}{2}\sum_n \sum_\alpha \sum_\mu \eta_\mu[\bar{\chi}_\alpha(n)\chi_\alpha(n+\hat{\mu}) - \bar{\chi}_\alpha(n)\chi_\alpha(n-\hat{\mu})] + \hat{M}\sum_n \sum_\alpha \bar{\chi}_\alpha(n)\chi_\alpha(n). \tag{10.85}$$

A solution of equation (10.84) is given by

$$T(n) = \gamma_1^{n_1}\gamma_2^{n_2} \ldots \gamma_d^{n_d}, \tag{10.86}$$

and

$$\eta_1(n) = 1, \;\; \eta_\mu(n) = (-1)^{n_1+n_2+\cdots+n_{\mu-1}}. \tag{10.87}$$

In equation (10.85) the index α runs over $\alpha = 1, 2, \ldots, 2^{\frac{d}{2}}$ but we can, in principle, let α run over any other number of possible values, viz $\alpha = 1, \ldots, k$. The minimal choice $k = 1$ corresponds to the action of the staggered fermions

$$S_F = \frac{1}{2}\sum_n \sum_\mu \eta_\mu(n)[\bar{\chi}(n)\chi(n+\hat{\mu}) - \bar{\chi}(n)\chi(n-\hat{\mu})] + \hat{M}\sum_n \bar{\chi}(n)\chi(n). \tag{10.88}$$

The coordinate n_μ of every lattice site can be written in the form $n_\mu = 2N_\mu + \rho_\mu$ where $2N_\mu$ are the coordinates of the origin of the hypercube to which the site belongs and ρ_μ are such that $\rho_\mu = 0$ or 1. Every hypercube will have 2^d sites and there are $N^d/2^d$ hypercubes (here N is the number of sites in every direction). We can write the action as

$$S_F = \frac{1}{2}\sum_{N,\rho} \sum_\mu \eta_\mu(\rho)[\bar{\chi}(2N+\rho)\chi(2N+\rho+\hat{\mu}) - \bar{\chi}(2N+\rho)\chi(2N+\rho-\hat{\mu})] + \hat{M}\sum_{N,\rho} \bar{\chi}(2N+\rho)\chi(2N+\rho). \tag{10.89}$$

[1] The fields $\hat{\psi}(n)$ and $\bar{\hat{\psi}}(n)$ are considered independent fields.

In the above we have used the fact that $\eta_\mu(2N+\rho)=\eta_\mu(\rho)$. We propose the relabeling

$$\chi(2N+\rho)=\chi_\rho(N). \tag{10.90}$$

The vector N labels now a site of an effective lattice with lattice spacing $2a$. Since there are 2^d vectors ρ we will think of ρ as an index which goes over 2^d values. Next, we need to deduce how to label $\chi(2N+\rho\pm\hat{\mu})$. If $\rho+\hat{\mu}$ is a vector of type ρ which means that the components of $\rho+\hat{\mu}$ are either 1 or 0 then $2N+\rho+\hat{\mu}$ labels a site within the hypercube of origin $2N$. In this case $\chi(2N+\rho+\hat{\mu})=\chi_{\rho+\hat{\mu}}(N)$. If $\rho+\hat{\mu}$ is not a vector of type ρ then $\rho-\hat{\mu}$ will be a vector of type ρ and in this case $\chi(2N+\rho+\hat{\mu})=\chi_{\rho-\hat{\mu}}(N+\hat{\mu})$. We write

$$\chi(2N+\rho+\hat{\mu})=\sum_{\rho'}\Big[\delta_{\rho+\hat{\mu},\rho'}\chi_{\rho'}(N)+\delta_{\rho-\hat{\mu},\rho}\chi_{\rho'}(N+\hat{\mu})\Big]. \tag{10.91}$$

Similarly

$$\chi(2N+\rho-\hat{\mu})=\sum_{\rho'}\Big[\delta_{\rho-\hat{\mu},\rho}\chi_{\rho'}(N)+\delta_{\rho+\hat{\mu},\rho}\chi_{\rho'}(N-\hat{\mu})\Big]. \tag{10.92}$$

The action of the staggered fermions becomes

$$\begin{aligned}S_F=&\frac{1}{2}\sum_{N,\rho,\rho'}\sum_\mu\eta_\mu(\rho)\Big[\delta_{\rho-\hat{\mu},\rho}\bar{\chi}_\rho(N)\hat{\partial}^R_\mu\chi_{\rho'}(N)+\delta_{\rho+\hat{\mu},\rho}\bar{\chi}_\rho(N)\hat{\partial}^L_\mu\chi_{\rho'}(N)\Big]\\&+\hat{M}\sum_{N,\rho}\bar{\chi}_\rho(N)\chi_\rho(N).\end{aligned} \tag{10.93}$$

Note that $\hat{\partial}^R_\mu=\hat{\partial}_\mu+\frac{1}{2}\bar{\partial}_\mu$ and $\hat{\partial}^L_\mu=\hat{\partial}_\mu-\frac{1}{2}\bar{\partial}_\mu$. Thus

$$S_F=\frac{1}{2}\sum_{N,\rho,\rho'}\bar{\chi}_\rho(N)\left[\sum_\mu\Gamma^\mu_{\rho\rho'}\hat{\partial}_\mu+\frac{1}{2}\sum_\mu\Gamma^{5\mu}_{\rho\rho'}\bar{\partial}_\mu+2\hat{M}\delta_{\rho\rho'}\right]\chi_{\rho'}(N). \tag{10.94}$$

In above the matrices Γ^μ and $\Gamma^{5\mu}$ are defined by

$$\begin{aligned}\Gamma^\mu_{\rho\rho'}&=\eta_\mu(\rho)\big(\delta_{\rho-\hat{\mu},\rho'}+\delta_{\rho+\hat{\mu},\rho'}\big)\\\Gamma^{5\mu}_{\rho\rho'}&=\eta_\mu(\rho)\big(\delta_{\rho-\hat{\mu},\rho'}-\delta_{\rho+\hat{\mu},\rho'}\big).\end{aligned} \tag{10.95}$$

Let us first assume that $\nu>\mu$, then we can check that $\eta_\nu(\rho\pm\hat{\mu})=-\eta_\nu(\rho)$ and $\eta_\mu(\rho\pm\hat{\nu})=\eta_\mu(\rho)$ and hence $\{\Gamma^\mu,\Gamma^\nu\}_{\rho\rho'}=0$. Similarly we can check that $\{\Gamma^\mu,\Gamma^\nu\}_{\rho\rho'}=0$ for $\mu>\nu$.

For $\mu=\nu$ we have $\eta_\mu(\rho\pm\hat{\mu})=\eta_\mu(\rho)$ and $\delta_{\rho\pm\hat{\mu},\rho'}\delta_{\rho'\pm\hat{\mu},\beta}=0$ and thus $(\Gamma^\mu\Gamma^\mu)_{\rho\beta}=\delta_{\rho-\hat{\mu},\rho'}\delta_{\rho'+\hat{\mu},\beta}+\delta_{\rho+\hat{\mu},\rho}\delta_{\rho'-\hat{\mu},\beta}$. Clearly for every fixed (ρ,β) with $\rho=\beta$ either $\delta_{\rho-\hat{\mu},\rho'}\delta_{\rho'+\hat{\mu},\beta}=1$ or $\delta_{\rho+\hat{\mu},\rho'}\delta_{\rho'-\hat{\mu},\beta}=1$. For $\rho\neq\beta$ we get 0. Thus $(\Gamma^\mu\Gamma^\mu)_{\rho\beta}=\delta_{\rho,\beta}$.

Hence we obtain

$$\{\Gamma^{\mu}, \Gamma^{\nu}\}_{\rho\rho'} = 2\delta_{\mu\nu}\delta_{\rho\rho'}. \tag{10.96}$$

Similar arguments lead to the results

$$\{\Gamma^{5\mu}, \Gamma^{5\nu}\}_{\rho\rho'} = -2\delta_{\mu\nu}\delta_{\rho\rho'}, \tag{10.97}$$

and

$$\{\Gamma^{\mu}, \Gamma^{5\nu}\}_{\rho\rho'} = 0. \tag{10.98}$$

We observe that Γ^{μ} and $\Gamma^{5\mu}$ satisfy the same algebra as $\gamma_{\mu} \otimes \mathbf{1}$ and $\gamma_5 \otimes \gamma_{\mu}\gamma_5$ where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. The first factor acts in Dirac space, whereas the second factor acts in flavor space. Thus Γ^{μ} and $\Gamma^{5\mu}$ are unitarily equivalent to $\gamma_{\mu} \otimes \mathbf{1}$ and $\gamma_5 \otimes \gamma_{\mu}\gamma_5$ respectively. We want to construct this unitary transformation explicitly. We define the 4×4 matrices

$$T_{\rho} = \gamma_1^{\rho_1}\gamma_2^{\rho_2}\gamma_3^{\rho_3}\gamma_4^{\rho_4}. \tag{10.99}$$

We introduce the 16×16 matrix

$$U_{\alpha\beta,\rho} = \frac{1}{2}(T_{\rho})_{\alpha\beta}. \tag{10.100}$$

The rows are labeled by the pair (α, β) which takes 16 values since α and β run over the values one to four, whereas the columns are labeled by the index ρ which runs over the 16 sites of a hypercube. We note that $(U^{+})_{\rho,\alpha\beta} = U^{*}_{\alpha\beta,\rho} = \frac{1}{2}(T_{\rho})^{*}_{\alpha\beta}$. The matrix U is unitary. Indeed we can check that

$$\begin{aligned}(U^{+}U)_{\rho\rho'} &= \frac{1}{4}\,\mathrm{Tr}\; T_{\rho}^{+}T_{\rho'} \\ &= \frac{1}{4}\,\mathrm{Tr}\; \gamma_4^{\rho_4+\rho_4'}\gamma_3^{\rho_3}\gamma_2^{\rho_2}\gamma_1^{\rho_1+\rho_1'}\gamma_2^{\rho_2'}\gamma_3^{\rho_3'}.\end{aligned} \tag{10.101}$$

Recall that the indices ρ_{μ} and ρ_{μ}' can only take the values 0 and 1. Since the trace of any product of distinct gamma matrix is zero we can conclude that

$$(U^{+}U)_{\rho\rho'} = \delta_{\rho\rho'}. \tag{10.102}$$

Next, we use the explicit representation $\gamma_i = ((\gamma_i)_{11}, (\gamma_i)_{12}, (\gamma_i)_{21}, (\gamma_i)_{22}) = (0, -i\sigma_i, i\sigma_i, 0)$ and $\gamma_4 = ((\gamma_4)_{11}, (\gamma_4)_{12}, (\gamma_4)_{21}, (\gamma_4)_{22}) = (\mathbf{1}, 0, 0, -\mathbf{1})$ to show that

$$\begin{aligned}(UU^{+})_{\alpha\beta,\alpha'\beta'} &= \frac{1}{4}\sum_{\rho}(T_{\rho})_{\alpha\beta}\left(T_{\rho}^{+}\right)_{\beta'\alpha'} \\ &= \delta_{\alpha\alpha'}\delta_{\beta\beta'}.\end{aligned} \tag{10.103}$$

We introduce the following fields

$$\begin{aligned}\hat{\psi}_{\alpha\beta}(N) &= \mathcal{N}_0 \sum_{\rho} U_{\alpha\beta,\rho}\chi_{\rho}(N)\\ \bar{\hat{\psi}}_{\alpha\beta}(N) &= \mathcal{N}_0 \sum_{\rho} \bar{\chi}_{\rho}(N)(U^{+})_{\rho,\alpha\beta}.\end{aligned} \tag{10.104}$$

Equivalently we have

$$\begin{aligned}\chi_{\rho}(N) &= \frac{1}{\mathcal{N}_0}\sum_{\alpha\beta}(U^{+})_{\rho,\alpha\beta}\hat{\psi}_{\alpha\beta}(N)\\ \bar{\chi}_{\rho}(N) &= \frac{1}{\mathcal{N}_0}\sum_{\alpha\beta}\bar{\hat{\psi}}_{\alpha\beta}(N)U_{\alpha\beta,\rho}.\end{aligned} \tag{10.105}$$

The index α is a Dirac index, whereas index β is a flavor index. By inserting these last two equations into equation (10.94) we obtain the action of the staggered fermions

$$\begin{aligned}S_F = &\frac{1}{2\mathcal{N}_0^2}\sum_{N,\alpha,\beta,\alpha',\beta'}\bar{\hat{\psi}}_{\alpha\beta}(N)\\ &\times\left[\sum_{\mu}\Lambda^{\mu}_{\alpha\beta,\alpha'\beta'}\hat{\partial}_{\mu} + \frac{1}{2}\sum_{\mu}\Lambda^{5\mu}_{\alpha\beta,\alpha'\beta'}\bar{\partial}_{\mu} + 2\hat{M}\delta_{\alpha\alpha'}\delta_{\beta\beta'}\right]\hat{\psi}_{\alpha'\beta'}(N)\end{aligned} \tag{10.106}$$

$$\Lambda^{\mu}_{\alpha\beta,\alpha'\beta'} = \sum_{\rho\rho'} U_{\alpha\beta,\rho}\Gamma^{\mu}_{\rho\rho'}(U^{+})_{\rho',\alpha'\beta'} \tag{10.107}$$

$$\Lambda^{5\mu}_{\alpha\beta,\alpha'\beta'} = \sum_{\rho\rho'} U_{\alpha\beta,\rho}\Gamma^{5\mu}_{\rho\rho'}(U^{+})_{\rho',\alpha'\beta'}. \tag{10.108}$$

Let us fix $\mu = 1$. Then $\Gamma^{1}_{\rho\rho'} = \delta_{\rho,\rho'+\hat{1}} + \delta_{\rho+\hat{1},\rho'}$. We compute

$$\begin{aligned}\Lambda^{1}_{\alpha\beta,\alpha'\beta'} &= \frac{1}{4}\sum_{\rho,\rho'}(T_{\rho'+\hat{1}})_{\alpha\beta}(T_{\rho'})^{+}_{\beta'\alpha}\delta_{\rho,\rho'+\hat{1}} + \frac{1}{4}\sum_{\rho,\rho'}(T_{\rho})_{\alpha\beta}(T_{\rho+\hat{1}})^{+}_{\beta'\alpha'}\delta_{\rho+\hat{1},\rho'}\\ &= \frac{1}{4}\sum_{\rho'}(T_{\rho'+\hat{1}})_{\alpha\beta}(T_{\rho'})^{+}_{\beta'\alpha}\delta_{\rho_1',0} + \frac{1}{4}\sum_{\rho}(T_{\rho})_{\alpha\beta}(T_{\rho+\hat{1}})^{+}_{\beta'\alpha'}\delta_{\rho_1,0}\\ &= \frac{1}{4}\sum_{\rho'}(T_{\rho'+\hat{1}})_{\alpha\beta}(T_{\rho'})^{+}_{\beta'\alpha'}.\end{aligned} \tag{10.109}$$

We can write $T_{\rho'+\hat{1}} = \gamma_1 T_{\rho'}$ and hence we obtain

$$\Lambda^{1}_{\alpha\beta,\alpha'\beta'} = (\gamma_1)_{\alpha\alpha'}\delta_{\beta\beta'}. \tag{10.110}$$

Similarly, we can show for the other values of μ that

$$\Lambda^{\mu}_{\alpha\beta,\alpha'\beta'} = (\gamma_\mu)_{\alpha\alpha'}\delta_{\beta\beta'}. \tag{10.111}$$

Let us fix $\mu = 1$ again. This time we have $\Gamma^{51}_{\rho\rho'} = \delta_{\rho,\rho'+\hat{1}} - \delta_{\rho+\hat{1},\rho'}$. We have

$$\begin{aligned}
\Lambda^{51}_{\alpha\beta,\alpha'\beta'} &= \frac{1}{4}\sum_{\rho,\rho'}(T_{\rho'+\hat{1}})_{\alpha\beta}(T_{\rho'})^{+}_{\beta'\alpha}\delta_{\rho,\rho'+\hat{1}} - \frac{1}{4}\sum_{\rho,\rho'}(T_{\rho})_{\alpha\beta}(T_{\rho+\hat{1}})^{+}_{\beta'\alpha'}\delta_{\rho+\hat{1},\rho'} \\
&= \frac{1}{4}\sum_{\rho'}(T_{\rho'+\hat{1}})_{\alpha\beta}(T_{\rho'})^{+}_{\beta'\alpha}\delta_{\rho_1',0} - \frac{1}{4}\sum_{\rho}(T_{\rho})_{\alpha\beta}(T_{\rho+\hat{1}})^{+}_{\beta'\alpha'}\delta_{\rho_1,0} \\
&= \frac{1}{4}\sum_{\rho'}(-1)^{\rho_1'}(T_{\rho'+\hat{1}})_{\alpha\beta}(T_{\rho'})^{+}_{\beta'\alpha'}.
\end{aligned} \tag{10.112}$$

In the representation $\gamma_i = (0, -i\sigma_i, i\sigma_i, 0)$ and $\gamma_4 = (\mathbf{1}, 0, 0, -\mathbf{1})$ we compute $\gamma_5 = (0, -\mathbf{1}, -\mathbf{1}, 0)$. We check that $\gamma_1 = \gamma_5\gamma_1^*\gamma_5$, $\gamma_1^*\gamma_\mu = -\gamma_\mu\gamma_1^*$, $\mu \neq 1$ and $\gamma_1^*\gamma_1 = \gamma_1\gamma_1^* = -\mathbf{1}$. Thus, we can write $T_{\rho'+\hat{1}} = \gamma_1 T_{\rho'} = (-1)^{\rho_1'}\gamma_5 T_{\rho'}\gamma_1^*\gamma_5$ and $(T_{\rho+\hat{1}})^{+} = -(-1)^{\rho_1}\gamma_1^*\gamma_5(T_\rho)^{+}\gamma_5$. Hence, we get

$$\begin{aligned}
\Lambda^{51}_{\alpha\beta,\alpha'\beta'} &= \frac{1}{4}(\gamma_5)_{\alpha\gamma}\left(\gamma_1^*\gamma_5\right)_{\lambda\beta}\sum_{\rho'}\left(T_{\rho'}\right)_{\gamma\lambda}\left(T_{\rho'}\right)^{+}_{\beta'\alpha'} \\
&= (\gamma_5)_{\alpha\alpha'}\left(\gamma_1^*\gamma_5\right)_{\beta\beta'}.
\end{aligned} \tag{10.113}$$

Similarly, we can show for the other values of μ that

$$\Lambda^{5\mu}_{\alpha\beta,\alpha'\beta'} = (\gamma_5)_{\alpha\alpha'}\left(\gamma_\mu^*\gamma_5\right)_{\beta\beta'}. \tag{10.114}$$

In the naive continuum limit we have $\hat{\psi} = b^{\frac{3}{2}}\psi$, $\hat{M} = bM$, $\hat{\partial}_\mu = b\partial_\mu$ and $\bar{\partial}_\mu = b^2\tilde{\partial}_\mu$ where $\sum_\mu \tilde{\partial}_\mu = \partial^2$ in the same way that $\sum_\mu \bar{\partial}_\mu = \hat{\partial}^2$. The b is the lattice spacing of the blocked lattice, i.e. $b = 2a$. Thus, the action becomes in this limit

$$\begin{aligned}
S_F = \frac{b^4}{2\mathcal{N}_0^2}\Bigg(&\sum_{x,\mu}\bar{\psi}(x)\left[(\gamma^\mu \otimes \mathbf{1})\partial_\mu + \frac{b}{2}\left(\gamma_5 \otimes \gamma_\mu^*\gamma_5\right)\tilde{\partial}_\mu\right]\psi(x) \\
&+ 2M\sum_x \bar{\psi}(x)(\mathbf{1}\otimes\mathbf{1})\psi(x)\Bigg).
\end{aligned} \tag{10.115}$$

In the above $x = Nb$ and the sum over x runs over all hypercubes of the blocked lattice. The second term vanishes in the limit $b \longrightarrow 0$ and, therefore, the expression will reduce to

$$S_F = \frac{1}{2\mathcal{N}_0^2}\int d^4x \sum_f \bar{\psi}^f(x)\left[\sum_\mu \gamma^\mu\partial_\mu + M_0\right]\psi^f(x). \tag{10.116}$$

Clearly we must have $2\mathcal{N}_0^2 = 1$ and $M_0 = 2M = \hat{M}/a$.

Finally, we calculate the two-point function in this case as follows. We go to the momentum space by Fourier transform, viz

$$\psi(x) = \psi(Nb) = \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} \frac{d^4p}{(2\pi)^4}\tilde{\psi}(p)e^{ip\cdot Nb} \tag{10.117}$$

$$\bar{\psi}(x) = \bar{\psi}(Nb) = \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} \frac{d^4p}{(2\pi)^4}\bar{\tilde{\psi}}(p)e^{-ip\cdot Nb}. \tag{10.118}$$

We compute

$$\partial_\mu\psi(x) = \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} \frac{d^4p}{(2\pi)^4}\frac{i}{b}\sin(bp_\mu)\tilde{\psi}(p)e^{ip\cdot Nb} \tag{10.119}$$

$$\tilde{\partial}_\mu\psi(x) = \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} \frac{d^4p}{(2\pi)^4}\left(-\frac{2}{b^2}\right)(1-\cos(bp_\mu))\tilde{\psi}(p)e^{ip\cdot Nb}. \tag{10.120}$$

The action S_F in momentum space will read as follows

$$\begin{aligned} S_F = \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} \frac{d^4p}{(2\pi)^4}\bar{\tilde{\psi}}(p)\Bigg(\sum_\mu\Bigg[&(\gamma^\mu\otimes\mathbf{1})\frac{i}{b}\sin(bp_\mu) \\ &-\frac{2}{b}\left(\gamma_5\otimes\gamma_\mu^*\gamma_5\right)\sin^2\left(\frac{bp_\mu}{2}\right)\Bigg] + M_0\Bigg)\tilde{\psi}(p). \end{aligned} \tag{10.121}$$

In the above we have used the obvious equation

$$b^4\sum_N e^{i(p-p')\cdot Nb} = (2\pi)^4\delta_P^{(4)}(p-p'). \tag{10.122}$$

The $\delta_P^{(4)}$ is the periodic delta function. We note that

$$\begin{aligned} &\left(\sum_\mu\left[(\gamma^\mu\otimes\mathbf{1})\frac{i}{b}\sin(bp_\mu) - \frac{2}{b}\left(\gamma_5\otimes\gamma_\mu^*\gamma_5\right)\sin^2\left(\frac{bp_\mu}{2}\right)\right] + M_0\right) \\ &\times\left(-\sum_\mu\left[(\gamma^\mu\otimes\mathbf{1})\frac{i}{b}\sin(bp_\mu) - \frac{2}{b}\left(\gamma_5\otimes\gamma_\mu^*\gamma_5\right)\sin^2\left(\frac{bp_\mu}{2}\right)\right] + M_0\right) \\ &= \sum_\mu\left(\frac{1}{b^2}\sin^2(bp_\mu) + \frac{4}{b^2}\sin^4\left(\frac{bp_\mu}{2}\right)\right) + M_0^2 \\ &= \sum_\mu\frac{4}{b^2}\sin^2\left(\frac{bp_\mu}{2}\right) + M_0^2. \end{aligned} \tag{10.123}$$

The propagator in momentum space is therefore given by

$$S(p) = \frac{-\sum_\mu \left[(\gamma^\mu \otimes \mathbf{1}) \frac{i}{b} \sin(bp_\mu) - \frac{2}{b} \left(\gamma_5 \otimes \gamma_\mu^* \gamma_5 \right) \sin^2 \left(\frac{bp_\mu}{2} \right) \right] + M_0}{\sum_\mu \frac{4}{b^2} \sin^2 \left(\frac{bp_\mu}{2} \right) + M_0^2}. \tag{10.124}$$

Because of the division by 2 in $\sin^2\left(\frac{bp_\mu}{2}\right)$ this propagator has the correct continuum limit. Indeed, for $b \longrightarrow 0$ this expression reduces to

$$S(p) = \frac{-i \sum_\mu (\gamma^\mu \otimes \mathbf{1}) p_\mu + M_0}{p^2 + M_0^2}. \tag{10.125}$$

10.3 Gauge fields on the lattice

10.3.1 Continuum considerations: quantum electrodynamics

A field theory is a dynamical system with N degrees of freedom where $N \longrightarrow \infty$. The classical description is given in terms of the Lagrangian and the action, while the quantum description is given in terms of the Feynman path integral and the correlation functions. In a scalar field theory the basic field has spin $j = 0$ with respect to Lorentz transformations. Scalar field theories are relevant to critical phenomena. In gauge theories the basic fields have spin $j = 1$ (gauge vector fields) and spin $j = 1/2$ (fermions) and they are relevant to particle physics. The requirement of renormalizability severely restricts the set of quantum field theories to only a few possible models. For example, quantum electrodynamics is a renormalizable field theory given by the action

$$S_{\text{QED}} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\gamma^\mu \partial_\mu - M \right) \psi - e \bar{\psi} \gamma_\mu \psi A^\mu \right]. \tag{10.126}$$

The γ^μ are the famous 4×4 Dirac gamma matrices which appear in any theory containing a spin 1/2 field. They satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ where $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The electromagnetic field is given by the $U(1)$ gauge vector field A^μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ while the fermion (electron) field is given by the spinor field ψ with mass M. The spinor ψ is a 4-component field and $\bar{\psi} = \psi^+ \gamma^0$. The interaction term is proportional to the electric charge e given by the last term $-e\bar{\psi}\gamma_\mu \psi A^\mu$. The Euler–Lagrange classical equations of motion, derived from the above action, are precisely the Maxwell equations $\partial_\mu F^{\mu\nu} = j^\nu$ with $j^\mu = e\bar{\psi}\gamma^\mu\psi$ and the Dirac equation $\left(i\gamma^\mu \partial_\mu - m - e\gamma_\mu A^\mu \right) \psi = 0$. The above theory is also invariant under the following $U(1)$ gauge transformations

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda, \ \psi \longrightarrow \exp(-ie\Lambda)\psi, \ \bar{\psi} \longrightarrow \bar{\psi} \exp(ie\Lambda). \tag{10.127}$$

The Feynman path integral is

$$Z = \int \mathcal{D}A^{\mu}\mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp(iS_{\text{QED}}). \tag{10.128}$$

Before we can study this theory, or any other gauge theory, numerically using the Monte Carlo method we need to:

1. Rotate to Euclidean signature in order to convert the theory into a statistical field theory.
2. Regularize the UV behavior of the theory by putting it on a lattice.

As a consequence, we obtain an ordinary statistical system accessible to ordinary sampling techniques such as the Metropolis algorithm.

We start by discussing a little further the above action.

The free fermion action in Minkowski spacetime is

$$S_F = \int d^4x \bar{\psi}(x)\big(i\gamma^{\mu}\partial_{\mu} - M\big)\psi(x). \tag{10.129}$$

This action is invariant under the global $U(1)$ transformation $\psi(x) \longrightarrow G\psi(x)$ and $\bar{\psi}(x) \longrightarrow \bar{\psi}(x)G^{-1}$ where $G = \exp(i\Lambda)$. The symmetry $U(1)$ can be made local (i.e. G becomes a function of x) by replacing the ordinary derivative ∂_{μ} with the covariant derivative $D_{\mu} = \partial_{\mu} + ieA_{\mu}$ where the $U(1)$ gauge field A_{μ} is the electromagnetic 4-vector potential. The action becomes

$$S_F = \int d^4x \bar{\psi}(x)\big(i\gamma^{\mu}D_{\mu} - M\big)\psi(x). \tag{10.130}$$

This action is invariant under

$$\psi \longrightarrow G(x)\psi, \;\; \bar{\psi} \longrightarrow \bar{\psi}G^{-1}(x) \tag{10.131}$$

provided we also transform the covariant derivative and the gauge field as follows

$$D_{\mu} \longrightarrow GD_{\mu}G^{-1} \iff A_{\mu} \longrightarrow G(x)A_{\mu}G^{-1}(x) - \frac{i}{e}G(x)\partial_{\mu}G^{-1}(x). \tag{10.132}$$

Since A_{μ} and $G(x) = \exp(i\Lambda(x))$ commute the transformation law of the gauge field reduces to $A_{\mu} \longrightarrow A_{\mu} - \frac{1}{e}\partial_{\mu}\Lambda$. The dynamics of the gauge field A_{μ} is given by the Maxwell action

$$S_G = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}, \;\; F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{10.133}$$

This action is also invariant under the local $U(1)$ gauge symmetry $A_{\mu} \longrightarrow A_{\mu} - \frac{1}{e}\partial_{\mu}\Lambda$. The total action is

$$S_{\text{QED}} = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + \int d^4x \bar{\psi}(x)\big(i\gamma^{\mu}\partial_{\mu} - M\big)\psi(x). \tag{10.134}$$

The Euclidean action S_F^{Eucl} is obtained by (i) making the replacement $x_0 \longrightarrow -ix_4$ wherever x_0 appears explicitly, (ii) substituting $\psi^E(x) = \psi(\vec{x}, x_4)$ for $\psi(x) = \psi(\vec{x}, t)$, (iii) making the replacements $A^0 \longrightarrow iA_4$ and $D^0 \longrightarrow iD_4$ and (iv) multiplying the obtained expression by $-i$. Since in Euclidean space the Lorentz group is replaced by the four-dimensional rotation group, we introduce new γ-matrices γ_μ^E as follows $\gamma_4^E = \gamma^0$, $\gamma_i^E = -i\gamma^i$. They satisfy $\{\gamma_\mu^E, \gamma_\nu^E\} = 2\delta_{\mu\nu}$. The fermion Euclidean action is then

$$S_F^{\text{Eucl}} = \int d^4x \bar{\psi}^E(x)\Big(\gamma_\mu^E D_\mu + M\Big)\psi^E(x). \tag{10.135}$$

Similarly, the Euclidean action S_G^{Eucl} is obtained by (i) making the replacement $x_0 \longrightarrow -ix_4$ wherever x_0 appears explicitly, (ii) making the replacement $A^0 \longrightarrow iA_4$ and (iii) multiplying the obtained expression by $-i$. We can check that $F_{\mu\nu}F^{\mu\nu}$ where $\mu, \nu = 0, 1, 2, 3$ will be replaced with $F_{\mu\nu}^2$ where $\mu = 1, 2, 3, 4$. The Euclidean gauge action is given by

$$S_G^{\text{Eucl}} = \frac{1}{4}\int d^4x F_{\mu\nu}^2. \tag{10.136}$$

The full Euclidean action is then

$$S_{QED}^{\text{Eucl}} = \frac{1}{4}\int d^4x F_{\mu\nu}^2 + \int d^4x \bar{\psi}^E(x)\Big(\gamma_\mu^E \partial_\mu + M\Big)\psi^E(x). \tag{10.137}$$

10.3.2 Fermion action on the lattice with $U(1)$ gauge symmetry

We start from the free fermion action on the lattice

$$\begin{aligned} S_F^W = &(\hat{M} + 4r)\sum_n \sum_\alpha \bar{\hat{\psi}}_\alpha(n)\hat{\psi}_\alpha(n) \\ &- \frac{1}{2}\sum_n \sum_\alpha \sum_\beta \sum_\mu \Big[(r + \gamma_\mu)_{\alpha\beta}\bar{\hat{\psi}}_\alpha(n + \hat{\mu})\hat{\psi}_\beta(n) + (r - \gamma_\mu)_{\alpha\beta}\bar{\hat{\psi}}_\alpha(n)\hat{\psi}_\beta(n + \hat{\mu})\Big]. \end{aligned} \tag{10.138}$$

This action has the following global $U(1)$ symmetry

$$\hat{\psi}_\alpha(n) \longrightarrow G\hat{\psi}_\alpha(n), \;\; \bar{\hat{\psi}}_\alpha(n) \longrightarrow \bar{\hat{\psi}}_\alpha(n)G^{-1}. \tag{10.139}$$

The phase $G = \exp(i\Lambda)$ is an element of $U(1)$. By requiring the theory to be invariant under local $U(1)$ symmetry by allowing G to depend on the lattice site we arrive at a gauge invariant fermion action on the lattice. The problem lies in how we can make the bilinear fermionic terms (the second and third terms) in the above action gauge invariant.

We go back to the continuum formulation and see how this problem is solved. In the continuum the fermionic bilinear $\bar{\psi}(x)\psi(y)$ transforms under a local $U(1)$ transformation as follows

$$\bar{\psi}(x)\psi(y) \longrightarrow \bar{\psi}(x)G^{-1}(x)G(y)\psi(y). \tag{10.140}$$

This bilinear can be made gauge covariant by inserting the Schwinger line integral

$$U(x, y) = \exp\left(ie \int_x^y dz_\mu A_\mu(z)\right). \tag{10.141}$$

This transforms as

$$U(x, y) \longrightarrow G(x)U(x, y)G^{-1}(y). \tag{10.142}$$

Therefore, the fermionic bilinear

$$\bar{\psi}(x)U(x, y)\psi(y) = \bar{\psi}(x)e^{ie\int_x^y dz_\mu A_\mu(z)}\psi(y) \tag{10.143}$$

is $U(1)$ gauge invariant. For $y = x + \epsilon$ we have

$$U(x, x + \epsilon) = e^{ie\epsilon_\mu A_\mu(x)}. \tag{10.144}$$

We conclude that in order to get local $U(1)$ gauge invariance we replace the second and third bilinear fermionic terms in the above action as follows

$$\begin{aligned} \bar{\hat{\psi}}(n)(r - \gamma_\mu)\hat{\psi}(n + \hat{\mu}) &\longrightarrow \bar{\hat{\psi}}(n)(r - \gamma_\mu)U_{n,n+\hat{\mu}}\hat{\psi}(n + \hat{\mu}) \\ \bar{\hat{\psi}}(n + \hat{\mu})(r - \gamma_\mu)\hat{\psi}(n) &\longrightarrow \bar{\hat{\psi}}(n + \hat{\mu})(r - \gamma_\mu)U_{n+\hat{\mu},n}\hat{\psi}(n). \end{aligned} \tag{10.145}$$

We obtain the action

$$\begin{aligned} S_F^W = (\hat{M} + 4r)&\sum_n \sum_\alpha \bar{\hat{\psi}}_\alpha(n)\hat{\psi}_\alpha(n) \\ &- \frac{1}{2}\sum_n \sum_\alpha \sum_\beta \sum_\mu \Big[(r + \gamma_\mu)_{\alpha\beta}\bar{\hat{\psi}}_\alpha(n + \hat{\mu})U_{n+\hat{\mu},n}\hat{\psi}_\beta(n) \\ &+ (r - \gamma_\mu)_{\alpha\beta}\bar{\hat{\psi}}_\alpha(n)U_{n,n+\hat{\mu}}\hat{\psi}_\beta(n + \hat{\mu})\Big]. \end{aligned} \tag{10.146}$$

The $U(1)$ element $U_{n,n+\hat{\mu}}$ lives on the lattice link connecting the two points n and $n + \hat{\mu}$. This link variable is, therefore, a directed quantity given explicitly by

$$\begin{aligned} U_{n,n+\hat{\mu}} &= e^{i\phi_\mu(n)} \equiv U_\mu(n), \\ U_{n+\hat{\mu},n} &= U^+_{n,n+\hat{\mu}} = e^{-i\phi_\mu(n)} \equiv U^+_\mu(n). \end{aligned} \tag{10.147}$$

The second equality is much clearer in the continuum formulation but on the lattice it is needed for the reality of the action. The phase $\phi_\mu(n)$ belongs to the compact interval $[0, 2\pi]$. Alternatively, we can work with $A_\mu(n)$ defined through

$$\phi_\mu(n) = eaA_\mu(n). \tag{10.148}$$

10.3.3 $U(1)$ gauge action on the lattice

Let us now consider the product of link variables around the smallest possible closed loop on the lattice, i.e. a plaquette. For a plaquette in the μ–ν plane we have

$$U_P \equiv U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^+(n+\hat{\nu})U_\nu^+(n). \tag{10.149}$$

The links are path-ordered. We can immediately compute

$$U_P \equiv U_{\mu\nu}(n) = e^{iea^2F_{\mu\nu}(n)}, \tag{10.150}$$

where

$$F_{\mu\nu} = \frac{1}{a}\Big[A_\nu(n+\hat{\mu}) - A_\nu(n) - A_\mu(n+\hat{\nu}) + A_\mu(n)\Big]. \tag{10.151}$$

In other words, in the continuum limit $a \longrightarrow 0$ we have

$$\frac{1}{e^2}\sum_n\sum_{\mu<\nu}\left[1 - \frac{1}{2}\Big(U_{\mu\nu}(n) + U_{\mu\nu}^+(n)\Big)\right] = \frac{a^4}{4}\sum_n\sum_{\mu,\nu}F_{\mu\nu}^2. \tag{10.152}$$

The $U(1)$ gauge action on the lattice is therefore

$$S_G = \frac{1}{e^2}\sum_P\left[1 - \frac{1}{2}\Big(U_p + U_p^+\Big)\right]. \tag{10.153}$$

10.3.4 $SU(N)$ symmetry on the lattice

We extend to $SU(N)$ symmetry by considering N Dirac fields with action

$$\begin{aligned} S_F^W = {} & (\hat{M} + 4r)\sum_{a=1}^N\sum_n\sum_\alpha \bar{\hat{\psi}}_\alpha^a(n)\hat{\psi}_\alpha^a(n) \\ & - \frac{1}{2}\sum_{a=1}^N\sum_n\sum_\alpha\sum_\beta\sum_\mu\Big[(r+\gamma_\mu)_{\alpha\beta}\bar{\hat{\psi}}_\alpha^a(n+\hat{\mu})\hat{\psi}_\alpha^a(n) \\ & + (r-\gamma_\mu)_{\alpha\beta}\bar{\hat{\psi}}_\alpha^a(n)\hat{\psi}_\alpha^a(n+\hat{\mu})\Big]. \end{aligned} \tag{10.154}$$

The global $SU(N)$ symmetry reads

$$\hat{\psi}_\alpha^a(n) \longrightarrow G^{ab}\hat{\psi}_\alpha^b(n),\ \ \bar{\hat{\psi}}_\alpha^a(n) \longrightarrow \bar{\hat{\psi}}_\alpha^b(n)(G^{-1})^{ba}. \tag{10.155}$$

The $N \times N$ constant matrix G is now an element of $SU(N)$. As before, in order to get local $SU(N)$ gauge invariance, we replace the second and third bilinear fermionic terms in the above action as follows

$$\begin{aligned} \bar{\hat{\psi}}^a(n)(r-\gamma_\mu)\hat{\psi}^a(n+\hat{\mu}) &\longrightarrow \bar{\hat{\psi}}^a(n)(r-\gamma_\mu)U_{n,n+\hat{\mu}}^{ab}\hat{\psi}^b(n+\hat{\mu}) \\ \bar{\hat{\psi}}^a(n+\hat{\mu})(r-\gamma_\mu)\hat{\psi}^a(n) &\longrightarrow \bar{\hat{\psi}}^a(n+\hat{\mu})(r-\gamma_\mu)U_{n+\hat{\mu},n}^{ab}\hat{\psi}^b(n). \end{aligned} \tag{10.156}$$

We obtain the action

$$\begin{aligned} S_F^W = (\hat{M} + 4r) \sum_n \sum_\alpha \bar{\hat{\psi}}_\alpha^a(n) \hat{\psi}^a{}_\alpha(n) \\ - \frac{1}{2} \sum_n \sum_\alpha \sum_\beta \sum_\mu \Big[(r + \gamma_\mu)_{\alpha\beta} \bar{\hat{\psi}}_\alpha^a(n + \hat{\mu}) U^{ab}_{n+\hat{\mu},n} \hat{\psi}^b{}_\beta(n) \\ + (r - \gamma_\mu)_{\alpha\beta} \bar{\hat{\psi}}_\alpha^a(n) U^{ab}_{n,n+\hat{\mu}} \hat{\psi}^b{}_\beta(n + \hat{\mu}) \Big]. \end{aligned} \tag{10.157}$$

The link variables $U_\mu(n) = U_{n,n+\hat{\mu}}$ are elements of $SU(N)$ in the N-dimensional fundamental representation. The matrix-valued link variable $U_\mu(n)$ is explicitly given by

$$U_{n,n+\hat{\mu}} = e^{i\phi_\mu(n)} \equiv U_\mu(n), \;\; U_{n+\hat{\mu},n} = U^+_{n,n+\hat{\mu}} = e^{-i\phi_\mu(n)} \equiv U^+_\mu(n). \tag{10.158}$$

The phase $\phi_\mu(n)$ is now a hermitian matrix which belongs to the Lie algebra of $SU(N)$. As before, we can also work with $A_\mu(n)$ defined through

$$\phi_\mu(n) = gaA_\mu(n), \;\; A_\mu(n) = \sum_{B=1}^{N^2-1} A_\mu{}^B \frac{\lambda^B}{2}. \tag{10.159}$$

The $N \times N$ matrices $\frac{\lambda^A}{2}$ are the generators of the fundamental representation of the Lie algebra of $SU(N)$. They satisfy (where f_{ABC} are the completely antisymmetric structure constants of the group)

$$\left[\frac{\lambda^A}{2}, \frac{\lambda^B}{2} \right] = i f_{ABC} \frac{\lambda^C}{2}. \tag{10.160}$$

Also

$$\mathrm{Tr}\, \lambda^A \lambda^B = 2\delta_{AB}. \tag{10.161}$$

The local $SU(N)$ transformation reads

$$\begin{aligned} \hat{\psi}^a_\alpha(n) &\longrightarrow G^{ab}(n) \hat{\psi}^b_\alpha(n), \\ \bar{\hat{\psi}}^a_\alpha(n) &\longrightarrow \bar{\hat{\psi}}^b_\alpha(n) (G^{-1}(n))^{ba}, \end{aligned} \tag{10.162}$$

and

$$\begin{aligned} U_\mu(n) &\longrightarrow G(n) U_\mu(n) G^{-1}(n + \hat{\mu}), \\ U^+_\mu(n) &\longrightarrow G(n + \hat{\mu}) U^+_\mu(n) G(n). \end{aligned} \tag{10.163}$$

The $SU(N)$ elements $G(n)$ can be written as $G(n) = \exp(i\Lambda(n))$ where $\Lambda(n)$ are $N \times N$ matrices in the Lie algebra of $SU(N)$.

Finally, the $SU(N)$ gauge action on the lattice by analogy with the $U(1)$ theory takes the form

$$S_G = \frac{2N}{g^2} \sum_P \left[1 - \frac{1}{2N} \operatorname{Tr} \left(U_p + U_p^+ \right) \right]. \tag{10.164}$$

The plaquette variable is given by

$$U_P \equiv U_{\mu\nu}(n) = U_\mu(n) U_\nu(n + \hat{\mu}) U_\mu^+(n + \hat{\nu}) U_\nu^+(n). \tag{10.165}$$

By using the Campbell–Baker–Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,\, B]+\cdots} \tag{10.166}$$

we find that

$$\begin{aligned} U_P = e^{iga^2 F_{\mu\nu}(n)},\ F_{\mu\nu}(n) = \partial_\mu A_\nu(n) - \partial_\nu A_\mu(n) \\ + ig[A_\mu(n),\, A_\nu(n)] + O(a). \end{aligned} \tag{10.167}$$

We compute the action

$$S_G = \frac{1}{2} \int d^4x \operatorname{Tr} F_{\mu\nu}^2 = \frac{1}{4} \int d^4x \left(F_{\mu\nu}^B \right)^2 \tag{10.168}$$

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu,\, A_\nu] = \sum_{B=1}^{N^2-1} F_{\mu\nu}^B \frac{\lambda^B}{2}, \\ F_{\mu\nu}^B &= \partial_\mu A_\nu^B - \partial_\nu A_\mu^B - g f_{BCD} A_\mu^C A_\nu^D. \end{aligned} \tag{10.169}$$

The continuum gauge action is invariant under the transformation

$$A_\mu \longrightarrow G(x) A G^{-1}(x) - \frac{i}{g} G(x) \partial_\mu G^{-1}(x). \tag{10.170}$$

10.4 Quenched quantum electrodynamics on the lattice

10.4.1 Quenched approximation

As another relatively simple example we will consider quenched quantum electrodynamics on the lattice. The partition function on a lattice Λ is given by

$$Z = \int \mathcal{D}U\ \mathcal{D}\bar{\psi} \mathcal{D}\psi\ e^{-S_G[U] - S_F[U, \bar{\psi}, \psi]}. \tag{10.171}$$

The measures are defined by

$$\mathcal{D}U = \prod_{n \in \Lambda} \prod_{\mu=1}^{4} dU_\mu(n),\ \mathcal{D}\bar{\psi} = \prod_{n \in \Lambda} d\bar{\psi}(n),\ \mathcal{D}\psi = \prod_{n \in \Lambda} d\psi(n). \tag{10.172}$$

The plaquette and the link variables are given by

$$U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^+(n+\hat{\nu})U_\nu^+(n), \;\; U_\mu(n) = e^{i\phi_\mu(n)}. \tag{10.173}$$

The action of a $U(1)$ gauge theory on a lattice is given by (with $\beta = 1/e^2$)

$$\begin{aligned} S_G[U] &= \beta \sum_{n\in\Lambda} \sum_{\mu<\nu} \Big[1 - \frac{1}{2}\big(U_{\mu\nu}(n) + U_{\mu\nu}^+(n)\big)\Big] \\ &= \beta \sum_{n\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\big[1 - U_{\mu\nu}(n)\big]. \end{aligned} \tag{10.174}$$

The action of fermions coupled to a $U(1)$ gauge field on a lattice is given by

$$S_F[U, \bar{\hat{\psi}}, \hat{\psi}] = \sum_\alpha \sum_\beta \sum_n \sum_m \bar{\hat{\psi}}_\alpha(n)\mathcal{D}_{\alpha\beta}(U)_{n,m}\hat{\psi}_\beta(m), \tag{10.175}$$

where

$$\begin{aligned} \mathcal{D}_{\alpha\beta}(U)_{n,m} = \hat{M}\delta_{\alpha\beta}\delta_{n,m} &- \frac{1}{2}(\gamma_\mu)_{\alpha\beta}\delta_{n,m+\hat{\mu}}U_{n+\hat{\mu},n} \\ &+ \frac{1}{2}(\gamma_\mu)_{\alpha\beta}\delta_{m,n+\hat{\mu}}U_{n,n+\hat{\mu}}. \end{aligned} \tag{10.176}$$

Using the result

$$\int \mathcal{D}\bar{\hat{\psi}}\mathcal{D}\hat{\psi}\; e^{-\sum_\alpha \sum_\beta \sum_n \sum_m \bar{\hat{\psi}}_\alpha(n)\mathcal{D}_{\alpha\beta}(U)_{n,m}\hat{\psi}_\beta(m)} = \det \mathcal{D}_{\alpha\beta}(U)_{n,m}, \tag{10.177}$$

the partition function becomes

$$Z = \int \mathcal{D}U \; \det \mathcal{D}_{\alpha\beta}(U)_{n,m}\, e^{-S_G[U]}. \tag{10.178}$$

At this stage we will make the approximation that we can set the determinal equal to 1, i.e. the quantum electrodynamics partition function will be approximated by

$$Z = \int \mathcal{D}U \; e^{-S_G[U]}. \tag{10.179}$$

This is called the quenched approximation.

10.4.2 Wilson loop and Creutz ratio

The first observable we would like to measure is the expectation value of the action which after dropping the constant term is given by

$$\langle S_G[U]\rangle = -\beta \sum_{n\in\Lambda} \sum_{\mu<\nu} \langle \mathrm{Re}\; U_{\mu\nu}(n)\rangle. \tag{10.180}$$

The specific heat is the corresponding second moment, viz

$$C_v = \langle S_G[U]^2 \rangle - \langle S_G[U] \rangle^2. \tag{10.181}$$

We will also measure the expectation value of the so-called Wilson loop, which has a length I in one of the spatial directions (say 1), and a width J in the temporal direction 4. This rectangular loop C is defined by

$$\begin{aligned} W_C[U] = {} & S(n, n + I\hat{1}) T(n + I\hat{1}, n + I\hat{1} + J\hat{4}) \\ & \times S^+(n + J\hat{4}, n + I\hat{1} + J\hat{4}) T^+(n, n + J\hat{4}). \end{aligned} \tag{10.182}$$

The Wilson lines are

$$\begin{aligned} S(n, n + I\hat{1}) &= \prod_{i=0}^{I-1} U_1(n + i\hat{1}) \\ S(n + J\hat{4}, n + I\hat{1} + J\hat{4}) &= \prod_{i=0}^{I-1} U_1(n + i\hat{1} + J\hat{4}). \end{aligned} \tag{10.183}$$

The temporal transporters are

$$\begin{aligned} T(n + I\hat{1}, n + I\hat{1} + J\hat{4}) &= \prod_{j=0}^{J-1} U_4(n + I\hat{1} + j\hat{4}) \\ T(n, n + J\hat{4}) &= \prod_{j=0}^{J-1} U_4(n + j\hat{4}). \end{aligned} \tag{10.184}$$

The expectation value of $W_C[U]$ will be denoted by

$$W[I, J] = \frac{\int \mathcal{D}U \; W_C[U] \, e^{-S_G[U]}}{\int \mathcal{D}U \; e^{-S_G[U]}}. \tag{10.185}$$

By using the fact that under $\phi_\mu(n) \longrightarrow -\phi_\mu(n)$, the partition function is invariant while the Wilson loop changes its orientation, i.e. $W_C[U] \longrightarrow W_C[U]^+$, we obtain

$$W[I, J] = \langle \mathrm{Re} \; W_C[U] \rangle. \tag{10.186}$$

It is almost obvious that in the continuum limit

$$W[I, J] \longrightarrow W[R, T] = \left\langle \exp\left(ie \oint_C dx_\mu A_\mu \right) \right\rangle. \tag{10.187}$$

The loop C is now a rectangular contour with spatial length $R = Ia$ and timelike length $T = Ja$. This represents the probability amplitude for the process of creating an infinitely heavy, i.e. static, quark–antiquark[2] pair at time $t = 0$ which are

[2] For $U(1)$ we should really speak of an electron–positron pair.

separated by a distance R, then allowing them to evolve in time and then eventually annihilate after a long time T.

The precise meaning of the expectation value (10.186) is as follows

$$\langle \mathcal{O} \rangle = \frac{1}{L} \sum_{i=1}^{L} \left(\frac{1}{N^3 N_T} \sum_n \text{Re } W_C[U_i] \right). \tag{10.188}$$

In other words, we also take the average over the lattice which is necessary in order to reduce noise in the measurement of the Creutz ratio (see below).

The above Wilson loop is the order parameter of the pure $U(1)$ gauge theory. For large time T we expect the behavior

$$W[R,\, T \longrightarrow \infty] \longrightarrow e^{-V(R)T} = e^{-aV(R)J}, \tag{10.189}$$

where $V(R)$ is the static quark–antiquark potential. For strong coupling (small β) we can show that the potential is linear, viz

$$V(R) = \sigma R. \tag{10.190}$$

The constant σ is called the string tension from the fact that the force between the quark and the antiquark can be modeled by the force in a string attached to the quark and antiquark. For a linear potential the Wilson loop follows an area law $W[R,\, T] = \exp(-\sigma A)$ with $A = a^2 IJ$. This behavior is typical in a confining phase which occurs at high temperature.

For small coupling (large β, low temperature) the lattice $U(1)$ gauge field becomes weakly coupled and as a consequence we expect the Coulomb potential to dominate the static quark–antiquark potential, viz

$$V(R) = \frac{Z}{R}. \tag{10.191}$$

Hence, for large R the quark and antiquark become effectively free and their energy is simply the sum of their self-energies. The Wilson loop in this case follows a perimeter law $W[R,\, T] = \exp(-2\epsilon T)$.

In summary, for a rectangular $R \times T$ Wilson loop with perimeter $P = 2(R + T)$ and area $A = RT$ we expect the behavior

$$W[R,\, T] = e^{-\sigma A}, \text{ confinement phase} \tag{10.192}$$

$$W[R,\, T] = e^{-\epsilon P}, \text{ Coulomb phase.} \tag{10.193}$$

In general the Wilson loop will behave as

$$W[R,\, T] = e^{-B-\sigma A-\epsilon P}. \tag{10.194}$$

The perimeter piece actually dominates for any fixed size loop. To measure the string tension we must, therefore, eliminate the perimeter behavior which can be achieved using the so-called Creutz ratio defined by

$$\chi(I, J) = -\ln \frac{W[I, J]W[I-1, J-1]}{W[I, J-1]W[I-1, J]}. \tag{10.195}$$

For large loops clearly

$$\chi(I, J) = a^2\sigma. \tag{10.196}$$

This should hold especially in the confinement phase, whereas in the Coulomb phase, we should expect $\chi(I, J) \sim 0$.

The 1×1 Wilson loop $W(1, 1)$ is special since it is related to the average action per plaquette. We have

$$W[1, 1] = \left\langle \text{Re } U_1(n)U_4(n + \hat{1})U_4^+(n)U_1^+(n + \hat{4}) \right\rangle. \tag{10.197}$$

Next, we compute straightforwardly

$$-\frac{\partial \ln Z}{\partial \beta} = \sum_n \sum_{\mu<\nu} \langle [1 - \text{Re } U_{\mu\nu}(n)] \rangle. \tag{10.198}$$

Clearly all the planes $\mu\nu$ are equivalent and thus we should have

$$\begin{aligned} -\frac{\partial \ln Z}{\partial \beta} &= 6 \sum_n \langle [1 - \text{Re } U_{14}(n)] \rangle \\ &= 6 \sum_n \left\langle \left[1 - \text{Re } U_1(n)U_4(n + \hat{1})U_4^+(n)U_1^+(n + \hat{4}) \right] \right\rangle. \end{aligned} \tag{10.199}$$

Note that there are N^3N_T lattice sites. Each site corresponds to four plaquettes in every plane $\mu\nu$ and thus it corresponds to 4×6 plaquettes in all. Each plaquette in a plane $\mu\nu$ corresponds to four sites and thus to avoid overcounting we must divide by four. In summary, we have $4 \times 6 \times N^3 \times N_T/4$ plaquettes in total. Six is therefore the ratio of the number of plaquettes to the number of sites.

We have then

$$-\frac{1}{6N^3N_T}\frac{\partial \ln Z}{\partial \beta} = 1 - \frac{1}{N^3N_T} \sum_n \left\langle \text{Re } U_1(n)U_4(n + \hat{1})U_4^+(n)U_1^+(n + \hat{4}) \right\rangle. \tag{10.200}$$

We can now observe that all lattice sites n are the same under the expectation value, namely

$$-\frac{1}{6N^3N_T}\frac{\partial \ln Z}{\partial \beta} = 1 - \left\langle \text{Re } U_1(n)U_4(n + \hat{1})U_4^+(n)U_1^+(n + \hat{4}) \right\rangle. \tag{10.201}$$

This is the average action per plaquette (the internal energy) denoted by

$$P = -\frac{1}{6N^3N_T}\frac{\partial \ln Z}{\partial \beta} = 1 - W[1, 1]. \tag{10.202}$$

10.4.3 The Metropolis algorithm

The action of pure $U(1)$ gauge theory, the corresponding partition function and the measure of interest are given on a lattice Λ, respectively, by (with $\beta = 1/e^2$)

$$S_G[U] = \beta \sum_{n\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\,[1 - U_{\mu\nu}(n)] \tag{10.203}$$

$$Z = \int \mathcal{D}U\ e^{-S_G[U]} \tag{10.204}$$

$$\mathcal{D}U = \prod_{n\in\Lambda} \prod_{\mu=1}^{4} dU_\mu(n). \tag{10.205}$$

The vacuum expectation value of any observable $\mathcal{O} = \mathcal{O}(U)$ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U\ \mathcal{O}\ e^{-S_G[U]}. \tag{10.206}$$

For $U(1)$ gauge theory we can write

$$U_\mu(n) = e^{i\phi_\mu(n)}. \tag{10.207}$$

Hence

$$\mathcal{D}U = \prod_{n\in\Lambda} \prod_{\mu=1}^{4} d\phi_\mu(n). \tag{10.208}$$

We will use the Metropolis algorithm to solve this problem. This goes as follows. Starting from a given gauge field configuration, we choose a lattice point n and a direction μ, and change the link variable there, which is $U_\mu(n)$, to $U_\mu(n)'$. This link is shared by six plaquettes. The corresponding variation of the action is

$$\Delta S_G[U_\mu(n)] = S_G[U'] - S_G[U]. \tag{10.209}$$

The gauge field configurations U and U' differ only by the value of the link variable $U_\mu(n)$. We need to isolate the contribution of $U_\mu(n)$ to the action S_G. Note the fact that $U^+_{\mu\nu} = U_{\nu\mu}$. We write

$$S_G[U] = \beta \sum_{n\in\Lambda} \sum_{\mu<\nu} 1 - \frac{\beta}{2} \sum_{n\in\Lambda} \sum_{\mu<\nu} \Big(U_{\mu\nu}(n) + U^+_{\mu\nu}(n) \Big). \tag{10.210}$$

The second term is

$$-\frac{\beta}{2} \sum_{n\in\Lambda} \sum_{\mu<\nu} U_{\mu\nu}(n) = -\frac{\beta}{2} \sum_{n\in\Lambda} \sum_{\mu<\nu} U_\mu(n) U_\nu(n+\hat{\mu}) U^+_\mu(n+\hat{\nu}) U^+_\nu(n). \tag{10.211}$$

In the μ–ν plane, the link variable $U_\mu(n)$ appears twice corresponding to the two lattice points n and n–$\hat{\nu}$. For every μ there are three relevant planes. The six relevant terms are, therefore, given by

$$-\frac{\beta}{2}\sum_{n\in\Lambda}\sum_{\mu<\nu}U_{\mu\nu}(n)\;\longrightarrow\;-\frac{\beta}{2}\sum_{\nu\neq\mu}\Big(U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^+(n+\hat{\nu})U_\nu^+(n) + U_\mu^+(n)U_\nu^+(n-\hat{\nu})U_\mu(n-\hat{\nu})U_\nu(n-\hat{\nu}+\hat{\mu})\Big)+\cdots \tag{10.212}$$

By adding the complex conjugate terms we obtain

$$-\frac{\beta}{2}\sum_{n\in\Lambda}\sum_{\mu<\nu}(U_{\mu\nu}(n)+U_{\mu\nu}^+(n))\longrightarrow-\frac{\beta}{2}\Big(U_\mu(n)\mathcal{A}_\mu(n) + U_\mu^+(n)\mathcal{A}_\mu^+(n)\Big)+\cdots \tag{10.213}$$

The $\mathcal{A}_\mu(n)$ is the sum over the six so-called staples which are the products over the other three link variables which together with $U_\mu(n)$ make up the six plaquettes which share $U_\mu(n)$. Explicitly we have

$$\mathcal{A}_\mu(n)=\sum_{\nu\neq\mu}\Big(U_\nu(n+\hat{\mu})U_\mu^+(n+\hat{\nu})U_\nu^+(n) + U_\nu^+(n+\hat{\mu}-\hat{\nu})U_\mu^+(n-\hat{\nu})U_\nu(n-\hat{\nu})\Big). \tag{10.214}$$

We have then the result

$$-\frac{\beta}{2}\sum_{n\in\Lambda}\sum_{\mu<\nu}\Big(U_{\mu\nu}(n)+U_{\mu\nu}^+(n)\Big)\longrightarrow-\beta\,\mathrm{Re}(U_\mu(n)\mathcal{A}_\mu(n))+\cdots \tag{10.215}$$

We compute then

$$\Delta S_G[U_\mu(n)]=S_G[U']-S_{G[U]} =-\beta(U_\mu(n)'-U_\mu(n))\mathcal{A}_\mu(n). \tag{10.216}$$

Having computed the variation $\Delta S_G[U_\mu(n)]$, next we inspect its sign. If this variation is negative then the proposed change $U_\mu(n)\longrightarrow U_\mu(n)'$ will be accepted (classical mechanics). If the variation is positive, we compute the Boltzmann probability

$$\exp(-\Delta S_G[U_\mu(n)])=\exp(\beta(U_\mu(n)'-U_\mu(n))\mathcal{A}_\mu(n)). \tag{10.217}$$

The proposed change $U_\mu(n)\longrightarrow U_\mu(n)'$ will be accepted according to this probability (quantum mechanics). In practice we will pick a uniform random number r between 0 and 1 and compare it with $\exp(-\Delta S_G[U_\mu(n)])$. If $\exp(-\Delta S_G[U_\mu(n)])<r$ we accept this change otherwise we reject it.

We go through the above steps for every link in the lattice which constitutes one Monte Carlo step. Typically equilibration (thermalization) is reached after a large number of Monte Carlo steps at which point we can start taking measurements based on the formula (10.206) written as

$$\langle \mathcal{O} \rangle = \frac{1}{L} \sum_{i=1}^{L} \mathcal{O}_i, \;\; \mathcal{O}_i = \mathcal{O}(U_i). \tag{10.218}$$

The L configurations $U_i = \{U_\mu(n)\}_i$ are L thermalized gauge field configurations distributed according to $\exp(-S_G[U])$.

The error bars in the different measurements will be estimated using the jackknife method. We can also compute auto-correlation time and take it into account by separating the measured gauge field configurations U_i by at least one unit of auto-correlation time.

Let us also comment on how we choose the proposed configurations $U_\mu(n)'$. The custom is to take $U_\mu(n)' = XU_\mu(n)$ where X is an element in the gauge group (which is here $U(1)$) near the identity. In order to maintain a symmetric selection probability, X should be drawn randomly from a set of $U(1)$ elements which also contains X^{-1}. For $U(1)$ gauge groups we have $X = \exp(i\phi)$ where $\phi \in [0, 2\pi]$. In principle, the acceptance rate can be maintained around 0.5 by tuning appropriately the angle ϕ. Reunitarization of $U_\mu(n)'$ may also be applied to reduce rounding errors.

The final technical remark is with regard to boundary conditions. In order to reduce edge effects we usually adopt periodic boundary conditions, i.e.

$$\begin{aligned} U_\mu(N, n_2, n_3, n_4) &= U_\mu(0, n_2, n_3, n_4) \\ U_\mu(n_1, N, n_3, n_4) &= U_\mu(n_1, 0, n_3, n_4) \\ U_\mu(n_1, n_2, N, n_4) &= U_\mu(n_1, n_2, n, 0, n_4) \\ U_\mu(n_1, n_2, n_3, N_T) &= U_\mu(n_1, n_2, n_3, 0). \end{aligned} \tag{10.219}$$

This means in particular that the lattice is actually a four-dimensional torus. In the actual code this is implemented by replacing $i \pm 1$ by $\mathrm{ip}(i)$ and $\mathrm{im}(i)$, $\mathrm{ipT}(i)$ and $\mathrm{imT}(i)$, respectively, which are defined by

```
do i=1,N
   ip(i)=i+1
   im(i)=i-1
enddo
   ip(N)=1
   im(1)=N
do i=1,NT
   ipT(i)=i+1
   imT(i)=i-1
enddo
   ipT(NT)=1
   imT(1)=NT
```

A code written along the above lines can be found in the last chapter of [10].

10.4.4 Some numerical results

1. We run simulations for $N = 3, 4, 8, 10, 12$ with the coupling constant in the range $\beta = 2, \ldots, 12$. We use typically 2^{14} thermalization steps and 2^{14} measurements steps.
2. We measure the specific heat (figure 10.3). We observe a peak in the specific heat at around $\beta = 1$. The peak grows with N which signals a critical behavior typical of second-order transition.
3. The simplest order parameter is the action per plaquette P, defined in equation (10.202), which is shown in figure 10.4. We observe good agreement between the high-temperature and low-temperature expansions of P on one hand and the corresponding observed behavior in the strong coupling and weak-coupling regions, respectively, on the other hand. We note that the high-temperature and low-temperature expansions of the pure $U(1)$ gauge field are given by

$$P = 1 - \frac{\beta}{2} + O(\beta^3), \quad \text{high } T \tag{10.220}$$

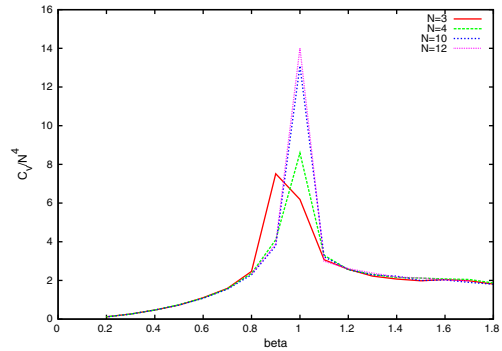

Figure 10.3. The specific heat.

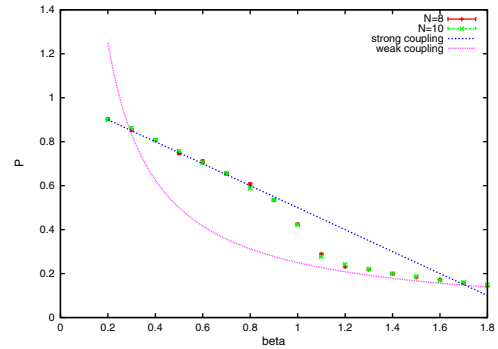

Figure 10.4. The action per plaquette.

$$P = 1 - \frac{1}{4\beta} + O(1/\beta^2), \text{ low } T. \tag{10.221}$$

We do not observe a clear-cut discontinuity in P which is, in any case, consistent with the conclusion that this phase is second order. We note that for higher $U(N)$ the transition is first order [4].

A related object to P is the total action shown in figure 10.5.

4. More powerful order parameters are the Wilson loops which are shown in figure 10.6. We observe that the Wilson loop in the strong-coupling region averages to zero very quickly as we increase the size of the loop. This may be explained by an area law behavior. In the weak-coupling region, the evolution as a function of the area is much slower. The demarcation between the two phases becomes very sharp (possibly a jump) for large loops at $\beta = 1$.
5. Calculating the expectation value of the Wilson loop and then extracting the string tension is very difficult since the perimeter law is more often dominant. The Creutz ratios (figure 10.7) allow us to derive the string tension in a direct way without measuring the Wilson loop. The string tension is the coefficient

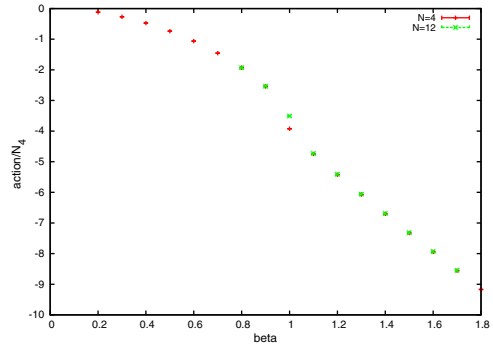

Figure 10.5. The total action.

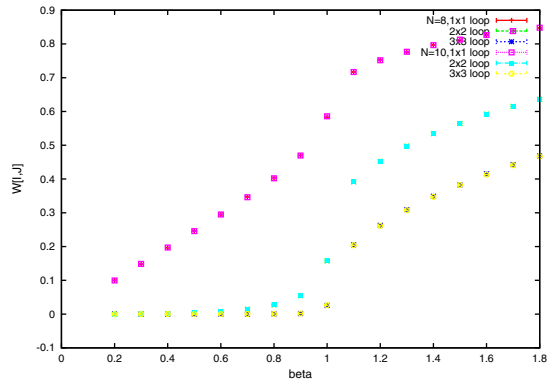

Figure 10.6. The Wilson loop.

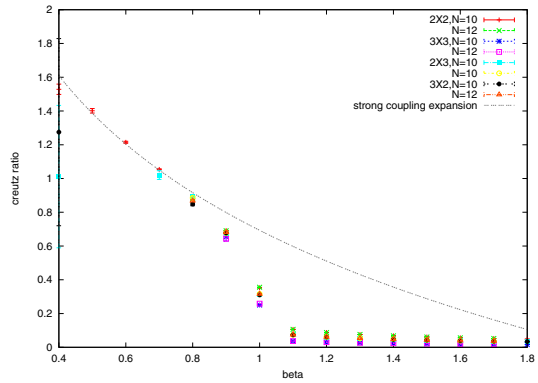

Figure 10.7. The Creutz ratios.

of the linearly rising part of the potential for large (infinite) separations of a quark–antiquark pair in the absence of pair production processes. In this way, we hope to measure the physical string tension in a narrow range of the coupling constant.

We observe that the string tension in the weak-coupling regime is effectively independent of the coupling constant and it is essentially zero. In the strong-coupling regime we reproduce the strong-coupling behavior

$$\sigma = -\ln \frac{\beta}{2}. \tag{10.222}$$

10.4.5 Coulomb and confinement phases

The physics of the compact $U(1)$ theory is clearly different in the weak- and strong-coupling regions. This can be understood from the fact that there is a phase transition as a function of the bare coupling constant. The compact $U(1)$ theory at weak-coupling is not confining and contains no glueballs but simply the photons of the free Maxwell theory. One speaks of a Coulomb phase at weak-coupling and a confining phase at strong-coupling. In the Coulomb phase photons are massless and the static potential has the standard Coulomb form

$$V = -\frac{e^2}{4\pi r} + \text{constant}, \tag{10.223}$$

whereas in the confinement phase photons become massive and the potential is linearly confining at large distances

$$V = \sigma r. \tag{10.224}$$

There is a phase transition at a critical coupling $\beta \approx 1$ at which the string tension $\sigma(\beta)$ vanishes in the Coulomb phase. In the confinement phase topological configurations are important, such as monopoles and glueballs.

The strong-coupling expansion is an expansion in powers of $1/g^2$. It has the advantage over the weak-coupling expansion that it has a non-zero radius of convergence. A lot of effort has been put into using it as a method of computation similar to the high-temperature or the hopping parameter expansion for scalar field theories. One has to be able to tune on the values of the coupling constant where the theory exhibits continuum behavior. This turns out to be difficult for gauge theories. However, a very important aspect of the strong-coupling expansion is that it gives insight into the qualitative behavior of the theory such as confinement and the particle spectrum.

The strong-coupling expansion of compact $U(1)$ theory shows explicitly that the theory is confining, i.e. the potential is linear with a string tension given by (with $a_1 = \beta/2$)

$$\sigma = -\ln a_1 - 2(d-2)a_1^4 + \cdots. \tag{10.225}$$

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Chapter 11

The Wilson and functional renormalization group equations

The renormalization group equation is a central tool of perturbative and non-perturbative quantum field theory which is vital for a proper understanding of the renormalizability of the theory and its phase diagram. The Wilson approach [1] to the renormalization group equation is in our opinion the most profound description of the true nature and final goals of quantum field theory. In this chapter, and after a careful review of the original Wilson approach, we describe in great detail the functional renormalization group equation which is an exact non-perturbative formulation of the Wilson renormalization group equation. The original literature on the functional renormalization group equation includes Polchinski [2] (Polchinski's equation for the effective action) and Wetterich [3] (Wetterich's equation for the average action). See also [4].

11.1 Wilson renormalization group approach

The primary reference for this section is the classic paper by Wilson and Kogut [1]. The article [5] was very useful and we follow the presentation of [6].

11.1.1 The basic idea

As an example we consider the Landau theory given by the Hamiltonian

$$E[m] = \int d^dx \mathcal{E}(m(x)),\ \mathcal{E} = \frac{1}{2}(\vec{\partial} m(x))^2 + \sum_{n=1}^{\infty} K_n m^n(x) + \cdots. \tag{11.1}$$

The field $m(x)$ is the order parameter (magnetization) and the dot stands for higher derivative terms The coupling constants K_n depend on the cutoff Λ. The most important example for us is the Landau–Wilson model given by

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$$\mathcal{E} = \frac{1}{2}(\vec{\partial} m(x))^2 + \frac{1}{2} r_0 m^2(x) + u_0 m^4(x). \tag{11.2}$$

The renormalization group transformation consists of three steps:

- **Coarse-graining**: We integrate in the partition function over the high momentum modes.
- **Rescaling**: We rescale the momenta back to the original domain in order to restore the original coarse-grained 'picture'.
- **Normalization**: We rescale the fields appropriately which is necessary since the relative size of the fluctuations of the rescaled magnetization is in general different from the original.

These three steps define a renormalization group transformation which, in general, is not invertible and thus we are really dealing with a semigroup structure. By repeating these steps we get then highly non-linear equations which are the renormalization group equations.

The partition function of the Landau theory in momentum space is of the form

$$Z = \int \prod_{|\vec{k}|<\Lambda}{}' [d\tilde{m}(\vec{k})][d\tilde{m}^\dagger(\vec{k})] \exp(-E[\tilde{m}(\vec{k})]). \tag{11.3}$$

The prime indicates that we integrate only over half of the momentum space. The energy functional in momentum space reads (with $K_2 = r_0/2$)

$$E[\tilde{m}] = \frac{1}{2} \int d^d k (\vec{k}^2 + r_0) |\tilde{m}(\vec{k})|^2 + \cdots. \tag{11.4}$$

We now apply the above stated three steps of the renormalization group transformation.

- **Coarse-graining (integration)**: We integrate in the partition function over all momentum modes which satisfy $\Lambda/b \leqslant |\vec{k}| \leqslant \Lambda$ with $b > 1$. Thus the new energy functional should be defined by the expression

$$\exp(-E'[\tilde{m}(\vec{k})]) = \int \prod_{\frac{\Lambda}{b}<|\vec{k}|<\Lambda}{}' [d\tilde{m}(\vec{k})][d\tilde{m}^\dagger(\vec{k})] \exp(-E[\tilde{m}(\vec{k})]). \tag{11.5}$$

  Clearly, the new Hamiltonian E' depends only on the fields $\tilde{m}(\vec{k})$ for which $|\vec{k}| < \Lambda/b$. This new Hamiltonian is generally of the same form as the original one but with different coupling constants, viz

$$E'[\tilde{m}] = \frac{1}{2} \int_{|\vec{k}|<\frac{\Lambda}{b}} d^d k (A\vec{k}^2 + \tilde{r}_0) |\tilde{m}(\vec{k})|^2 + \cdots. \tag{11.6}$$

  In fact, E' will also contain additional terms which did not exist in the original action and which will be dropped (Wilson truncation). The partition function becomes

$$Z = \int \prod_{|\vec{k}|<\frac{\Lambda}{b}}{}' [d\tilde{m}(\vec{k})][d\tilde{m}^{\dagger}(\vec{k})]\exp(-E'[\tilde{m}(\vec{k})]). \tag{11.7}$$

- **Rescaling**: We rescale the momenta back to the original domain, i.e. we restore the cutoff back to Λ, as $k_i \longrightarrow k_i' = bk_i$. The action becomes

$$E'[\tilde{m}] = \frac{b^{-d}}{2}\int_{|\vec{k}|<\Lambda} d^d k\left(\frac{A}{b^2}\vec{k}^2 + \tilde{r}_0\right)\left|\tilde{m}\left(\frac{\vec{k}}{b}\right)\right|^2 + \cdots. \tag{11.8}$$

- **Normalization**: In order to restore the canonical normalization of the kinetic term we rescale the fields as

$$\tilde{m}'(\vec{k}) = \sqrt{\frac{A}{b^{d+2}}}\, m\left(\frac{\vec{k}}{b}\right). \tag{11.9}$$

  The energy becomes

$$E'[\tilde{m}'(\vec{k})] = \frac{1}{2}\int_{|\vec{k}|<\Lambda} d^d k(\vec{k}^2 + r_0')|\tilde{m}'(k)|^2 + \cdots, \tag{11.10}$$

  where the renormalized mass is given by

$$r_0' = \frac{b^2}{A}\tilde{r}_0. \tag{11.11}$$

  The partition function becomes

$$Z' = \int \prod_{|\vec{k}|<\Lambda}{}' [d\tilde{m}'(\vec{k})][d\tilde{m}'^{\dagger}(\vec{k})]\exp(-E'[\tilde{m}'(\vec{k})]). \tag{11.12}$$

In summary, the coupling constants $K = (K_2, K_3, \ldots)$ in the original Hamiltonian E will be mapped under the renormalization group transformation to new coupling-constant $K' = (K_2', K_3', \ldots)$ in the renormalized Hamiltonian E'. The coupling constants K' are called renormalized coupling constants. We also say that the coupling constants of the theory 'flow' under a change of the energy scale. If we denote the renormalization group transformation by R then this mapping can be written as ($K = K^{(0)}$, $K' = K^{(1)}$)

$$K^{(1)} = R(K^{(0)}). \tag{11.13}$$

This step can be iterated an arbitrary number of times. The renormalization group transformation is the same for all iterations and as a consequence we can write after $i + 1$ iterations the renormalization group transformation

$$K^{(i+1)} = R(K^{(i)}). \tag{11.14}$$

By inspection, since the partition functions (11.3) and (11.12) are given by the same function except for the replacement $K \longrightarrow K'$, the free energy should satisfy the relation

$$f(K) = f_0(K) + f(K'). \tag{11.15}$$

Next, since the volumes (numbers of degrees of freedom) before and after the renormalization group transformation are related by $(\Lambda'/\Lambda)^d = b^{-d}$, the free energy per degree of freedom must satisfy the relation

$$F(K) = F_0(K) + b^{-d}F(K'). \tag{11.16}$$

11.1.2 Formalism

We are interested in the most general properties of the following renormalization group transformation

$$K^{(i+1)} = R(K^{(i)}). \tag{11.17}$$

A fixed point K^* in the coupling-constant space is a point which does not move, i.e. it is invariant, under the successive action of the above renormalization group transformation. Thus, we must have

$$K^* = R(K^*). \tag{11.18}$$

Many points K in the coupling-constant space will approach the fixed point under a large number of iterations of the renormalization group transformations. We write this as

$$K^{(i)} \longrightarrow K^*, \; i \longrightarrow \infty. \tag{11.19}$$

The fixed point K^* describes the system at the critical temperature $T = T_c$. At this point the system becomes scale invariant, i.e. it becomes invariant under a change of length scale. This can also be expressed by the fact that correlation length ξ (inverse mass) diverges at the fixed point.

Now we linearize the above renormalization group equation around the fixed point K^* by writing

$$K_\alpha^{(i+1)} - K_\alpha^* = (K^{(i)} - K^*)_\alpha = W_{\alpha\beta}\Big(K_\beta^{(i)} - K_\beta^*\Big). \tag{11.20}$$

The linearized renormalization group transformation is given by

$$W_{\alpha\beta} = \frac{\partial R_\alpha}{\partial K_\beta}|_{K=K^*}. \tag{11.21}$$

The origin of the coupling-constant space is chosen to be K^*. The left eigenvectors ϕ of the renormalization group matrix W are defined by

$$\phi_\beta W_{\beta\alpha} = \lambda\phi_\alpha. \tag{11.22}$$

For each left eigenvector ϕ we define a scaling field v by the relation

$$v^{(i)} = \sum_{\alpha} \phi_{\alpha}(K^{(i)} - K^{*})_{\alpha}. \tag{11.23}$$

It satisfies, under the renormalization group transformation, the crucial property

$$v^{(i+1)} = \lambda v^{(i)}. \tag{11.24}$$

We say that v scales under the renormalization group transformation with a factor of λ. Thus, the scaling field v increases if $\lambda > 1$ and decreases if $\lambda < 1$. In other words, the eigenvalue λ associated with the left eigenvector ϕ is what controls the flow behavior of the corresponding scaling field. Recall that under the renormalization group transformation the length scales as $L \longrightarrow bL$, i.e. it increases since $b > 1$. Thus, if the eigenvalue λ increases under the renormalization group transformation then we may write $\lambda = b^{y}$ where $y > 0$, whereas for $y < 0$ the eigenvalue λ decreases. The exponent y is called the dimension of the scaling field v. The scaling fields are in some sense the proper coupling constants, whereas their exponents are their quantum mass dimensions.

Hence, we will choose in the coupling-constant space the origin at K^{*}, whereas the preferred directions are given by the scaling fields v. The fundamental equation (11.16) reads in terms of the scaling fields

$$F(v_1, v_2, \ldots) = F_0(v_1, v_2, \ldots) + b^{-d} F(\lambda_1 v_1, \lambda_2 v_2, \ldots). \tag{11.25}$$

The function F_0 is regular at the fixed point (critical temperature), whereas the second term gives the singular part of the free energy. Some definitions are now in order.

- For $\lambda < 1$ the scaling field is called irrelevant (decreases in the RG transformation), whereas for $\lambda > 1$ it is called relevant (increases in the RG transformation). Thus, if we want to remain on a fixed point we must set the relevant scaling fields to zero. The case $\lambda = 1$ corresponds to marginal fields.
- The fixed point lies on a hypersurface called the critical surface in the coupling-constant space. The critical surface is defined by setting all the relevant scaling fields to zero. Thus, starting from any point on the critical surface we will approach the fixed point under a large number of iterations of the renormalization group transformation, whereas starting from any point off the critical surface we will diverge from the fixed point. All actions (physical systems) located on the critical surface define collectively a so-called universality class.
- The eigenvalues λ determine the critical exponents y. All systems in the same universality class have the same critical exponents.

11.1.3 Critical exponents

The correlation length ξ behaves around the critical temperature as

$$\xi \sim |t|^{-1/D_t},\ D_t = \frac{1}{\nu}, \tag{11.26}$$

where ν is the mass critical exponent and D_t is the temperature critical exponent. The gap (order parameter) critical exponent β and the magnetic field critical exponent D_h are given by

$$M|_{H=0} \sim |t|^{\beta},\ \beta = \frac{d - D_h}{D_t}. \tag{11.27}$$

The anomalous exponent η is defined by the behavior of the 2-point function at the critical temperature given by

$$\Gamma(x) \sim \frac{1}{r^p} \exp(-r/\xi),\ p = d - 2 + \eta = 2d - 2D_h. \tag{11.28}$$

There are three more critical exponents which can be determined solely in terms of D_t and D_h (scaling relations). These are α (specific heat), γ (susceptibility), δ (magnetization at non-zero magnetic field).

11.1.4 The Landau–Wilson model

In the above differential equations we change the variables as

$$x = \frac{r}{\Lambda^2},\ y = C_d \frac{u}{\Lambda^{4-d}}. \tag{11.29}$$

We then obtain the RG equations

$$\frac{dx}{d\tau} = 2x + 12y - 12xy,\ \frac{dy}{d\tau} = \epsilon y - 36y^2. \tag{11.30}$$

The fixed point (x_*, y_*) is defined by

$$0 = 2x_* + 12y_* - 12x_*y_*,\ 0 = \epsilon y_* - 36y_*^2. \tag{11.31}$$

We have the two solutions

$$\begin{aligned} &(x_*, y_*) = (0, 0), \text{ Gaussian} \\ &(x_*, y_*) = (-\epsilon/6, \epsilon/36), \text{ Wilson–Fisher} \end{aligned} \tag{11.32}$$

Linearizing the RG transformation around the fixed point by writing $x = x_* + \delta x$ and $y = y_* + \delta y$ we obtain the differential equations

$$\frac{d\delta x}{d\tau} = (2 - 12y_*)\delta x + (12 - 12x_*)\delta y,\ \frac{d\delta y}{d\tau} = (\epsilon - 72y_*)\delta y. \tag{11.33}$$

The RG matrix W is given by

$$W = \begin{pmatrix} 2 - 12y_* & 12 - 12x_* \\ 0 & \epsilon - 72y_* \end{pmatrix} \tag{11.34}$$

with eigenvalues given respectively by $\lambda_1 = 2 - 12y_*$ and $\lambda_2 = \epsilon - 72y_*$. The corresponding left eigenvectors are given by

$$\phi_1 = \begin{pmatrix} a \\ \frac{12(1-x_*)a}{2-\epsilon+60y_*} \end{pmatrix}, \lambda_1 = 2 - 12y_* \tag{11.35}$$

$$\phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda_1 = \epsilon - 72y_*. \tag{11.36}$$

The scaling fields are given, respectively, by

$$v_1 = a \cdot \delta x + \frac{12(1-x_*)a}{2-\epsilon+60y_*} \cdot \delta y, \lambda_1 = 2 - 12y_* \tag{11.37}$$

$$v_2 = 0 \cdot \delta x + 1 \cdot \delta y, \lambda_2 = \epsilon - 72y_*. \tag{11.38}$$

These are given explicitly as follows.

- **Gaussian fixed point** $(x_*, y_*) = (0, 0)$**:**

$$\begin{aligned} v_1 &= \delta x + 6\delta y, \lambda_1 = 2 \\ v_2 &= \delta y, \lambda = \epsilon. \end{aligned} \tag{11.39}$$

- **Wilson–Fisher fixed point** $(x_*, y_*) = (-\epsilon/6, \epsilon/36)$**:**

$$\begin{aligned} v_1 &= \delta x + (6-\epsilon)\delta y, \lambda_1 = 2 - \frac{\epsilon}{3} \\ v_2 &= \delta y, \lambda_2 = -\epsilon. \end{aligned} \tag{11.40}$$

The scaling fields satisfy

$$\frac{dv_i}{d\tau} = \lambda_i v_i. \tag{11.41}$$

The solution is given by

$$v_i = v_0 b^{\lambda_i}. \tag{11.42}$$

Let us assume now that $d < 4$ and close to 4 ($\epsilon > 0$ and small). In this case we have $\lambda_1 = 2 - \epsilon/3 > 1$ and thus v_1 is a relevant scaling field, whereas $\lambda_2 = -\epsilon < 1$ and thus v_2 is an irrelevant scaling field.

The critical surface corresponding to the Wilson–Fisher fixed point is determined by setting the relevant scaling field, i.e. v_1, to zero. This is equivalent to the line in the x–y plane given by

$$y - y_* = -\frac{1}{6-\epsilon}(x - x_*). \tag{11.43}$$

To the linear order of ϵ this line passes by the other trivial fixed point at the origin. This line or critical surface represents our first preferred direction in the x–y plane.

The critical surface $v_1 = 0$ represents the critical temperature $T = T_C$. This can be seen as follows. The other independent preferred direction should be obtained by setting the irrelevant variable v_2 to 0. This gives $\delta y = 0$ (the direction is parallel to the x-axis) and $v_1 = \delta x = \delta r/\Lambda^2$ (this is the direction of increase or decrease of the temperature away from the critical value T_c).

The critical surface $v_1 = 0$ corresponds, therefore, precisely to the critical temperature $T = T_C$. Hence, starting at a point A with $v_1 < 0$, i.e. with $T < T_C$, the system under RG will approach the low-temperature phase where the symmetry is spontaneously broken. Whereas starting at a point B with $v_1 > 0$, i.e. with $T > T_C$, the system will approach under successive RG transformations the high-temperature phase.

The mass critical exponent is directly deduced to be (v_1 is the relevant scaling field identified as the temperature and it behaves as $v_1 \sim b^{\lambda_i}$)

$$D_t = \lambda_1 = 2 - \frac{\epsilon}{3} \Rightarrow \nu = \frac{1}{D_t} = \frac{1}{2} + \frac{\epsilon}{12}. \tag{11.44}$$

The magnetic field scaling field was shown in the exercise to be given by the mean-field value:

$$D_h = \frac{d+2}{2}. \tag{11.45}$$

From these two critical exponents we can calculate all other exponents as we have shown in the previous section.

The last remark concerns the dimensions $d > 4$. In this case the two scaling fields are both relevant and therefore the Wilson–Fisher fixed point is unstable. The theory is actually dominated by mean-field.

11.2 The Wilson approximate recursion formulas

11.2.1 Kadanoff–Wilson phase space analysis

We start by describing a particular phase space cell decomposition due to Wilson [7, 8] which is largely motivated by Kadanoff block spins [9, 10]. See also [1, 11] and [12, 13].

We assume a hard cutoff 2Λ. Thus, if $\phi(x)$ is the field (spin) variable and $\tilde{\phi}(k)$ is its Fourier transform we will assume that $\tilde{\phi}(k)$ is zero for $k > 2\Lambda$.

We expand the field as

$$\phi(x) = \sum_{\vec{m}} \sum_{l=0}^{\infty} \psi_{\vec{m}l}(x)\phi_{\vec{m}l}. \tag{11.46}$$

The wavefunctions $\psi_{\vec{m}l}(x)$ satisfy the orthonormality condition

$$\int d^d x \psi^*_{\vec{m}_1 l_1}(x)\psi_{\vec{m}_2 l_2}(x) = \delta_{\vec{m}_1 \vec{m}_2}\delta_{l_1 l_2}. \tag{11.47}$$

The Fourier transform $\tilde{\psi}_{\vec{m}l}(k)$ is defined by

$$\tilde{\psi}_{\vec{m}l}(k) = \int d^d x \psi_{\vec{m}l}(x) e^{ikx}. \tag{11.48}$$

The interpretation of l and $\vec{m}$ is as follows. We decompose momentum space into thin spherical shells, i.e. logarithmically as

$$\frac{1}{2^l} \leqslant \frac{|k|}{\Lambda} \leqslant \frac{1}{2^{l-1}}. \tag{11.49}$$

The functions $\tilde{\psi}_{\vec{m}l}(k)$ for a fixed l are non-zero only inside the shell l, i.e. for $1/2^l \leqslant |k|/\Lambda \leqslant 1/2^{l-1}$. We will assume furthermore that the functions $\tilde{\psi}_{\vec{m}l}(k)$ are constant within its shell and satisfy the normalization condition

$$\int \frac{d^d k}{(2\pi)^d} |\tilde{\psi}_{\vec{m}l}(k)|^2 = 1. \tag{11.50}$$

The functions $\tilde{\psi}_{\vec{m}l}(k)$ and $\psi_{\vec{m}l}(x)$ for a fixed l and a fixed $\vec{m}$ should be thought of as minimal wave packets, i.e. if Δk is the width of $\tilde{\psi}_{\vec{m}l}(k)$ and Δx is the width of $\psi_{\vec{m}l}(x)$ then one must have by the uncertainty principle the requirement $\Delta x \Delta k = (2\pi)^d$. Thus, for each shell we divide position space into blocks of equal size each with volume inversely proportional to the volume of the corresponding shell. The volume of the lth momentum shell is proportional to R^d where $R = 1/2^l$, viz $\Delta k = (2\pi)^d 2^{-ld} w$ where w is a constant. Hence, the volume of the corresponding position space box is $\Delta x = 2^{ld} w^{-1}$. The functions $\psi_{\vec{m}l}(x)$ are non-zero (constant) only inside this box by construction. This position space box is characterized by the index $\vec{m}$ as is obvious from the normalization condition

$$\int d^d x |\psi_{\vec{m}l}(x)|^2 = \int_{\vec{x} \in \text{box } \vec{m}} d^d x |\psi_{\vec{m}l}(x)|^2 = 1. \tag{11.51}$$

In other words, we have

$$\int d^d x = \sum_{\vec{m}} \int_{\vec{x} \in \text{box } \vec{m}} d^d x. \tag{11.52}$$

The normalization conditions in momentum and position spaces lead to the relations

$$|\tilde{\psi}_{\vec{m}l}(k)| = 2^{ld/2} w^{-1/2} \tag{11.53}$$

$$|\psi_{\vec{m}l}(x)| = 2^{-ld/2} w^{1/2}. \tag{11.54}$$

Obviously $|\tilde{\psi}_{\vec{m}l}(0)| = 0$. Thus $\int d^d x \psi_{\vec{m}l}(x) = \tilde{\psi}_{\vec{m}l}(0) = 0$ and as a consequence we will assume that $\psi_{\vec{m}l}(x)$ is equal to $+2^{-ld/2} w^{1/2}$ in one half of the box and $-2^{-ld/2} w^{1/2}$ in the other half.

The meaning of the index $\vec{m}$ which labels the position space boxes can be clarified further by the following argument. By an appropriate scale transformation in

momentum space we can scale the momenta such that the lth shell becomes the largest shell $l = 0$. Clearly the correct scale transformation is $k \longrightarrow 2^l k$ since $1 \leqslant |2^l k|/\Lambda \leqslant 2$. This corresponds to a scale transformation in position space of the form $x \longrightarrow x/2^l$. We obtain, therefore, the relation $\psi_{\vec{m}l}(x) = \psi_{\vec{m}0}(x/2^l)$. Next we perform an appropriate translation in position space to bring the box $\vec{m}$ to the box $\vec{0}$. This is clearly given by the translation $\vec{x} \longrightarrow \vec{x} - a_0\vec{m}$. We obtain, therefore, the relation

$$\psi_{\vec{m}l}(x) = \psi_{\vec{0}0}(x/2^l - a_0\vec{m}). \tag{11.55}$$

The functions $\tilde{\psi}_{\vec{m}l}(k)$ and $\psi_{\vec{m}l}(x)$ correspond to a single degree of freedom in phase space occupying a volume $(2\pi)^d$, i.e. a single cell in phase space is characterized by l and $\vec{m}$. Each momentum shell l corresponds to a lattice in the position space with a lattice spacing given by

$$a_l = (\Delta x)^{1/d} = 2^l a_0,\ a_0 = w^{-1/d}. \tag{11.56}$$

The largest shell $l = 0$ corresponds to a lattice spacing a_0 and each time l is increased by 1 the lattice spacing gets doubled which is the original spin blocking idea of Kadanoff.

We are interested in integrating out only the $l = 0$ modes. We write then

$$\phi(x) = \sum_{\vec{m}} \psi_{\vec{m}0}(x)\phi_{\vec{m}0} + \phi_1(x) \tag{11.57}$$

$$\phi_1(x) = \sum_{\vec{m}} \sum_{l=1}^{\infty} \psi_{\vec{m}l}(x)\phi_{\vec{m}l}. \tag{11.58}$$

From the normalization (16.22) and the scaling law (11.55) we have

$$\psi_{\vec{m}l}(x) = 2^{-d/2}\psi_{\vec{m}l-1}(x/2). \tag{11.59}$$

We define $\phi'(x/2)$ by

$$\phi_1(x) = 2^{-d/2}\alpha_0\phi'(x/2). \tag{11.60}$$

In other words, we have

$$\phi'(x/2) = \sum_{\vec{m}} \sum_{l=1}^{\infty} \psi_{\vec{m}l-1}(x/2)\phi'_{\vec{m}l-1},\ \phi'_{\vec{m}l-1} = \alpha_0^{-1}\phi_{\vec{m}l}. \tag{11.61}$$

We have then

$$\phi(x) = \sum_{\vec{m}} \psi_{\vec{m}0}(x)\phi_{\vec{m}0} + 2^{-d/2}\alpha_0\phi'(x/2). \tag{11.62}$$

11.2.2 Recursion formulas

We will be interested in actions of the form

$$S_0(\phi(x)) = \frac{K}{2}\int d^dx R_0(\phi(x))\partial_\mu\phi(x)\partial^\mu\phi(x) + \int d^dx P_0(\phi(x)). \tag{11.63}$$

We will assume that P_0 and R_0 are even polynomials of the field and that $dR_0/d\phi$ is much smaller than P_0 for all relevant configurations. The partition function is

$$\begin{aligned} Z_0 &= \int \mathcal{D}\phi(x) e^{-S_0(\phi(x))} \\ &= \int \mathcal{D}\phi'(x/2)\int\prod_{\vec{m}} d\phi_{\vec{m}0} e^{-S_0(\phi(x))}. \end{aligned} \tag{11.64}$$

The degrees of freedom contained in the fluctuation $\phi'(x/2)$ correspond to momentum shells $l \geqslant 1$ and thus correspond to position space wave packets larger than the box $\vec{m}$ by at least a factor of 2. We can thus assume that $\phi'(x/2)$ is almost constant over the box $\vec{m}$. If x_0 is the center of the box $\vec{m}$ we can expand $\phi'(x/2)$ as

$$\begin{aligned} \phi'(x/2) = \phi'(x_0/2) &+ (x - x_0)^\mu \partial_\mu \phi'(x_0/2) \\ &+ \frac{1}{2}(x - x_0)^\mu (x - x_0)^\nu \partial_\mu\partial_\nu\phi'(x_0/2) + \cdots \end{aligned} \tag{11.65}$$

The partition function becomes

$$Z_0 = \int \mathcal{D}\phi'(x_0/2)\int\prod_{\vec{m}} d\phi_{\vec{m}0} e^{-S_0(\phi(x))}. \tag{11.66}$$

The kinetic term
We compute

$$\begin{aligned} \int d^dx R_0 \partial_\mu\phi(x)\partial^\mu\phi(x) = \int d^dx R_0 \Bigg[&\sum_{\vec{m},\vec{m}'} \partial_\mu\psi_{\vec{m}0}(x)\partial^\mu\psi_{\vec{m}'0}(x)\cdot\phi_{\vec{m}0}\phi_{\vec{m}'0} \\ &+ 2^{1-d/2}\alpha_0\sum_{\vec{m}}\phi_{\vec{m}0}\partial_\mu\psi_{\vec{m}0}(x)\partial_\mu\phi'(x/2) \\ &+ 2^{-d}\alpha_0^2\partial_\mu\phi'(x/2)\partial^\mu\phi'(x/2)\Bigg]. \end{aligned} \tag{11.67}$$

The integral $\int d^dx$ is $\sum_{\vec{m}}\int_{\vec{x}\in\text{box }\vec{m}}$. In the box $\vec{m}$ the function $R_0(\phi(x))$ can be replaced by the function $R_0(\psi_{\vec{m}0}(x)\phi_{\vec{m}0} + 2^{-d/2}\alpha_0\phi'(x/2))$. In this box $\psi_{\vec{m}0}(x)$ is approximated by $+w^{1/2}$ in one half of the box and by $-w^{1/2}$ in the other half, i.e. by a step function. Thus, the third term in the above equation can be approximated by (dropping also higher derivative corrections and defining $u_{\vec{m}} = 2^{-d/2}\alpha_0\phi'(x_0/2)$)

$$2^{-d-1}\alpha_0^2 w^{-1}\sum_{\vec{m}}\Big[R_0(w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})+R_0(-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})\Big] \times \partial_\mu\phi'(x_0/2)\partial^\mu\phi'(x_0/2). \tag{11.68}$$

The contribution of $\partial_\mu\psi_{\vec{m}0}(x)$ is, however, only appreciable when $\psi_{\vec{m}0}(x)=0$, i.e. at the center of the box. In the first and second terms of the above equation we can then replace $R_0(\phi(x))$ by $R_0(u_{\vec{m}})$. The second term in the above equation (11.67) vanishes by conservation of momentum. In the first term we can neglect all the coupling terms $\phi_{\vec{m}0}\phi_{\vec{m}'0}$ with $\vec{m}\neq\vec{m}'$ since $\psi_{\vec{m}'0}(x)$ is zero inside the box $\vec{m}$. We define the integral

$$\rho=\int d^dx\partial_\mu\psi_{\vec{m}0}(x)\partial^\mu\psi_{\vec{m}0}(x). \tag{11.69}$$

This is independent of $\vec{m}$ because of the relation (11.55). The first term becomes therefore $\rho\sum_{\vec{m}}R_0(u_{\vec{m}})\phi^2_{\vec{m}0}$. Equation (11.67) becomes

$$\begin{aligned}\int d^dxR_0\partial_\mu\phi(x)\partial^\mu\phi(x)=\rho\sum_{\vec{m}}R_0(u_{\vec{m}})\phi^2_{\vec{m}0}+2^{-d-1}\alpha_0^2w^{-1}\\ \times\sum_{\vec{m}}\Big[R_0(w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})+R_0(-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})\Big]\\ \times\partial_\mu\phi'(x_0/2)\partial^\mu\phi'(x_0/2).\end{aligned} \tag{11.70}$$

The interaction term

Next we want to compute

$$\int_{\vec{x}\in\text{box }\vec{m}}d^dxP_0(\phi^{(m)}(x))=\int_{\vec{x}\in\text{box }\vec{m}}d^dxP_0(\psi_{\vec{m}0}(x)\phi_{\vec{m}0}+2^{-d/2}\alpha_0\phi'(x/2)). \tag{11.71}$$

We note again that within the box $\vec{m}$ the field is given by

$$\phi^{(m)}(x)=\psi_{\vec{m}0}(x)\phi_{\vec{m}0}+2^{-d/2}\alpha_0\phi'(x/2). \tag{11.72}$$

Introduce $z_0=\psi_{\vec{m}0}(x)\phi_{\vec{m}0}+u_{\vec{m}}$. We have then

$$\begin{aligned}\int_{\vec{x}\in\text{box }\vec{m}}d^dxP_0(\phi^{(m)}(x))=\int_{\vec{x}\in\text{box }\vec{m}}d^dx\Bigg[P_0(z_0)\\ +\left((x-x_0)^\mu\partial_\mu u_{\vec{m}}+\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu\right)\\ \times\partial_\mu\partial_\nu u_{\vec{m}}\frac{dP_0}{dz}|_{z_0}\\ +\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu\partial_\mu u_{\vec{m}}\partial_\nu u_{\vec{m}}\frac{d^2P_0}{dz^2}|z_0+\cdots\Bigg].\end{aligned} \tag{11.73}$$

As stated earlier $\psi_{\vec{m}0}(x)$ is approximated by $+w^{1/2}$ in one half of the box and by $-w^{1/2}$ in the other half. Also, recall that the volume of the box is w^{-1}. The first term can be approximated by

$$\int_{\vec{x}\in \text{box } \vec{m}} d^d x P_0(z_0) = \frac{w^{-1}}{2}\Big[P_0(w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}}) + P_0(-w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}})\Big]. \tag{11.74}$$

We compute now the third term in equation (11.73). We start from the obvious identity

$$\begin{aligned}\int_{\text{box}} d^d x \frac{1}{2}(x - x_0)^\mu (x - x_0)^\nu = &\int_{\text{box}_+} d^d x \frac{1}{2}(x - x_0)^\mu (x - x_0)^\nu \\ &+ \int_{\text{box}_-} d^d x \frac{1}{2}(x - x_0)^\mu (x - x_0)^\nu.\end{aligned} \tag{11.75}$$

We will think of the box $\vec{m}$ as a sphere of volume w^{-1}. Thus

$$\int_{\text{box}} d^d x \frac{1}{2}(x - x_0)^\mu (x - x_0)^\nu = \frac{1}{2} V \eta^{\mu\nu}. \tag{11.76}$$

We have the definitions

$$V = \frac{1}{d}\int_{\text{box}} r^2 d^d x,\ w^{-1} = \int_{\text{box}} d^d x. \tag{11.77}$$

Explicitly we have

$$V = \frac{1}{d+2}\left(\frac{dw^{-1}}{\Omega_{d-1}}\right)^{(d+2)/d}\frac{\Omega_{d-1}}{d}. \tag{11.78}$$

The above identity becomes then

$$\begin{aligned}\frac{1}{2} V \eta^{\mu\nu} = &\int_{\text{box}_+} d^d x \frac{1}{2}(x - x_0)^\mu (x - x_0)^\nu \\ &+ \int_{\text{box}_-} d^d x \frac{1}{2}(x - x_0)^\mu (x - x_0)^\nu.\end{aligned} \tag{11.79}$$

The sum of these two integrals is rotational invariant. As stated before, we think of the box as a sphere divided into two regions of equal volume. The first region (the first half of the box) is a concentric smaller sphere, whereas the second region (the second half of the box) is a thin spherical shell. Both regions are spherically symmetric and thus we can assume that

$$\int_{\text{box}_\pm} d^d x \frac{1}{2}(x - x_0)^\mu (x - x_0)^\nu = \frac{1}{2} V_\pm \eta^{\mu\nu}. \tag{11.80}$$

Clearly $V = V_+ + V_-$. The integral of interest is

$$\mathcal{I}_1 = \int_{\vec{x}\in\text{box}} d^d x \frac{1}{2}(x - x_0)^\mu (x - x_0)^\nu \partial_\mu \partial_\nu u_{\vec{m}} \frac{dP_0}{dz}|_{z_0}. \tag{11.81}$$

We compute

$$\begin{aligned}\mathcal{I}_1 &= \frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial_\nu u_{\vec{m}}\int_{\text{box}_+} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu \\ &\quad + \frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial_\nu u_{\vec{m}}\int_{\text{box}_-} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu \\ &= \frac{V_+}{2}\frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial^\mu u_{\vec{m}} + \frac{V_-}{2}\frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial^\mu u_{\vec{m}} \\ &= 2^{-1-d/2}\alpha_0 V_+\frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial^\mu\phi'(x_0/2) \\ &\quad + 2^{-1-d/2}\alpha_0 V_-\frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial^\mu\phi'(x_0/2).\end{aligned} \tag{11.82}$$

Similarly, the fourth term in equation (11.73) is computed as follows. We are interested in the integral

$$\mathcal{I}_2 = \int_{\vec{x}\in\text{box }\vec{m}} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu\partial_\mu u_{\vec{m}}\partial_\nu u_{\vec{m}}\frac{d^2P_0}{dz^2}|z_0. \tag{11.83}$$

We compute

$$\begin{aligned}\mathcal{I}_2 &= \frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\partial_\nu u_{\vec{m}}\int_{\text{box}_+} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu \\ &\quad + \frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\partial_\nu u_{\vec{m}}\int_{\text{box}_-} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu \\ &= \frac{V_+}{2}\frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\partial^\mu u_{\vec{m}} + \frac{V_-}{2}\frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\partial^\mu u_{\vec{m}} \\ &= 2^{-1-d}\alpha_0^2 V_+\frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\phi'(x_0/2)\partial^\mu\phi'(x_0/2) \\ &\quad + 2^{-1-d}\alpha_0^2 V_-\frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\phi'(x_0/2)\partial^\mu\phi'(x_0/2).\end{aligned} \tag{11.84}$$

Finally, we compute the first term in equation (11.73), we have

$$\begin{aligned}\int_{\vec{x}\in\text{box }\vec{m}} d^dx(x-x_0)^\mu\partial_\mu u_{\vec{m}}\frac{dP_0}{dz}|_{z_0} &= \frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\int_{\text{box}_+} d^dx(x-x_0)^\mu \\ &\quad + \frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\int_{\text{box}_-} d^dx(x-x_0)^\mu.\end{aligned} \tag{11.85}$$

Clearly we must have

$$\int_{\text{box}_+} d^dx(x-x_0)^\mu + \int_{\text{box}_-} d^dx(x-x_0)^\mu = 0. \tag{11.86}$$

We will assume that both integrals vanish again by the same argument. The linear term is therefore 0, viz

$$\int_{\vec{x}\in \text{box}\,\vec{m}} d^d x (x - x_0)^\mu \partial_\mu u_{\vec{m}} \frac{dP_0}{dz}|_{z_0} = 0. \tag{11.87}$$

The final result is

$$\begin{aligned}
\int_{\vec{x}\in \text{box}\,\vec{m}} d^d x P_0(\phi^{(m)}(x)) = {} & \frac{w^{-1}}{2}\Big[P_0(w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}}) + P_0(-w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}})\Big] \\
& + 2^{-1-d/2}\alpha_0\Big[V_+ \frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}} \\
& + V_- \frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\Big]\partial_\mu\partial^\mu\phi'(x_0/2) \\
& + 2^{-1-d}\alpha_0^2\Big[V_+ \frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}} \\
& + V_- \frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\Big]\partial_\mu\phi'\partial^\mu\phi'(x_0/2).
\end{aligned} \tag{11.88}$$

The action

By putting all the previous results together we obtain the expansion of the action in the following form

$$\begin{aligned}
S_0(\phi(x)) = {} & \frac{K}{2}\int d^d x R_0(\phi(x))\partial_\mu\phi(x)\partial^\mu\phi(x) + \int d^d x P_0(\phi(x)) \\
= {} & \frac{K\rho}{2}\sum_{\vec{m}} R_0(u_{\vec{m}})\phi^2_{\vec{m}0} \\
& + 2^{-d-2}K\alpha_0^2 w^{-1}\sum_{\vec{m}}\Big[R_0(w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}}) \\
& + R_0(-w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}})\Big]\partial_\mu\phi'(x_0/2)\partial^\mu\phi'(x_0/2) \\
& + \frac{w^{-1}}{2}\sum_{\vec{m}}\Big(P_0(w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}}) + P_0(-w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}})\Big) \\
& + 2^{-1-d/2}\alpha_0\sum_{\vec{m}}\Big(V_+ \frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}} + V_- \frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\Big)\partial_\mu\partial^\mu\phi'(x_0/2) \\
& + 2^{-1-d}\alpha_0^2\sum_{\vec{m}}\Big(V_+ \frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}} \\
& + V_- \frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\Big)\partial_\mu\phi'(x_0/2)\partial^\mu\phi'(x_0/2).
\end{aligned} \tag{11.89}$$

The path integral

We need now to evaluate the path integral

$$Z = \int \prod_{\vec{m}} d\phi_{\vec{m}0} e^{-S_0(\phi(x))}. \tag{11.90}$$

We introduce the variables

$$u_{\vec{m}} \longrightarrow z_{\vec{m}} = \left(\frac{K\rho}{2w}\right)^{1/2} u_{\vec{m}} \tag{11.91}$$

$$\phi_{\vec{m}0} \longrightarrow y_{\vec{m}} = \left(\frac{K\rho}{2}\right)^{1/2} \phi_{\vec{m}0} \tag{11.92}$$

$$R_0 \longrightarrow W_0(x) = R_0\left(\left(\frac{2w}{K\rho}\right)^{1/2} x\right) \tag{11.93}$$

$$P_0 \longrightarrow Q_0(x) = w^{-1} P_0\left(\left(\frac{2w}{K\rho}\right)^{1/2} x\right). \tag{11.94}$$

We compute then

$$\begin{aligned} Z = \prod_{\vec{m}} \left(\frac{2}{K\rho}\right)^{1/2} \int dy_{\vec{m}} \exp\Bigg(& -y_{\vec{m}}^2 W_0(z_{\vec{m}}) \\ & - \frac{1}{2} Q_0(y_{\vec{m}} + z_{\vec{m}}) - \frac{1}{2} Q_0(-y_{\vec{m}} + z_{\vec{m}}) \\ & - 2^{-(3+d)/2} \alpha_0 (K\rho w)^{1/2} \Bigg[V_+ \frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}} + z_{\vec{m}}) \\ & + V_- \frac{dQ_0}{dy_{\vec{m}}}(-y_{\vec{m}} + z_{\vec{m}}) \Bigg] \partial_\mu \partial^\mu \phi'(x_0/2) \\ & - 2^{-d-2} \alpha_0^2 K\rho \Bigg[V_+ \frac{d^2 Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}} + z_{\vec{m}}) \\ & + V_- \frac{d^2 Q_0}{dy_{\vec{m}}^2}(-y_{\vec{m}} + z_{\vec{m}}) \Bigg] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2) \\ & - 2^{-d-2} \alpha_0^2 K w^{-1} \Big[W_0(y_{\vec{m}} + z_{\vec{m}}) \\ & + W_0(-y_{\vec{m}} + z_{\vec{m}}) \Big] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2) \Bigg). \end{aligned} \tag{11.95}$$

Define

$$
\begin{aligned}
M_0(z_{\vec{m}}) = \int dy_{\vec{m}} \exp\Bigg(& -y_{\vec{m}}^2 W_0(z_{\vec{m}}) \\
& - \frac{1}{2} Q_0(y_{\vec{m}} + z_{\vec{m}}) - \frac{1}{2} Q_0(-y_{\vec{m}} + z_{\vec{m}}) \\
& - 2^{-(3+d)/2} \alpha_0 (K\rho w)^{1/2} \Bigg[V_+ \frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}} + z_{\vec{m}}) \\
& + V_- \frac{dQ_0}{dy_{\vec{m}}}(-y_{\vec{m}} + z_{\vec{m}}) \Bigg] \partial_\mu \partial^\mu \phi'(x_0/2) \\
& - 2^{-d-2} \alpha_0^2 K\rho \Bigg[V_+ \frac{d^2 Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}} + z_{\vec{m}}) \\
& + V_- \frac{d^2 Q_0}{dy_{\vec{m}}^2}(-y_{\vec{m}} + z_{\vec{m}}) \Bigg] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2) \\
& - 2^{-d-2} \alpha_0^2 K w^{-1} \Big[W_0(y_{\vec{m}} + z_{\vec{m}}) \\
& + W_0(-y_{\vec{m}} + z_{\vec{m}}) \Big] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2) \Bigg).
\end{aligned}
\tag{11.96}
$$

In this equation $\phi'(x_0/2)$ is given in terms of $z_{\vec{m}}$ by

$$
\phi'(x_0/2) = \frac{2^{d/2}}{\alpha_0} \left(\frac{2w}{K\rho} \right)^{1/2} z_{\vec{m}}. \tag{11.97}
$$

The remaining dependence on the box $\vec{m}$ is only through the center of the box x_0. Then

$$
\begin{aligned}
Z &= \prod_{\vec{m}} \left(\frac{2}{K\rho} \right)^{1/2} M_0(z_{\vec{m}}) \\
&= \prod_{\vec{m}} \left(\frac{2}{K\rho} \right)^{1/2} \exp(\ln M_0(z_{\vec{m}})) \\
&= \exp\left(\sum_{\vec{m}} \ln \frac{M_0(z_{\vec{m}})}{I_0(0)} \right) \prod_{\vec{m}} \left(\frac{2}{K\rho} \right)^{1/2} I_0(0).
\end{aligned}
\tag{11.98}
$$

The function $I_0(z)$ is defined by

$$
I_0(z) = \int dy \exp\left(-y^2 W_0(z) - \frac{1}{2} Q_0(y + z) - \frac{1}{2} Q_0(-y + z) \right). \tag{11.99}
$$

In order to compute M_0 we will assume that the derivative terms are small and expand the exponential around the ultra local approximation. We compute

$$
\begin{aligned}
M_0(z_{\vec{m}}) = I_0(z_{\vec{m}})\Bigg[& 1 - 2^{-(3+d)/2}\alpha_0(K\rho w)^{1/2}V \\
& \times \left\langle \frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}} + z_{\vec{m}}) \right\rangle \partial_\mu \partial^\mu \phi'(x_0/2) \\
& - 2^{-d-1}\alpha_0^2 K \Bigg[\frac{\rho V}{2} \left\langle \frac{d^2 Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}} + z_{\vec{m}}) \right\rangle \\
& + w^{-1} \left\langle W_0(y_{\vec{m}} + z_{\vec{m}}) \right\rangle \Bigg] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2) + \cdots \Bigg].
\end{aligned}
\tag{11.100}
$$

The path integral (11.90) becomes

$$
\begin{aligned}
Z = \prod_{\vec{m}} \left(\frac{2}{K\rho} \right)^{1/2} I_0(0) \times \exp\Bigg(& \sum_{\vec{m}} \ln \frac{I_0(z_{\vec{m}})}{I_0(0)} - 2^{-(3+d)/2}\alpha_0(K\rho w)^{1/2}V \\
& \times \sum_{\vec{m}} \left\langle \frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}} + z_{\vec{m}}) \right\rangle \partial_\mu \partial^\mu \phi'(x_0/2) \\
& - 2^{-d-1}\alpha_0^2 K \sum_{\vec{m}} \Bigg[\frac{\rho V}{2} \left\langle \frac{d^2 Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}} + z_{\vec{m}}) \right\rangle \\
& + w^{-1} \left\langle W_0(y_{\vec{m}} + z_{\vec{m}}) \right\rangle \Bigg] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2) \Bigg).
\end{aligned}
\tag{11.101}
$$

We make the change of variable $x_0/2 \longrightarrow x$ (this means that the position space wave packet with $l = 1$ corresponding to the highest not integrated momentum will now fit into the box) to obtain

$$
\begin{aligned}
Z = \prod_{\vec{m}} \left(\frac{2}{K\rho} \right)^{1/2} I_0(0) \times \exp\Bigg(& \sum_{\vec{m}} \ln \frac{I_0(z_{\vec{m}})}{I_0(0)} - 2^{-(3+d)/2}\frac{\alpha_0(K\rho w)^{1/2}V}{4} \\
& \times \sum_{\vec{m}} \left\langle \frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}} + z_{\vec{m}}) \right\rangle \partial_\mu \partial^\mu \phi'(x) \\
& - 2^{-d-1}\frac{\alpha_0^2 K}{4} \sum_{\vec{m}} \Bigg[\frac{\rho V}{2} \left\langle \frac{d^2 Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}} + z_{\vec{m}}) \right\rangle \\
& + w^{-1} \left\langle W_0(y_{\vec{m}} + z_{\vec{m}}) \right\rangle \Bigg] \partial_\mu \phi'(x) \partial^\mu \phi'(x) \Bigg).
\end{aligned}
\tag{11.102}
$$

Now $z_{\vec{m}}$ is given by $z_{\vec{m}} = (K\rho/2w)^{1/2} 2^{-d/2} \alpha_0 \phi'(x)$. It is clear that $dQ_0(y_{\vec{m}} + z_{\vec{m}})/dy_{\vec{m}} = dQ_0(y_{\vec{m}} + z_{\vec{m}})/dz_{\vec{m}}$, etc. Recall that the volume of the box $\vec{m}$

is w^{-1} and thus we can make the identification $w\int d^dx = \sum_{\vec{m}}$. However, we have also made the rescaling $x_0 \longrightarrow 2x$ and hence we must make instead the identification $2^d w\int d^dx = \sum_{\vec{m}}$. We obtain

$$\begin{aligned} Z = \prod_{\vec{m}} \left(\frac{2}{K\rho}\right)^{1/2} I_0(0) \times \exp\Bigg(& 2^d w \int d^dx \ln \frac{I_0(z)}{I_0(0)} \\ & - 2^{-(3+d)/2}\frac{\alpha_0(K\rho w)^{1/2}V}{4} 2^d w \\ & \times \int d^dx \left\langle \frac{dQ_0}{dz}(z)\right\rangle \partial_\mu\partial^\mu\phi'(x) \\ & - 2^{-d-1}\frac{\alpha_0^2 K}{4} 2^d w \int d^dx \Bigg[\frac{\rho V}{2}\left\langle \frac{d^2Q_0}{dz^2}(z)\right\rangle \\ & + w^{-1}\langle W_0(z)\rangle\Bigg]\partial_\mu\phi'(x)\partial^\mu\phi'(x)\Bigg). \end{aligned} \tag{11.103}$$

Now z is given by $z = (K\rho/2w)^{1/2}2^{-d/2}\alpha_0\phi'(x)$. The expectation values $\langle O^n(z)\rangle$ are defined by

$$\begin{aligned} \langle O^n(z)\rangle = \frac{1}{I(z)}\int dy \left(\frac{O(y+z)+O(-y+z)}{2}\right)^n \\ \times \exp\left(-y^2 W_0(z) - \frac{1}{2}Q_0(y+z) - \frac{1}{2}Q_0(-y+z)\right). \end{aligned} \tag{11.104}$$

Next we derive in a straightforward way the formula

$$\begin{aligned} \frac{d}{dz}\left\langle \frac{dQ_0}{dz}(z)\right\rangle = \left\langle \frac{d^2Q_0}{dz^2}(z)\right\rangle - \left\langle \left(\frac{dQ_0}{dz}\right)^2(z)\right\rangle + \left\langle \frac{dQ_0}{dz}(z)\right\rangle^2 \\ + \frac{dW_0}{dz}\Bigg(\left\langle \frac{dQ_0}{dz}(z)\right\rangle \frac{1}{I_0(z)}\int dy y^2 e\cdots \\ - \frac{1}{I_0(z)}\int dy \frac{dQ_0}{dz}(y+z)y^2 e\cdots\Bigg). \end{aligned} \tag{11.105}$$

By integrating by part the second term in equation (11.103) we can see that the first term in equation (11.105) cancels the third term in equation (11.103). The last term can be neglected if we assume that dW_0/dz is much smaller than Q_0. We obtain then

$$
\begin{aligned}
Z = \prod_{\vec{m}} \left(\frac{2}{K\rho}\right)^{1/2} I_0(0) \times \exp\Bigg(& 2^d w \int d^d x \ln \frac{I_0(z)}{I_0(0)} \\
& - \frac{\alpha_0^2 K w}{8} \int d^d x \Bigg[\frac{\rho V}{2} \Bigg(\left\langle \left(\frac{dQ_0}{dz} \right)^2 (z) \right\rangle \\
& - \left\langle \frac{dQ_0}{dz}(z) \right\rangle^2 \Bigg) + w^{-1} \langle W_0(z) \rangle \Bigg] \\
& \times \partial_\mu \phi'(x) \partial^\mu \phi'(x) \Bigg).
\end{aligned}
\tag{11.106}
$$

The recursion formulas

The full path integral is therefore given by

$$
\begin{aligned}
Z_0 &= \int \mathcal{D}\phi'(x_0/2) \int \prod_{\vec{m}} d\phi_{\vec{m}0} e^{-S_0(\phi(x))} \\
&\propto \int \mathcal{D}\phi'(x) \int \exp\Bigg(2^d w \int d^d x \ln \frac{I_0(z)}{I_0(0)} \\
&\quad - \frac{\alpha_0^2 K w}{8} \int d^d x \Bigg[\frac{\rho V}{2} \Bigg(\left\langle \left(\frac{dQ_0}{dz} \right)^2 (z) \right\rangle \\
&\quad - \left\langle \frac{dQ_0}{dz}(z) \right\rangle^2 \Bigg) + w^{-1} \langle W_0(z) \rangle \Bigg] \partial_\mu \phi'(x) \partial^\mu \phi'(x) \Bigg).
\end{aligned}
\tag{11.107}
$$

We write this as

$$
Z_1 = \int \mathcal{D}\phi'(x) e^{-S_1(\phi'(x))}. \tag{11.108}
$$

The new action S_1 has the same form as the action S_0, viz

$$
S_1(\phi'(x)) = \frac{K}{2} \int d^d x R_1(\phi'(x)) \partial_\mu \phi'(x) \partial^\mu \phi'(x) + \int d^d x P_1(\phi'(x)). \tag{11.109}
$$

The new polynomials P_1 and R_1 (or equivalently Q_1 and W_1) are given in terms of the old ones P_0 and R_0 (or equivalently Q_0 and W_0) by the relations

$$
\begin{aligned}
W_1(2^{d/2}\alpha_0^{-1} z) &= R_1(\phi'(x)) \\
&= \frac{\alpha_0^2 C_d}{8} \Bigg(\left\langle \left(\frac{dQ_0}{dz} \right)^2 (z) \right\rangle - \left\langle \frac{dQ_0}{dz}(z) \right\rangle^2 \Bigg) + \frac{\alpha_0^2}{4} \langle W_0(z) \rangle
\end{aligned}
\tag{11.110}
$$

$$Q_1(2^{d/2}\alpha_0^{-1}z) = w^{-1}P_1(\phi'(x)) \\ = -2^d \ln \frac{I_0(z)}{I_0(0)}. \tag{11.111}$$

The constant C_d is given by

$$C_d = w\rho V. \tag{11.112}$$

Before we write down the recursion formulas we also introduce the notation

$$\phi^{(0)} = \phi,\ \phi^{(1)} = \phi'. \tag{11.113}$$

The above procedure can be repeated to integrate out the momentum shell $l = 1$ and get from S_1 to S_2. The modes with $l \geqslant 2$ will involve a new constant α_1 and instead of P_0, R_0, S_0 and I_0 we will have P_1, R_1, S_1 and I_1. Aside from this trivial relabeling, everything else will be the same including the constants w, V and C_d since $l = 1$ can be mapped to $l = 0$ due to the scaling $x_0 \longrightarrow 2x$ (see equations (11.59) and (11.61)). This whole process can be repeated an arbitrary number of times to get a renormalization group flow of the action given explicitly by the sequences $P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots \longrightarrow P_i$ and $R_0 \longrightarrow R_1 \longrightarrow R_2 \longrightarrow \cdots \longrightarrow R_i$. By assuming that $dR_i/d\phi^{(i)}$ is sufficiently small compared to P_i for all i the recursion formulas which relate the different operators at the renormalization group steps i and $i + 1$ are obviously given by

$$W_{i+1}(2^{d/2}\alpha_i^{-1}z) = R_{i+1}(\phi^{(i+1)}(x)) \\ = \frac{\alpha_i^2 C_d}{8}\left(\left\langle\left(\frac{dQ_i}{dz}\right)^2(z)\right\rangle - \left\langle\frac{dQ_i}{dz}(z)\right\rangle^2\right) + \frac{\alpha_i^2}{4}\langle W_i(z)\rangle \tag{11.114}$$

$$Q_{i+1}(2^{d/2}\alpha_i^{-1}z) = w^{-1}P_{i+1}(\phi^{(i+1)}(x)) \\ = -2^d \ln \frac{I_i(z)}{I_i(0)}. \tag{11.115}$$

The function $I_i(z)$ is given by the same formula (11.99) with the substitutions $I_0 \longrightarrow I_i$, $W_0 \longrightarrow W_i$ and $Q_0 \longrightarrow Q_i$.

The field $\phi^{(i+1)}$ and the variable z are related by $z = (K\rho/2w)^{1/2}2^{-d/2}\alpha_i\phi^{(i+1)}$. The full action at the renormalization group step i is

$$S_i(\phi^{(i)}(x)) = \frac{K}{2}\int d^dx R_i(\phi^{(i)}(x))\partial_\mu\phi^{(i)}(x)\partial^\mu\phi^{(i)}(x) + \int d^dx P_i(\phi^{(i)}(x)). \tag{11.116}$$

The constants α_i will be determined from the normalization condition

$$W_{i+1}(0) = 1. \tag{11.117}$$

Since Q is even this normalization condition is equivalent to

$$\frac{\alpha_i^2}{4}\langle W_i(z)\rangle|_{z=0} = 1. \tag{11.118}$$

The ultra local recursion formula
This corresponds to keeping in the expansion (11.65) only the first term. The resulting recursion formula is obtained from the above recursion formulas by dropping from equation (11.114) the fluctuation term

$$\left\langle \left(\frac{dQ_i}{dz}\right)^2(z)\right\rangle - \left\langle \frac{dQ_i}{dz}(z)\right\rangle^2. \tag{11.119}$$

The recursion formula (11.114) becomes

$$W_{i+1}(2^{d/2}\alpha_i^{-1}z) = \frac{\alpha_i^2}{4}\langle W_i(z)\rangle. \tag{11.120}$$

The solution of this equation together with the normalization condition (11.118) is given by

$$W_i = 1,\ \alpha_i = 2. \tag{11.121}$$

The remaining recursion formula is given by

$$Q_{i+1}(2^{d/2}2^{-1}z) = -2^d \ln \frac{I_i(z)}{I_i(0)}. \tag{11.122}$$

We state without proof that the use of this recursion formula is completely equivalent to the use in perturbation theory of the Polyakov–Wilson rules given by the approximations:

- We replace every internal propagator $1/(k^2 + r_0^2)$ by $1/(\Lambda^2 + r_0^2)$.
- We replace every momentum integral $\int_{\Lambda/2}^{\Lambda} d^dp/(2\pi)^d$ by the volume $c/4$ where $c = 4\Omega_{d-1}\Lambda^d(1 - 2^{-d})/(d(2\pi)^d)$.

11.2.3 The Wilson–Fisher fixed point within this scheme

Let us start with a ϕ^4 action given by

$$S_0[\phi_0] = \int d^dx\left(\frac{1}{2}(\partial_\mu\phi_0)^2 + \frac{1}{2}r_0\phi_0^2 + u_0\phi_0^4\right). \tag{11.123}$$

The Fourier transform of the field is given by

$$\phi_0(x) = \int_0^{\Lambda} \frac{d^dp}{(2\pi)^d}\tilde{\phi}_0(p)e^{ipx}. \tag{11.124}$$

We will decompose the field as

$$\phi_0(x) = \phi_1'(x) + \Phi(x). \tag{11.125}$$

The background field $\phi_1'(x)$ corresponds to the low frequency modes $\tilde{\phi}_0(p)$ where $0 \leqslant p \leqslant \Lambda/2$, whereas the fluctuation field $\Phi(x)$ corresponds to high frequency modes $\tilde{\phi}_0(p)$ where $\Lambda/2 < p \leqslant \Lambda$, viz

$$\phi_1'(x) = \int_0^{\Lambda/2} \frac{d^d p}{(2\pi)^d} \tilde{\phi}_0(p) e^{ipx}, \; \Phi(x) = \int_{\Lambda/2}^{\Lambda} \frac{d^d p}{(2\pi)^d} \tilde{\phi}_0(p) e^{ipx}. \tag{11.126}$$

The goal is integrate out the high frequency modes from the partition function. The partition function is given by

$$\begin{aligned} Z_0 &= \int \mathcal{D}\phi_0 e^{-S_0[\phi_0]} \\ &= \int \mathcal{D}\phi_1' \int \mathcal{D}\Phi e^{-S_0[\phi_1'+\Phi]} \\ &= \int \mathcal{D}\phi_1' e^{-S_1[\phi_1']}. \end{aligned} \tag{11.127}$$

The first goal is to determine the action $S_1[\phi_1']$. We have

$$\begin{aligned} e^{-S_1[\phi_1']} &= \int d\Phi e^{-S_0[\phi_1'+\Phi]} \\ &= e^{-S_0[\phi_1']} \int d\Phi e^{-S_0[\Phi]} e^{-u_0 \int d^d x [4\Phi^3 \phi_1' + 4\Phi \phi_1'^3 + 6\Phi^2 \phi_1'^2]}. \end{aligned} \tag{11.128}$$

We will expand in the field ϕ_1' up to the fourth power. We define expectation values with respect to the partition function

$$Z = \int d\Phi e^{-S_0[\Phi]}. \tag{11.129}$$

Let us also introduce

$$\begin{aligned} V_1 &= -4u_0 \int d^d x \Phi^3 \phi_1' \\ V_2 &= -6u_0 \int d^d x \Phi^2 \phi_1'^2 \\ V_3 &= -4u_0 \int d^d x \Phi \phi_1'^3. \end{aligned} \tag{11.130}$$

Then we compute

$$e^{-S_1[\phi_1']} = Ze^{-S_0[\phi_1']}\left\langle\left[1 + V_1 + V_2 + V_3 + \frac{1}{2}\left(V_1^2 + 2V_1V_2 + V_2^2 + 2V_1V_3\right) + \frac{1}{6}(V_1^3 + 3V_1^2V_2) + \frac{1}{24}V_1^4\right]\right\rangle. \tag{11.131}$$

By using the symmetry $\Phi \longrightarrow -\Phi$ we obtain

$$e^{-S_1[\phi_1']} = Ze^{-S_0[\phi_1']} \times \left\langle\left[1 + V_2 + \frac{1}{2}\left(V_1^2 + V_2^2 + 2V_1V_3\right) + \frac{1}{6}\left(3V_1^2V_2\right) + \frac{1}{24}V_1^4\right]\right\rangle. \tag{11.132}$$

The term $\langle V_1V_3\rangle$ vanishes by momentum conservation. We rewrite the different expectation values in terms of connected functions. We have

$$\begin{aligned}
\langle V_2\rangle &= \langle V_2\rangle_{\text{co}}\\
\left\langle V_1^2\right\rangle &= \left\langle V_1^2\right\rangle_{\text{co}}\\
\left\langle V_2^2\right\rangle &= \left\langle V_2^2\right\rangle_{\text{co}} + \langle V_2\rangle_{\text{co}}^2\\
\left\langle V_1^2V_2\right\rangle &= \left\langle V_1^2V_2\right\rangle_{\text{co}} + \left\langle V_1^2\right\rangle_{\text{co}}\langle V_2\rangle_{\text{co}}\\
\left\langle V_1^4\right\rangle &= \left\langle V_1^4\right\rangle_{\text{co}} + 3\left\langle V_1^2\right\rangle_{\text{co}}^2.
\end{aligned} \tag{11.133}$$

By using these results the partition function becomes

$$e^{-S_1[\phi_1']} = Ze^{-S_0[\phi_1']}e^{\langle V_2\rangle_{\text{co}} + \frac{1}{2}(\langle V_1^2\rangle_{\text{co}} + \langle V_2^2\rangle_{\text{co}}) + \frac{1}{2}\langle V_1^2V_2\rangle_{\text{co}} + \frac{1}{24}\langle V_1^4\rangle_{\text{co}}}. \tag{11.134}$$

In other words the partition function is expressible only in terms of irreducible connected functions. This is sometimes known as the cumulant expansion. The action $S_1[\phi_1']$ is given by

$$S_1\left[\phi_1'\right] = S_0\left[\phi_1'\right] - \langle V_2\rangle_{\text{co}} - \frac{1}{2}\left(\left\langle V_1^2\right\rangle_{\text{co}} + \left\langle V_2^2\right\rangle_{\text{co}}\right) - \frac{1}{2}\left\langle V_1^2V_2\right\rangle_{\text{co}} - \frac{1}{24}\left\langle V_1^4\right\rangle_{\text{co}}. \tag{11.135}$$

We need the propagator

$$\langle\Phi(x)\Phi(y)\rangle_{\text{co}} = \langle\Phi(x)\Phi(y)\rangle_0 - 12u_0 \times \int d^dz\langle\Phi(x)\Phi(z)\rangle_0\langle\Phi(y)\Phi(z)\rangle_0\langle\Phi(z)\Phi(z)\rangle_0 + O\left(u_0^2\right). \tag{11.136}$$

The free propagator is obviously given by

$$\langle \Phi(x)\Phi(y)\rangle_0 = \int_{\Lambda/2}^{\Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + r_0} e^{ip(x-y)}. \tag{11.137}$$

Thus

$$\begin{aligned} \langle \Phi(x)\Phi(y)\rangle_{\rm co} = & \left[\int \frac{d^d p_1}{(2\pi)^d} \frac{1}{p_1^{\,2} + r_0} - 12u_0 \right. \\ & \left. \times \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{1}{\left(p_1^{\,2} + r_0\right)^2 \left(p_2^{\,2} + r_0\right)} + O\left(u_0^2\right) \right] e^{ip_1(x-y)}. \end{aligned} \tag{11.138}$$

We can now compute

$$\begin{aligned} -\langle V_2\rangle_{\rm co} &= 6u_0 \int d^d x \phi_1'^2(x) \langle \Phi^2(x)\rangle_{\rm co} \\ &= 6u_0 \int \frac{d^d p}{(2\pi)^d} \left|\tilde{\phi}_1'(p)\right|^2 \left[\int \frac{d^d p_1}{(2\pi)^d} \frac{1}{p_1^{\,2} + r_0} \right. \\ &\quad \left. - 12u_0 \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{1}{\left(p_1^{\,2} + r_0\right)^2 \left(p_2^{\,2} + r_0\right)} + O\left(u_0^2\right) \right] \end{aligned} \tag{11.139}$$

$$\begin{aligned} -\frac{1}{2}\langle V_1^2\rangle_{\rm co} &= -8u_0^2 \int d^d x_1 d^d x_2 \phi_1'(x_1)\phi_1'(x_2) \langle \Phi^3(x_1)\Phi^3(x_2)\rangle_{\rm co} \\ &= -8u_0^2 \int d^d x_1 d^d x_2 \phi_1'(x_1)\phi_1'(x_2) \\ &\quad \times [6\langle \Phi(x_1)\Phi(x_2)\rangle_0^3 + 3\langle \Phi(x_1)\Phi(x_2)\rangle_0 \\ &\quad \times \langle \Phi(x_1)\Phi(x_1)\rangle_0 \langle \Phi(x_2)\Phi(x_2)\rangle_0] \\ &= -8u_0^2 \int \frac{d^d p}{(2\pi)^d} \left|\tilde{\phi}_1'(p)\right|^2 \left[6 \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \right. \\ &\quad \left. \times \frac{1}{\left(p_1^{\,2} + r_0\right)\left(p_2^{\,2} + r_0\right)((p + p_1 + p_2)^2 + r_0)} + O(u_0) \right]. \end{aligned} \tag{11.140}$$

The second term of the second equality of the above equation did not contribute because of momentum conservation. Next we compute

$$\begin{aligned}
-\frac{1}{2}\langle V_2^2\rangle_{\rm co} &= -18u_0^2\int d^dx_1 d^dx_2 \phi'^2_1(x_1)\phi'^2_1(x_2)\langle\Phi^2(x_1)\Phi^2(x_2)\rangle_{\rm co}\\
&= -36u_0^2\int d^dx_1 d^dx_2 \phi'^2_1(x_1)\phi'^2_1(x_2)\langle\Phi(x_1)\Phi(x_2)\rangle_0^2\\
&= -12u_0^2\int \frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_3}{(2\pi)^d}\tilde\phi_1'(p_1)\cdots\\
&\quad\times\tilde\phi_1'(p_3)\tilde\phi_1'(-p_1-p_2-p_3)\Big[\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r_0}\\
&\quad\times\frac{1}{(k+p_1+p_2)^2+r_0}+2\text{ permutations}+O(u_0)\Big].
\end{aligned}\tag{11.141}$$

The last two terms of the cumulant expansion are of order u_0^3 and u_0^4, respectively, which we are not computing. The action $S_1[\phi_1']$ reads then explicitly

$$\begin{aligned}
S_1[\phi_1'] &= \frac{1}{2}\int_0^{\Lambda/2}\frac{d^dp}{(2\pi)^d}|\tilde\phi_1'(p)|^2\Big\{p^2+r_0+12u_0\int\frac{d^dk_1}{(2\pi)^d}\frac{1}{k_1^2+r_0}\\
&\quad-144u_0^2\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{(k_1^2+r_0)^2(k_2^2+r_0)}-96u_0^2\\
&\quad\times\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{(k_1^2+r_0)(k_2^2+r_0)((p+k_1+k_2)^2+r_0)}+O(u_0^3)\Big\}\\
&\quad+\int_0^{\Lambda/2}\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_3}{(2\pi)^d}\tilde\phi_1'(p_1)\cdots\\
&\quad\times\tilde\phi_1'(p_3)\tilde\phi_1'(-p_1-p_2-p_3)\Big[u_0-12u_0^2\Big[\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r_0}\\
&\quad\times\frac{1}{(k+p_1+p_2)^2+r_0}+2\text{ permutations}\Big]+O\big(u_0^3\big)\Big].
\end{aligned}\tag{11.142}$$

The Fourier mode $\tilde\phi_1'(p)$ is of course equal $\tilde\phi_0(p)$ for $0\leqslant p\leqslant\Lambda/2$ and 0 otherwise. We scale now the field as

$$\tilde\phi_1'(p)=\alpha_0\tilde\phi_1(2p).\tag{11.143}$$

The action becomes

$$
\begin{aligned}
S_1[\phi_1] = \frac{1}{2}\alpha_0^2 2^{-d}\int_0^\Lambda \frac{d^dp}{(2\pi)^d}|\tilde{\phi}_1(p)|^2\Bigg\{&\frac{p^2}{4}+r_0\\
&+12u_0\int\frac{d^dk_1}{(2\pi)^d}\frac{1}{k_1^{\,2}+r_0}-144u_0^2\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\\
&\times\frac{1}{\left(k_1^{\,2}+r_0\right)^2\left(k_2^{\,2}+r_0\right)}-96u_0^2\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\\
&\times\frac{1}{\left(k_1^{\,2}+r_0\right)\left(k_2^{\,2}+r_0\right)\left(\left(\frac{1}{2}p+k_1+k_2\right)^2+r_0\right)}+O\left(u_0^3\right)\Bigg\}\\
&+\alpha_0^4 2^{-3d}\int_0^\Lambda\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_3}{(2\pi)^d}\tilde{\phi}_1(p_1)\cdots\\
&\times\tilde{\phi}_1(p_3)\tilde{\phi}_1(-p_1-p_2-p_3)\Bigg[u_0-12u_0^2\Bigg[\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r_0}\\
&\times\frac{1}{\left(k+\frac{1}{2}p_1+\frac{1}{2}p_2\right)^2+r_0}+2\text{ permutations}\Bigg]+O\left(u_0^3\right)\Bigg].
\end{aligned}
\tag{11.144}
$$

In the above equation the internal momenta k_i are still unscaled in the interval $[\Lambda/2, \Lambda]$. The one-loop truncation of this result is given by

$$
\begin{aligned}
S_1[\phi_1] = \frac{1}{2}\alpha_0^2 2^{-d}\int_0^\Lambda \frac{d^dp}{(2\pi)^d}|\tilde{\phi}_1(p)|^2\Bigg\{&\frac{p^2}{4}+r_0\\
&+12u_0\int\frac{d^dk_1}{(2\pi)^d}\frac{1}{k_1^{\,2}+r_0}+O\left(u_0^2\right)\Bigg\}\\
&+\alpha_0^4 2^{-3d}\int_0^\Lambda\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_3}{(2\pi)^d}\tilde{\phi}_1(p_1)\cdots\\
&\times\tilde{\phi}_1(p_3)\tilde{\phi}_1(-p_1-p_2-p_3)\Bigg[u_0-12u_0^2\Bigg[\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r_0}\\
&\times\frac{1}{\left(k+\frac{1}{2}p_1+\frac{1}{2}p_2\right)^2+r_0}+2\text{ permutations}\Bigg]+O\left(u_0^3\right)\Bigg].
\end{aligned}
\tag{11.145}
$$

We bring the kinetic term to the canonical form by choosing α_0 as

$$\alpha_0 = 2^{1+d/2}. \tag{11.146}$$

Furthermore, we truncate the interaction term in the action by setting the external momenta to zero since we are only interested in the renormalization group flow of the operators present in the original action. We get then

$$\begin{aligned} S_1[\phi_1] = \frac{1}{2}\int_0^\Lambda \frac{d^d p}{(2\pi)^d}|\tilde{\phi}_1(p)|^2(p^2 + r_1) \\ + u_1 \int_0^\Lambda \frac{d^d p_1}{(2\pi)^d} \cdots \frac{d^d p_3}{(2\pi)^d}\tilde{\phi}_1(p_1) \ldots \tilde{\phi}_1(p_3)\tilde{\phi}_1(-p_1 - p_2 - p_3). \end{aligned} \tag{11.147}$$

The new mass parameter r_1 and the new coupling-constant u_1 are given by

$$r_1 = 4r_0 + 48u_0 \int \frac{d^d k_1}{(2\pi)^d}\frac{1}{k_1^2 + r_0} + O\left(u_0^2\right) \tag{11.148}$$

$$u_1 = 2^{4-d}\left[u_0 - 36u_0^2 \int \frac{d^d k}{(2\pi)^d}\frac{1}{(k^2 + r_0)^2} + O\left(u_0^3\right)\right]. \tag{11.149}$$

Now we employ the Wilson–Polyakov rules corresponding to the ultra local Wilson recursion formula (11.122) consisting of making the following approximations:

- We replace every internal propagator $1/(k^2 + r_0^2)$ by $1/(\Lambda^2 + r_0^2)$.
- We replace every momentum integral $\int_{\Lambda/2}^{\Lambda} d^d p/(2\pi)^d$ by the volume $c/4$ where $c = 4\Omega_{d-1}\Lambda^d(1 - 2^{-d})/(d(2\pi)^d)$.

The mass parameter r_1 and the coupling-constant u_1 become

$$r_1 = 4\left[r_0 + 3c\frac{u_0}{\Lambda^2 + r_0} + O\left(u_0^2\right)\right] \tag{11.150}$$

$$u_1 = 2^{4-d}\left[u_0 - 9c\frac{u_0^2}{(\Lambda^2 + r_0)^2} + O\left(u_0^3\right)\right]. \tag{11.151}$$

This is the result of our first renormalization group step. Since the action $S_1[\phi_1]$ is of the same form as the action $S_0[\phi_0]$ the renormalization group calculation can be repeated without any change to go from r_1 and u_1 to a new mass parameter r_2 and a new coupling-constant u_2. This whole process can evidently be iterated an arbitrary number of times to define a renormalization group flow $(r_0, u_0) \longrightarrow (r_1, u_1) \longrightarrow \cdots (r_l, u_l) \longrightarrow (r_{l+1}, u_{l+1}) \cdots$. The renormalization group recursion equations relating (r_{l+1}, u_{l+1}) to (r_l, u_l) are given precisely by the above equations, viz

$$r_{l+1} = 4\left[r_l + 3c\frac{u_l}{\Lambda^2 + r_l}\right] \tag{11.152}$$

$$u_{l+1} = 2^{4-d}\left[u_l - 9c\frac{u_l^2}{(\Lambda^2 + r_l)^2}\right]. \tag{11.153}$$

The fixed points of the renormalization group equations are obviously defined by

$$r_* = 4\left[r_* + 3c\frac{u_*}{\Lambda^2 + r_*}\right] \tag{11.154}$$

$$u_* = 2^{4-d}\left[u_* - 9c\frac{u_*^2}{(\Lambda^2 + r_*)^2}\right]. \tag{11.155}$$

We find the solutions

$$\text{Gaussian fixed point} : r_* = 0,\ u_* = 0, \tag{11.156}$$

and (by assuming that u_* is sufficiently small)

$$\text{Wilson–Fisher fixed point} : r_* = -\frac{4cu_*}{\Lambda^2},\ u_* = \frac{\Lambda^4}{9c}(1 - 2^{d-4}). \tag{11.157}$$

For $\epsilon = 4 - d$ small the non-trivial (interacting) Wilson–Fisher fixed point approaches the trivial (free) Gaussian fixed point as

$$r_* = -\frac{4}{9}\Lambda^2\epsilon \ln 2,\ u_* = \frac{\Lambda^4}{9c}\epsilon \ln 2. \tag{11.158}$$

The value u_* controls the strength of the interaction of the low energy (infrared) physics of the system.

11.2.4 The critical exponents ν

In the Gaussian model the recursion formula reads simply $r_{l+1} = 4r_l$ and hence we have two possible solutions. At $T = T_c$ the mass parameter r_0 must be zero and hence $r_l = 0$ for all l, i.e. $r_0 = 0$ is a fixed point. For $T \neq T_c$ the mass parameter r_0 is non-zero and hence $r_l = 4^l r_0 \longrightarrow \infty$ for $l \longrightarrow \infty$ ($r_0 = \infty$ is the second fixed point). For T near T_c the mass parameter r_0 is linear in $T - T_c$.

In the ϕ^4 model the situation is naturally more complicated. We can be at the critical temperature $T = T_c$ without having the parameters r_0 and u_0 at their fixed point values. Indeed, as we have already seen, for any value u_0 there will be a critical value $r_{0c} = r_{0c}(u_0)$ of r_0 corresponding to $T = T_c$. At $T = T_c$ we have $r_l \longrightarrow r_*$ and $u_l \longrightarrow u_*$ for $l \longrightarrow \infty$. For $T \neq T_c$ we will have, in general, a different limit for large l.

The critical exponent ν can be calculated by studying the behavior of the theory only for T near T_c. As stated above $r_l(T_c) \longrightarrow r_*$ and $u_l(T_c) \longrightarrow u_*$ for $l \longrightarrow \infty$. From the analytic property of the recursion formulas we conclude that $r_l(T)$ and $u_l(T)$ are analytic functions of the temperature and hence near T_c we should have

$r_l(T) = r_l(T_c) + (T - T_c)r_l'(T_c) + \cdots$ and $u_l(T) = u_l(T_c) + (T - T_c)u_l'(T_c) + \cdots$ and as a consequence $r_l(T)$ and $u_l(T)$ are close to the fixed point values for sufficiently large l and sufficiently small $T - T_c$. We are thus led in a natural way to studying the recursion formulas only around the fixed point, i.e. to studying the linearized recursion formulas.

The linearized recursion formulas

Now we linearize the recursion formulas around the fixed point. We find without any approximation

$$\begin{aligned} r_{l+1} - r_* = &\left[4 - \frac{12cu_*}{(\Lambda^2 + r_l)(\Lambda^2 + r_*)}\right](r_l - r_*) \\ &+ \frac{12c}{\Lambda^2 + r_*}(u_l - u_*) + 12cu_*\left[\frac{1}{\Lambda^2 + r_l} - \frac{1}{\Lambda^2 + r_*}\right] \end{aligned} \tag{11.159}$$

$$\begin{aligned} u_{l+1} - u_* = &2^{4-d}\frac{9cu_*^2(2\Lambda^2 + r_* + r_l)}{(\Lambda^2 + r_l)^2(\Lambda^2 + r_*)^2}(r_l - r_*) \\ &+ 2^{4-d}\left[1 - \frac{9c}{(\Lambda^2 + r_l)^2}(u_l + u_*)\right](u_l - u_*). \end{aligned} \tag{11.160}$$

Keeping only linear terms we find

$$r_{l+1} - r_* = \left[4 - \frac{12cu_*}{(\Lambda^2 + r_*)^2}\right](r_l - r_*) + \frac{12c}{\Lambda^2 + r_*}(u_l - u_*) \tag{11.161}$$

$$u_{l+1} - u_* = 2^{4-d}\frac{18cu_*^2}{(\Lambda^2 + r_*)^3}(r_l - r_*) + 2^{4-d}\left[1 - \frac{18cu_*}{(\Lambda^2 + r_*)^2}\right](u_l - u_*). \tag{11.162}$$

This can be put into the matrix form

$$\begin{pmatrix} r_{l+1} - r_* \\ u_{l+1} - u_* \end{pmatrix} = M\begin{pmatrix} r_l - r_* \\ u_l - u_* \end{pmatrix}. \tag{11.163}$$

The matrix M is given by

$$\begin{aligned} M &= \begin{pmatrix} 4 - \dfrac{12cu_*}{(\Lambda^2 + r_*)^2} & \dfrac{12c}{\Lambda^2 + r_*} \\ 2^{4-d}\dfrac{18cu_*^2}{(\Lambda^2 + r_*)^3} & 2^{4-d}\left(1 - \dfrac{18cu_*}{(\Lambda^2 + r_*)^2}\right) \end{pmatrix} \\ &= \begin{pmatrix} 4 - \dfrac{4}{3}\epsilon \ln 2 & \dfrac{12c}{\Lambda^2}\left(1 + \dfrac{4}{9}\epsilon \ln 2\right) \\ 0 & 1 - \epsilon \ln 2 \end{pmatrix}. \end{aligned} \tag{11.164}$$

After n steps of the renormalization group we will have

$$\begin{pmatrix} r_{l+n} - r_* \\ u_{l+n} - u_* \end{pmatrix} = M^n \begin{pmatrix} r_l - r_* \\ u_l - u_* \end{pmatrix}. \tag{11.165}$$

In other words for large n the matrix M^n is completely dominated by the largest eigenvalue of M.

Let λ_1 and λ_2 be the eigenvalues of M with eigenvectors w_1 and w_2, respectively, such that $\lambda_1 > \lambda_2$. Clearly for $u_* = 0$ we have

$$\begin{aligned} \lambda_1 = 4,\ w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 = 1,\ w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{11.166}$$

The matrix M is not symmetric and thus diagonalization is achieved by an invertible (and not an orthogonal) matrix U. We write

$$M = UDU^{-1}. \tag{11.167}$$

The eigenvalues λ_1 and λ_2 can be determined from the trace and determinant which are given by

$$\lambda_1 + \lambda_2 = M_{11} + M_{22},\ \lambda_1\lambda_2 = M_{11}M_{22} - M_{12}M_{21}. \tag{11.168}$$

We obtain

$$\lambda_1 = 4 - \frac{4}{3}\epsilon \ln 2,\ \lambda_2 = 1 - \epsilon \ln 2. \tag{11.169}$$

The corresponding eigenvectors are

$$w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ w_2 = \begin{pmatrix} -\frac{4c}{\Lambda^2}\left(1 + \frac{5}{9}\epsilon \ln 2\right) \\ 1 \end{pmatrix}. \tag{11.170}$$

We write the equation $Mw_k = \lambda_k w_k$ as (with $(w_k)_j = w_{jk}$)

$$M_{ij}w_{jk} = \lambda_k w_{ik}. \tag{11.171}$$

The identity $M = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$ can be rewritten as

$$M_{ij} = \sum_k \lambda_k w_{ik} v_{kj} = \lambda_1 w_{i1} v_{1j} + \lambda_2 w_{i2} v_{2j}. \tag{11.172}$$

The vectors v_k are the eigenvectors of M^T with eigenvalues λ_k, respectively, viz (with $(v_k)_j = v_{kj}$)

$$M^T_{ij} v_{kj} = v_{kj} M_{ji} = \lambda_k v_{ki}. \tag{11.173}$$

We find explicitly

$$v_1 = \begin{pmatrix} 1 \\ \frac{4c}{\Lambda^2}\left(1 + \frac{5}{9}\epsilon \ln 2\right) \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{11.174}$$

The orthonormality condition is then

$$\sum_j v_{kj} w_{jl} = \delta_{kl}. \tag{11.175}$$

From the result (11.172) we deduce that

$$\begin{aligned} M_{ij}^n &= \lambda_1^{\,n} w_{i1} v_{1j} + \lambda_2^{\,n} w_{i2} v_{2j} \\ &\simeq \lambda_1^{\,n} w_{i1} v_{1j}. \end{aligned} \tag{11.176}$$

The linearized recursion formulas then take the form

$$r_{l+n} - r_* \simeq \lambda_1^{\,n} w_{11}(v_{11}(r_l - r_*) + v_{12}(u_l - u_*)) \tag{11.177}$$

$$u_{l+n} - u_* \simeq \lambda_1^{\,n} w_{21}(v_{11}(r_l - r_*) + v_{12}(u_l - u_*)). \tag{11.178}$$

Since $r_l = r_l(T)$ and $u_l = u_l(T)$ are close to the fixed point values for sufficiently large l and sufficiently small $T - T_c$ we conclude that $r_l - r_*$ and $u_l - u_*$ are both linear in $T - T_c$ and as a consequence

$$v_{11}(r_l - r_*) + v_{12}(u_l - u_*) = c_l(T - T_c). \tag{11.179}$$

The linearized recursion formulas become

$$r_{l+n} - r_* \simeq c_l \lambda_1^{\,n} w_{11}(T - T_c) \tag{11.180}$$

$$u_{l+n} - u_* \simeq c_l \lambda_1^{\,n} w_{21}(T - T_c). \tag{11.181}$$

The critical exponent ν

The correlation length corresponding to the initial action is given by

$$\xi_0(T) = X(r_0(T), u_0(T)). \tag{11.182}$$

After $l + n$ renormalization group steps the correlation length becomes

$$\xi_{l+n}(T) = X(r_{l+n}(T), u_{l+n}(T)). \tag{11.183}$$

At each renormalization group step we scale the momenta as $p \longrightarrow 2p$ which corresponds to scaling the distances as $x \longrightarrow x/2$. The correlation length is a measure of distance and thus one must have

$$X(r_{l+n}, u_{l+n}) = 2^{-l-n} X(r_0, u_0). \tag{11.184}$$

From equations (11.180) and (11.181) we have

$$\begin{aligned}(r_{l+n+1} - r_*)|_{T-T_c=\tau/\lambda_1} &= (r_{l+n} - r_*)|_{T-T_c=\tau}, \\ (u_{l+n+1} - u_*)|_{T-T_c=\tau/\lambda_1} &= (u_{l+n} - u_*)|_{T-T_c=\tau}.\end{aligned} \tag{11.185}$$

Hence

$$X(r_{l+n+1}, u_{l+n+1})|_{T=T_c+\tau/\lambda_1} = X(r_{l+n}, u_{l+n})|_{T=T_c+\tau}. \tag{11.186}$$

By using the two results (11.184) and (11.186) we obtain

$$2^{-l-n-1}\xi_0(T_c + \tau/\lambda_1) = 2^{-l-n}\xi_0(T_c + \tau). \tag{11.187}$$

We expect

$$\xi_0(T_c + \tau) \propto \tau^{-\nu}. \tag{11.188}$$

In other words

$$\frac{1}{2}\left(\frac{\tau}{\lambda_1}\right)^{-\nu} = \tau^{-\nu} \Leftrightarrow \lambda_1^{\nu} = 2 \Leftrightarrow \nu = \frac{\ln 2}{\ln \lambda_1}. \tag{11.189}$$

11.2.5 The critical exponent η

The ultra local recursion formula (11.122) used so far does not lead to a wavefunction renormalization since all momentum dependence of Feynman diagrams has been dropped and as a consequence the value of the anomalous dimension η within this approximation is 0. This can also be seen from the field scaling (11.143) with the choice (11.146) which are made at every renormalization group step and hence the wavefunction renormalization is independent of the momentum.

In any case, we can see from equation (11.145) that the wavefunction renormalization at the first renormalization group step is given by

$$Z = \frac{\alpha_0^2}{2^{2+d}}. \tag{11.190}$$

On the other hand, we have already established that the scaling behavior of $Z(\lambda)$ for small λ (the limit in which we approach the infrared stable fixed point) is λ^{η}. In our case $\lambda = 1/2$ and hence we must have

$$Z = 2^{-\eta}. \tag{11.191}$$

Let α_* be the fixed value of the sequence α_i. Then from the above two equations we obtain the formula

$$\eta = -\frac{\ln Z}{\ln 2} = d + 2 - \frac{2\ln \alpha_*}{\ln 2}. \tag{11.192}$$

As discussed above, since $\alpha_* = 2^{1+d/2}$ for the ultra local recursion formula (11.122) we get $\eta = 0$.

To incorporate a non-zero value of the critical exponent η we must go to the more accurate, yet more complicated, recursion formulas (11.114) and (11.115). The field scaling at each renormalization group step is a different number α_i. These numbers are determined from the normalization condition (11.118).

Recall that integrating out the momenta $1 \leqslant |k|/\Lambda \leqslant 2$ resulted in the field $\phi_1(x) = \sum_{\vec{m}} \sum_{l=1}^{\infty} \psi_{\vec{m}l}(x) \phi_{\vec{m}l}$ which was expressed in terms of the field $\phi'(x) = \phi^{(1)}$ which appears in the final action as $\phi_1(x) = 2^{-d/2} \alpha_0 \phi'(x/2)$. After n renormalization group steps we integrate out the momenta $2^{1-n} \leqslant |k|/\Lambda \leqslant 2$ which results in the field $\phi_n(x) = \sum_{\vec{m}} \sum_{l=n}^{\infty} \psi_{\vec{m}l}(x) \phi_{\vec{m}l}$. However, the action will be expressed in terms of the field $\phi' = \phi^{(n)}$ defined by

$$\phi_n(x) = 2^{-nd/2} \alpha_0 \alpha_1 \ \ldots \ \alpha_{n-1} \phi^{(n)}(x/2^n). \tag{11.193}$$

We are interested in the 2-point function

$$\langle \phi_{\vec{m}l} \phi_{\vec{m}'l'} \rangle = \frac{1}{Z} \int \prod_{l_1=0}^{\infty} \prod_{\vec{m}_1} d\phi_{\vec{m}_1 l_1} \phi_{\vec{m}l} \phi_{\vec{m}'l'} e^{-S_0[\phi]}. \tag{11.194}$$

Let us concentrate on the integral with $l_1 = l$ and $\vec{m}_1 = \vec{m}$ and assume that $l' > l$. We have then the integral

$$\ldots \int d\phi_{\vec{m}l} \phi_{\vec{m}l} \int \prod_{l_1=0}^{l-1} \prod_{\vec{m}_1} d\phi_{\vec{m}_1 l_1} e^{-S_0[\phi]} = \ \ldots \int d\phi_{\vec{m}l} \phi_{\vec{m}l} \ e^{-S_l[\phi']}. \tag{11.195}$$

We have $\phi' = \phi^{(l)}$ where $\phi^{(l)}$ contains the momenta $|k|/\Lambda \leqslant 1/2^{l-1}$. Since $\phi_{\vec{m}l}$ is not integrated we have $\phi_{\vec{m}l} = \alpha_0 \ \ldots \ \alpha_{l-1} (K\rho/2)^{-1/2} y_{\vec{m}}$ which is the generalization of $\phi_{\vec{m}l} = \alpha_0 \phi'_{\vec{m}l-1}$. We want now to further integrate $\phi_{\vec{m}l}$. The final result is similar to equation (11.95) except that we have an extra factor of $y_{\vec{m}}$ and $z_{\vec{m}}$ contains all the modes with $l_1 > l$. The integral thus clearly vanishes because it is odd under $y_{\vec{m}} \longrightarrow -\vec{y}_{\vec{m}}$.

We conclude that we must have $l' = l$ and $\vec{m}' = \vec{m}$ otherwise the above 2-point function vanishes. After a few more calculations we obtain

$$\begin{aligned} \langle \phi_{\vec{m}l} \phi_{\vec{m}'l'} \rangle = \delta_{ll'} \delta_{\vec{m}\vec{m}'} \alpha_0^2 \ \ldots \ \alpha_{l-1}^2 \left(\frac{K\rho}{2} \right)^{-1} \\ \times \frac{\displaystyle\int \prod_{l_1=l+1} \prod_{\vec{m}_1} d\phi_{\vec{m}_1 l_1} \prod_{\vec{m}_1} M_l(z_{\vec{m}_1}) \cdot R_l(z_{\vec{m}})}{\displaystyle\int \prod_{l_1=l+1} \prod_{\vec{m}_1} d\phi_{\vec{m}_1 l_1} \prod_{\vec{m}_1} M_l(z_{\vec{m}_1})}. \end{aligned} \tag{11.196}$$

The function M_l is given by the same formula (11.96) with the substitutions $M_0 \longrightarrow M_l$, $W_0 \longrightarrow W_l$ and $Q_0 \longrightarrow Q_l$. The variable $z_{\vec{m}}$ is given explicitly by

$$z_{\vec{m}} = \left(\frac{K\rho}{2w}\right)^{1/2} 2^{-d/2}\alpha_l \phi^{(l+1)}(x_0/2). \tag{11.197}$$

The function $R_l(z)$ is defined by

$$R_l(z) = M_l^{-1}(z) \int dy y^2 \exp(\ \dots\ ). \tag{11.198}$$

The exponent is given by the same exponent of equation (11.96) with the substitutions $M_0 \longrightarrow M_l$, $W_0 \longrightarrow W_l$ and $Q_0 \longrightarrow Q_l$.

An order of magnitude formula for the 2-point function can be obtained by replacing the function $R_l(z)$ by $R_l(0)$. We obtain then

$$\langle \phi_{\vec{m}l} \phi_{\vec{m}'l'} \rangle = \delta_{ll'}\delta_{\vec{m}\vec{m}'}\alpha_0^2 \ \dots\ \alpha_{l-1}^2 \left(\frac{K\rho}{2}\right)^{-1} R_l(0). \tag{11.199}$$

At a fixed point of the recursion formulas we must have

$$W_l \longrightarrow W_*,\ Q_l \longrightarrow Q_* \Leftrightarrow R_l \longrightarrow R_*, \tag{11.200}$$

and

$$\alpha_l \longrightarrow \alpha_*. \tag{11.201}$$

The 2-point function is, therefore, given by

$$\langle \phi_{\vec{m}l} \phi_{\vec{m}'l'} \rangle \propto \delta_{ll'}\delta_{\vec{m}\vec{m}'}\alpha_*^{2l} \left(\frac{K\rho}{2}\right)^{-1} R_*(0). \tag{11.202}$$

The modes $\langle \phi_{\vec{m}l} \rangle$ correspond to the momentum shell $2^{-l} \leqslant |k|/\Lambda \leqslant 2^{1-l}$, i.e. $k \sim \Lambda 2^{-l}$. On the other hand the 2-point function is expected to behave as

$$\langle \phi_{\vec{m}l} \phi_{\vec{m}'l'} \rangle \propto \delta_{ll'}\delta_{\vec{m}\vec{m}'} \frac{1}{k^{2-\eta}}, \tag{11.203}$$

where η is precisely the anomalous dimension. By substituting $k \sim \Lambda 2^{-l}$ in this last formula we obtain

$$\langle \phi_{\vec{m}l} \phi_{\vec{m}'l'} \rangle \propto \delta_{ll'}\delta_{\vec{m}\vec{m}'} \Lambda^{\eta-2} 2^{l(2-\eta)}. \tag{11.204}$$

By comparing the l-dependent bits in equations (11.202) and (11.204) we find that the anomalous dimension is given by

$$2^{2-\eta} = \alpha_*^2 \Rightarrow \eta = 2 - \frac{2\ln \alpha_*}{\ln 2}. \tag{11.205}$$

11.3 Generating functionals

In this section we will re-derive the various generating functionals of quantum field theory following the presentation and notation of [4].

11.3.1 The generating functional $\mathcal{G}[J]$

For simplicity, we will only deal with a bosonic real field ϕ. We start from the action and the path integral

$$S[\phi] = S_0[\phi] + S_1[\phi] \tag{11.206}$$

$$Z = \int d\phi \exp(-S[\phi]). \tag{11.207}$$

$S_1[\phi]$ is the interaction part while $S_0[\phi]$ is the free Gaussian part which is of the form

$$S_0[\phi] = -\frac{1}{2}(\phi,\, G_0^{-1}\phi) = -\frac{1}{2}\int_\alpha \int_\beta \phi_\alpha (G_0^{-1})_{\alpha\beta}\phi_\beta. \tag{11.208}$$

Obviously, $\int_\alpha$ denotes integration over the continuous components and summation over the discrete components of the label α, while the Gaussian propagator G_0 is a real symmetric matrix in label space. We will also use the free path integral

$$Z_0 = \int d\phi \exp(-S_0[\phi]). \tag{11.209}$$

The n-point Green's functions are defined by

$$G^{(n)}_{\alpha_1\ldots\alpha_n} = \langle \phi_{\alpha_n} \ldots \phi_{\alpha_1}\rangle = \frac{\int d\phi \exp(-S[\phi])\phi_{\alpha_n}\ldots\phi_{\alpha_1}}{\int d\phi \exp(-S[\phi])}. \tag{11.210}$$

The corresponding generating functional is given in terms of a current J by the formula

$$\mathcal{G}[J] = \frac{\int d\phi \exp(-S[\phi] + (J,\, \phi))}{\int d\phi \exp(-S[\phi])}. \tag{11.211}$$

Indeed, we compute

$$G^{(n)}_{\alpha_1\ldots\alpha_n} = \frac{\delta^n \mathcal{G}[J]}{\delta J_{\alpha_n}\ldots\delta J_{\alpha_1}}|_{J=0}. \tag{11.212}$$

Or equivalently

$$\mathcal{G}[J] = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\alpha_1}\cdots\int_{\alpha_n} G^{(n)}_{\alpha_1\ldots\alpha_n} J_{\alpha_1}\ldots J_{\alpha_n}. \tag{11.213}$$

11.3.2 The generating functional $\mathcal{G}_c[J]$

The generating functional for connected Green's functions is defined by

$$\mathcal{G}_c[J] = \ln\frac{Z}{Z_0}\mathcal{G}[J],\ \mathcal{G}[J] = \frac{Z_0}{Z}\exp(\mathcal{G}_c[J]). \tag{11.214}$$

Equivalently

$$\mathcal{G}_c[J] = \ln\left(\frac{\int d\phi \exp(-S[\phi] + (J,\,\phi))}{\int d\phi \exp(-S_0[\phi])}\right). \tag{11.215}$$

The connected n-point Green's functions are then given by

$$G^{(n)}_{c\alpha_1\ldots\alpha_n} = \frac{\delta^n \mathcal{G}_c[J]}{\delta J_{\alpha_n}\ldots\delta J_{\alpha_1}}|_{J=0}. \tag{11.216}$$

Or equivalently

$$\mathcal{G}_c[J] = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\alpha_1}\cdots\int_{\alpha_n} G^{(n)}_{c\alpha_1\ldots\alpha_n} J_{\alpha_1}\ldots J_{\alpha_n}. \tag{11.217}$$

Using the language of probability theory, the full Green's functions $G^{(n)}_{\alpha_1\ldots\alpha_n}$ play the role of moments of the probability distribution defined by the Euclidean action, whereas the connected Green's functions $G^{(n)}_{c,\alpha_1\ldots\alpha_n}$ play the role of cumulants.

We can easily check that

$$\mathcal{G}_{0,c}[J] = \ln\left(\frac{\int d\phi \exp(-S_0[\phi] + (J,\,\phi))}{\int d\phi \exp(-S_0[\phi])}\right) = -\frac{1}{2}(J,\,G_0 J). \tag{11.218}$$

Furthermore, we compute (with the substitution $\phi \longrightarrow \delta/\delta J$)

$$\begin{aligned}
\exp(\mathcal{G}_c[J]) &= \frac{Z}{Z_0}\mathcal{G}[J] \\
&= \frac{\int d\phi \exp(-S_0[\phi] - S_{\rm I}[\phi] + (J,\,\phi))}{\int d\phi \exp(-S_0[\phi])} \\
&= \exp\left(-S_{\rm I}\left[\frac{\delta}{\delta J}\right]\right)\frac{\int d\phi \exp(-S_0[\phi] + (J,\,\phi))}{\int d\phi \exp(-S_0[\phi])} \\
&= \exp\left(-S_{\rm I}\left[\frac{\delta}{\delta J}\right]\right)\exp(\mathcal{G}_{0,c}[J]) \\
&= \exp\left(-S_{\rm I}\left[\frac{\delta}{\delta J}\right]\right)\exp\left(-\frac{1}{2}(J,\,G_0 J)\right).
\end{aligned} \tag{11.219}$$

Thus if $G_0 \longrightarrow 0$, which will be our initial condition for the functional renormalization equation, we have

$$\mathcal{G}_c[J] \longrightarrow 0. \tag{11.220}$$

As it turns out, rewritten in this form, this is not a good initial condition.

11.3.3 The generating functional $\mathcal{G}_{ac}[\bar{\phi}]$

The generating functional for the amputated connected Green's functions is a functional depending on an auxiliary field $\bar{\phi}$ given by the formula

$$\exp(\mathcal{G}_{ac}[\bar{\phi}]) = \frac{\int d\phi \exp(-S_0[\phi] - S_1[\phi + \bar{\phi}])}{Z_0}. \tag{11.221}$$

As before, we define the amputated connected n-point Green's functions by

$$G^{(n)}_{ac\alpha_1\ldots\alpha_n} = \frac{\delta^n \mathcal{G}_{ac}[\bar{\phi}]}{\delta\bar{\phi}_{\alpha_n}\ldots\delta\bar{\phi}_{\alpha_1}}|_{\bar{\phi}=0}. \tag{11.222}$$

Or equivalently

$$\mathcal{G}_{ac}[\bar{\phi}] = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\alpha_1}\cdots\int_{\alpha_n} G^{(n)}_{ac\alpha_1\ldots\alpha_n}\bar{\phi}_{\alpha_1}\ldots\bar{\phi}_{\alpha_n}. \tag{11.223}$$

In the above path integral (11.221) we change the variable as $\phi' = \phi + \bar{\phi}$, use the property $G_0^T = G_0$ and we choose

$$\bar{\phi} = -G_0 J. \tag{11.224}$$

We obtain

$$\begin{aligned}\exp(\mathcal{G}_{ac}[\bar{\phi}]) &= \exp\left(\frac{1}{2}(J,\, G_0 J)\right)\frac{\int d\phi' \exp(-S_0[\phi'] - S_1[\phi'] + (J,\, \phi'))}{Z_0}\\ &= \exp\left(\frac{1}{2}(J,\, G_0 J)\right)\exp(\mathcal{G}_c[J]).\end{aligned} \tag{11.225}$$

We deduce

$$\mathcal{G}_c[J] = \mathcal{G}_{ac}[\bar{\phi} = -G_0 J] - \frac{1}{2}(J,\, G_0 J) \tag{11.226}$$

$$\mathcal{G}_{ac}[\bar{\phi}] = \mathcal{G}_c[J = -G_0^{-1}\bar{\phi}] + \frac{1}{2}(\bar{\phi},\, G_0^{-1}\bar{\phi}). \tag{11.227}$$

We can easily check that

$$\mathcal{G}_{0,c}[J] = -\frac{1}{2}(J,\, G_0 J) = -\frac{1}{2}(\bar{\phi},\, G_0^{-1}\bar{\phi}). \tag{11.228}$$

Thus,

$$\mathcal{G}_{ac}[\bar{\phi}] = (\mathcal{G}_c[J] - \mathcal{G}_{0,c}[J])|_{J=-G_0^{-1}\bar{\phi}}. \tag{11.229}$$

Thus, for a free field theory the generating functional of the amputated connected Green's functions vanishes identically.

Further, we compute

$$\begin{aligned}
\exp(\mathcal{G}_{ac}[\bar{\phi}]) &= \frac{\int d\phi \exp(-S_0[\phi])\exp(-S_1[\phi + \bar{\phi}])}{Z_0} \\
&= \frac{\int d\phi \exp(-S_0[\phi])\exp\left(-S_1\left[\frac{\delta}{\delta J}\right]\right)\exp((J, \phi + \bar{\phi}))|_{J=0}}{Z_0} \\
&= \exp\left(-S_1\left[\frac{\delta}{\delta J}\right]\right) \\
&= \frac{\int d\phi \exp\left(\frac{1}{2}(\phi, G_0^{-1}\phi)\right)\exp((J, \phi))\exp((J, \bar{\phi}))|_{J=0}}{Z_0} \\
&= \exp\left(-S_1\left[\frac{\delta}{\delta J}\right]\right)\exp(G_{0,c}[J])\exp((J, \bar{\phi}))|_{J=0} \\
&= \exp\left(-S_1\left[\frac{\delta}{\delta J}\right]\right)\exp\left(-\frac{1}{2}\left(\frac{\delta}{\delta\bar{\phi}}, G_0\frac{\delta}{\delta\bar{\phi}}\right)\right)\exp((J, \bar{\phi}))|_{J=0} \\
&= \exp\left(-\frac{1}{2}\left(\frac{\delta}{\delta\bar{\phi}}, G_0\frac{\delta}{\delta\bar{\phi}}\right)\right)\exp(-S_1[\bar{\phi}]).
\end{aligned} \tag{11.230}$$

Thus, the would-be initial condition $G_0 \longrightarrow 0$ becomes in terms of $\mathcal{G}_{ac}[\bar{\phi}]$ given by the good starting prescription

$$\mathcal{G}_{ac}[\bar{\phi}] \longrightarrow -S_1[\bar{\phi}],\ G_0 \longrightarrow 0. \tag{11.231}$$

11.3.4 The generating functional $\Gamma[\bar{\phi}']$

There are obviously some connected and amputated diagrams which can be separated into two parts by cutting a single line corresponding to a Gaussian propagator. These diagrams are called one-line reducible. Thus, we can decompose the n-point connected Green's functions $G^{(n)}_{c,\,\alpha_1\ldots\alpha_n}$ into irreducible one-point vertices $\Gamma^{(n)}_{\alpha_1\ldots\alpha_n}$ which by definition cannot be separated into two parts by cutting a single propagator. These vertices are generated from a generating functional $\Gamma[\bar{\phi}']$ known as the effective action, as follows

$$\Gamma^{(n)}_{\alpha_1\ldots\alpha_n} = \frac{\delta^n\Gamma[\bar{\phi}']}{\delta\bar{\phi}'_{\alpha_n}\ldots\delta\bar{\phi}'_{\alpha_1}}|_{\bar{\phi}'=0} \tag{11.232}$$

$$\Gamma[\bar{\phi}'] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha_1} \cdots \int_{\alpha_n} \Gamma^{(n)}_{\alpha_1 \ldots \alpha_n} \bar{\phi}'_{\alpha_1} \ldots \bar{\phi}'_{\alpha_n}. \tag{11.233}$$

The generating functional $\Gamma[\bar{\phi}']$ is the Legendre transform of the generating functional $G_c[J]$ given explicitly by

$$\Gamma[\bar{\phi}'] = \mathcal{L}[\bar{\phi}'] - S_0[\bar{\phi}'] \tag{11.234}$$

$$\mathcal{L}[\bar{\phi}'] = (\bar{\phi}', J) - \mathcal{G}_c[J] \tag{11.235}$$

$$\Rightarrow \Gamma[\bar{\phi}'] = (\bar{\phi}', J) - \mathcal{G}_c[J] - S_0[\bar{\phi}']. \tag{11.236}$$

The variable $\bar{\phi}'$ is the so-called classical field defined by

$$\bar{\phi}'_\alpha = \langle \phi_\alpha \rangle = \frac{\delta \mathcal{G}_c}{\delta J_\alpha}. \tag{11.237}$$

We note the equation of motion

$$\frac{\delta \mathcal{L}}{\delta \bar{\phi}'_\alpha} = J_\alpha. \tag{11.238}$$

By using this equation of motion and the chain rule again we obtain

$$\frac{\delta}{\delta \bar{\phi}'_\alpha} = \int_\beta \frac{\delta^2 \mathcal{L}}{\delta \bar{\phi}'_\alpha \delta \bar{\phi}'_\beta} \frac{\delta}{\delta J_\beta} \leftrightarrow \frac{\delta}{\delta \bar{\phi}} = \left[\frac{\delta}{\delta \bar{\phi}'} \otimes \frac{\delta}{\delta \bar{\phi}'} \mathcal{L} \right] \frac{\delta}{\delta J}. \tag{11.239}$$

By applying this operator equation to the definition of the classical field (11.237) we obtain the identity

$$1 = \left[\frac{\delta}{\delta \bar{\phi}'} \otimes \frac{\delta}{\delta \bar{\phi}'} \mathcal{L} \right] \left[\frac{\delta}{\delta J} \otimes \frac{\delta}{\delta J} \mathcal{G}_c \right]. \tag{11.240}$$

We can express the effective action $\Gamma[\bar{\phi}']$ in terms of the generating functional $\mathcal{G}_{ac}[\bar{\phi}']$ as follows. On one hand, we have

$$\Gamma[\bar{\phi}'] = (\bar{\phi}', J) - \mathcal{G}_c[J] - S_0[\bar{\phi}']. \tag{11.241}$$

On the other hand, we have

$$\mathcal{G}_{ac}[\bar{\phi}] = \mathcal{G}_c[J = -G_0^{-1}\bar{\phi}] + \frac{1}{2}(\bar{\phi}, G_0^{-1}\bar{\phi}). \tag{11.242}$$

We compute the classical field

$$\begin{aligned} \bar{\phi}'_\alpha &= \frac{\delta \mathcal{G}_c}{\delta J_\alpha} \\ &= \bar{\phi}_\alpha - G_{0,\alpha\beta} \frac{\delta \mathcal{G}_{ac}}{\delta \bar{\phi}_\beta}. \end{aligned} \tag{11.243}$$

By replacing with equation (11.242) in equation (11.241), and then using equation (11.243), we get

$$\begin{aligned}\Gamma[\bar{\phi}'] &= \left(\bar{\phi}',\, J\right) - \mathcal{G}_{ac}[\bar{\phi}] + \frac{1}{2}\left(\bar{\phi},\, G_0^{-1}\bar{\phi}\right) + \frac{1}{2}\left(\bar{\phi}',\, G_0^{-1}\bar{\phi}'\right) \\ &= \frac{1}{2}\left(\frac{\delta\mathcal{G}_{ac}}{\delta\bar{\phi}},\, G_0\frac{\delta\mathcal{G}_{ac}}{\delta\bar{\phi}}\right) - \mathcal{G}_{ac}[\bar{\phi}].\end{aligned} \tag{11.244}$$

Thus, the would-be initial condition $G_0 \longrightarrow 0$ becomes in terms of $\Gamma[\bar{\phi}]$ given by the good starting prescription

$$\Gamma[\bar{\phi}] \longrightarrow S_{\rm I}[\bar{\phi}],\; G_0 \longrightarrow 0. \tag{11.245}$$

11.3.5 The 2-point function and regularization

We want now to show explicitly that $\mathcal{G}_c[J]$ generates indeed connected Green's functions, and that $\mathcal{G}_{ac}[\bar{\phi}]$ generates the amputated connected Green's function, and that $\Gamma[\bar{\phi}']$ generates the irreducible vertices. For simplicity, we will show this for the case of the 2-point function.

The connected 2-point function $G^{(2)}_{c,\alpha_1\alpha_2}$ is given in terms of the full 2-point function $G^{(2)}_{\alpha_1\alpha_2}$ by

$$\begin{aligned}G^{(2)}_{c,\alpha_1\alpha_2} &= \langle\phi_{\alpha_2}\phi_{\alpha_1}\rangle - \langle\phi_{\alpha_2}\rangle\langle\phi_{\alpha_1}\rangle \\ &= G^{(2)}_{\alpha_1\alpha_2} - G^{(1)}_{\alpha_2}G^{(1)}_{\alpha_1}.\end{aligned} \tag{11.246}$$

Thus, the connected function is indeed equal to the full function minus the disconnected piece. For the free theory we obtain

$$\begin{aligned}G^{(2)}_{0,c\alpha_1\alpha_2} &= \langle\phi_{\alpha_2}\phi_{\alpha_1}\rangle_0 \\ &= G^{(2)}_{0,\alpha_1\alpha_2} \\ &= -G_{0,\alpha_1\alpha_2}.\end{aligned} \tag{11.247}$$

This last equation can be rewritten as

$$G_{0,\alpha_1\alpha_2} = -\frac{\delta^2\mathcal{G}_{0,c}[J]}{\delta J_{\alpha_2}\delta J_{\alpha_1}}|_{J=0} = -G^{(2)}_{0,c\alpha_1\alpha_2} \leftrightarrow G_0 = -\frac{\delta}{\delta J}\otimes\frac{\delta}{\delta J}\mathcal{G}_{0,c}[J]. \tag{11.248}$$

By analogy, the exact propagator will be defined by

$$G_{\alpha_1\alpha_2} = -\frac{\delta^2\mathcal{G}_c[J]}{\delta J_{\alpha_2}\delta J_{\alpha_1}}|_{J=0} = -G^{(2)}_{c\alpha_1\alpha_2} \leftrightarrow G = -\frac{\delta}{\delta J}\otimes\frac{\delta}{\delta J}\mathcal{G}_c[J]. \tag{11.249}$$

This satisfies the same symmetry property as G_0, i.e. $G^T = G$. We parametrize the difference between the exact propagator G and the free propagator G_0 by a self-energy Σ as

$$\begin{aligned} G^{-1} &= G_0^{-1} - \Sigma \Rightarrow G = G_0 + G_0 \Sigma G \Rightarrow G \\ &= G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + \cdots \end{aligned} \tag{11.250}$$

Next, by taking the second functional derivative of equation (11.227), we compute

$$\begin{aligned} \frac{\delta^2}{\delta \bar{\phi}_{\alpha_2} \delta \bar{\phi}_{\alpha_1}} \mathcal{G}_{ac} &= \left(G_0^{-1}\right)_{\beta_1 \alpha_1} \left(G_0^{-1}\right)_{\beta_2 \alpha_2} \frac{\delta^2 \mathcal{G}_c}{\delta J_{\beta_2} \delta J_{\beta_1}} + \left(G_0^{-1}\right)_{\alpha_1 \alpha_2} \\ &= -\left(G_0^{-1} G G_0^{-1}\right)_{\alpha_1 \alpha_2} + \left(G_0^{-1}\right)_{\alpha_1 \alpha_2}. \end{aligned} \tag{11.251}$$

Or equivalently

$$\begin{aligned} G_{ac}^{(2)} &= -G_0^{-1} G G_0^{-1} + G_0^{-1} \\ &= -\Sigma - \Sigma G_0 \Sigma - \Sigma G_0 \Sigma G_0 \Sigma - \cdots \end{aligned} \tag{11.252}$$

Alternatively, we write

$$G = G_0 - G_0 G_{ac}^{(2)} G_0. \tag{11.253}$$

This shows explicitly that $G_{ac}^{(2)}$ is the amputated 2-point function.

Now, by using equation (13.115) in equation (11.240) with $\bar{\phi} = \bar{\phi}'$, we obtain the formula

$$\begin{aligned} -G_{\alpha_1 \alpha_2}^{-1} &= \frac{\delta^2 \mathcal{L}}{\delta \bar{\phi}_{\alpha_2} \delta \bar{\phi}_{\alpha_1}} |_{\bar{\phi}=0} \\ &= \Gamma_{\alpha_1 \alpha_2}^{(2)} - \left(G_0^{-1}\right)_{\alpha_1 \alpha_2} \Rightarrow \Gamma^{(2)} = G_0^{-1} - G^{-1} = \Sigma. \end{aligned} \tag{11.254}$$

This shows that the 2-point vertex is exactly given by the self-energy, i.e. it is fully irreducible.

We will need to regularize the theory by regularizing the free propagator G_0 in the Gaussian action $S_0[\phi]$. For example, by means of a momentum cutoff Λ. Towards this end, we make the substitution

$$G_0 \longrightarrow G_{0,\Lambda}. \tag{11.255}$$

We require that

$$G_{0,\Lambda} \longrightarrow G_0, \ \Lambda \longrightarrow 0 \tag{11.256}$$

$$G_{0,\Lambda} \longrightarrow 0, \ \Lambda \longrightarrow \infty. \tag{11.257}$$

We think of Λ as an infrared cutoff, i.e. we obtain the original theory when we remove the cutoff $\Lambda \longrightarrow 0$. In other words, the fluctuations with momenta below the cutoff $|k| \leqslant \Lambda$ cannot propagate and thus they have still to be integrated out, while fluctuations with momenta above the cutoff $|k| \geqslant \Lambda$ have already been integrated out. This is reflected in the fact that we are going to calculate the derivative with respect to the cutoff Λ of various generating functionals, viz $\partial_\Lambda \mathcal{G}_c$, $\partial_\Lambda \mathcal{G}_{ac}$ and $\partial_\Lambda \Gamma$,

starting from their initial values at $\Lambda = \infty$. The above condition can be implemented by a sharp momentum cutoff given by

$$G_{0,\Lambda}(k) = \theta(|k| - \Lambda)G_0(k). \tag{11.258}$$

Obviously, for $\Lambda \longrightarrow 0$ we obtain the original theory, viz $G_{0,\Lambda} = G_0$, while for $\Lambda \longrightarrow \infty$ we obtain $G_{0,\Lambda} = 0$, i.e. all modes are frozen. A smooth momentum cutoff is then given by

$$G_{0,\Lambda}(k) = \theta_\Lambda(|k| - \Lambda)G_0(k) \tag{11.259}$$

$$\theta_\Lambda \longrightarrow 1, \Lambda \longrightarrow 0; \theta_\Lambda \longrightarrow 0, \Lambda \longrightarrow \infty. \tag{11.260}$$

In other words, θ_Λ is a smoothed step function with a width ϵ, viz

$$\theta_\Lambda(|k| - \Lambda) = \theta_\epsilon(|k| - \Lambda), \lim \theta_\epsilon(|k| - \Lambda) = \theta(|k| - \Lambda), \epsilon \longrightarrow 0. \tag{11.261}$$

We have already shown that the initial conditions for the generating functionals $\mathcal{G}_c[J]$, $\mathcal{G}_{ac}[\bar{\phi}]$ and $\Gamma[\bar{\phi}]$ are given by equations (12.273), (12.226) and (12.275). These equations can be rewritten as

$$\mathcal{G}_{c,\Lambda}[J] \longrightarrow 0, \Lambda \longrightarrow \infty \tag{11.262}$$

$$\mathcal{G}_{ac,\Lambda}[\bar{\phi}] \longrightarrow -S_{\rm I}[\bar{\phi}], \Lambda \longrightarrow \infty \tag{11.263}$$

$$\Gamma_\Lambda[\bar{\phi}] \longrightarrow S_{\rm I}[\bar{\phi}], \Lambda \longrightarrow \infty. \tag{11.264}$$

From equation (11.243), it is also obvious that the classical field behaves in this limit as

$$\bar{\phi}' \longrightarrow \bar{\phi}, \Lambda \longrightarrow \infty. \tag{11.265}$$

11.4 The functional renormalization group

The original literature on the functional renormalization group equation, which is an exact non-perturbative formulation of the Wilson renormalization group equation, includes Polchinski [2] and Wetterich [3]. See also [14–16] and [17, 18]. Again, here we will follow the presentation of [4].

11.4.1 The FRG equations

The goal now is to derive the functional renormalization group equations (FRG) obeyed by the various generating functionals. Towards this end, we compute the derivative with respect to the cutoff Λ of $\mathcal{G}_\Lambda[J]$, $\mathcal{G}_{c,\Lambda}[J]$, $\mathcal{G}_{ac,\Lambda}[\bar{\phi}]$ and $\Gamma_\Lambda[\phi']$. The computation of the derivative with respect to the cutoff Λ of $\mathcal{G}_\Lambda[J]$ is straightforward. We find (with the replacement $\phi \longrightarrow \delta/\delta J$)

$$\partial_\Lambda \mathcal{G}_\Lambda[J] = \left[\frac{1}{2}\left(\frac{\delta}{\delta J},\, \partial_\Lambda G_{0,\Lambda}^{-1}\frac{\delta}{\delta J}\right) - \partial_\Lambda \ln Z_\Lambda\right]\mathcal{G}_\Lambda[J]. \tag{11.266}$$

Next, we recall the defining equation

$$\mathcal{G}_\Lambda[J] = \frac{Z_{0,\Lambda}}{Z_\Lambda}\exp(\mathcal{G}_{c,\Lambda}[J]). \tag{11.267}$$

The derivative with respect to the cutoff Λ of this equation gives

$$\partial_\Lambda \mathcal{G}_\Lambda[J] = \left[\partial_\Lambda \mathcal{G}_{c,\Lambda}[J] + \partial_\Lambda \ln\frac{Z_{0,\Lambda}}{Z_\Lambda}\right]\mathcal{G}_\Lambda[J]. \tag{11.268}$$

By acting with the operator given by the first term in the right-hand side of equation (11.266) on $\mathcal{G}_\Lambda[J]$ we obtain

$$\begin{aligned}\frac{1}{2}\left(\frac{\delta}{\delta J},\, \partial_\Lambda G_{0,\Lambda}^{-1}\frac{\delta}{\delta J}\right)\mathcal{G}_\Lambda[J] = \Bigg[&\frac{1}{2}\left(\frac{\delta \mathcal{G}_{c,\Lambda}}{\delta J},\, \partial_\Lambda G_{0,\Lambda}^{-1}\frac{\delta \mathcal{G}_{c,\Lambda}}{\delta J}\right)\\ &+ \frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\left(\frac{\delta}{\delta J}\otimes\frac{\delta}{\delta J}\mathcal{G}_{c,\Lambda}\right)^T\Bigg]\mathcal{G}_\Lambda[J].\end{aligned} \tag{11.269}$$

By putting the above two equations together we obtain

$$\begin{aligned}\partial_\Lambda \mathcal{G}_{c,\Lambda}[J] = &\frac{1}{2}\left(\frac{\delta \mathcal{G}_{c,\Lambda}}{\delta J},\, \partial_\Lambda G_{0,\Lambda}^{-1}\frac{\delta \mathcal{G}_{c,\Lambda}}{\delta J}\right)\\ &+ \frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\left(\frac{\delta}{\delta J}\otimes\frac{\delta}{\delta J}\mathcal{G}_{c,\Lambda}\right)^T - \partial_\Lambda \ln Z_{0,\Lambda}.\end{aligned} \tag{11.270}$$

From equation (11.230) we have

$$\exp(\mathcal{G}_{ac,\Lambda}[\bar\phi]) = \exp\left(-\frac{1}{2}\left(\frac{\delta}{\delta\bar\phi},\, G_{0,\Lambda}\frac{\delta}{\delta\bar\phi}\right)\right)\exp(-S_{\mathrm{I}}[\bar\phi]). \tag{11.271}$$

By differentiating both sides with respect to Λ we obtain

$$\begin{aligned}\partial_\Lambda \mathcal{G}_{ac,\Lambda}\cdot\exp(\mathcal{G}_{ac,\Lambda}) &= -\frac{1}{2}\left(\frac{\delta}{\delta\bar\phi},\, \partial_\Lambda G_{0,\Lambda}\frac{\delta}{\delta\bar\phi}\right)\exp(\mathcal{G}_{ac,\Lambda})\\ &= \Bigg[-\frac{1}{2}\left(\frac{\delta \mathcal{G}_{ac,\Lambda}}{\delta\bar\phi},\, \partial_\Lambda G_{0,\Lambda}\frac{\delta \mathcal{G}_{ac,\Lambda}}{\delta\bar\phi}\right)\\ &\quad -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}\left(\frac{\delta}{\delta\bar\phi}\otimes\frac{\delta}{\delta\bar\phi}\mathcal{G}_{ac,\Lambda}\right)^T\Bigg]\exp(\mathcal{G}_{ac,\Lambda}).\end{aligned} \tag{11.272}$$

We get then the functional renormalization group equation

$$\begin{aligned}\partial_\Lambda \mathcal{G}_{ac,\Lambda} = &-\frac{1}{2}\left(\frac{\delta \mathcal{G}_{ac,\Lambda}}{\delta\bar{\phi}},\, \partial_\Lambda G_{0,\Lambda}\frac{\delta \mathcal{G}_{ac,\Lambda}}{\delta\bar{\phi}}\right)\\ &-\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}\left(\frac{\delta}{\delta\bar{\phi}}\otimes\frac{\delta}{\delta\bar{\phi}}\mathcal{G}_{ac,\Lambda}\right)^T.\end{aligned} \tag{11.273}$$

This is the Polchinski equation. Indeed, this FRG equation is identical to the Polchinski equation for the Wilsonian effective action [2].

To obtain the FRG equation for the generating functional $\Gamma_\Lambda[\bar{\phi}']$ we start from the definitions

$$\Gamma_\Lambda[\bar{\phi}'] = \mathcal{L}_\Lambda[\bar{\phi}'] + \frac{1}{2}\left(\bar{\phi}',\, G_{0,\Lambda}^{-1}\bar{\phi}'\right) \tag{11.274}$$

$$\mathcal{L}_\Lambda[\bar{\phi}'] = (J_\Lambda,\, \bar{\phi}') - \mathcal{G}_{c,\Lambda}[J_\Lambda]. \tag{11.275}$$

Of course, J and $\bar{\phi}$ are conjugate variables

$$\bar{\phi}' = \frac{\delta \mathcal{G}_{c,\Lambda}[J]}{\delta J}. \tag{11.276}$$

We will assume that J depends on the cutoff, i.e. $J = J_\Lambda[\bar{\phi}']$, in such a way that $\bar{\phi}'$ is independent of Λ. By taking the derivative with respect to the cutoff Λ we obtain

$$\partial_\Lambda \Gamma_\Lambda[\bar{\phi}'] = \partial_\Lambda \mathcal{L}_\Lambda[\bar{\phi}'] + \frac{1}{2}\left(\bar{\phi}',\, \partial_\Lambda G_{0,\Lambda}^{-1}\bar{\phi}'\right) \tag{11.277}$$

$$\partial_\Lambda \mathcal{L}_\Lambda[\bar{\phi}'] = (\partial_\Lambda J_\Lambda,\, \bar{\phi}') - \partial_\Lambda \mathcal{G}_{c,\Lambda} - \int_\beta \partial_\Lambda (J_\Lambda)_\beta \frac{\delta \mathcal{G}_{c,\Lambda}}{\delta (J_\Lambda)_\beta}. \tag{11.278}$$

The second term arises from the explicit dependence of $\mathcal{G}_{c,\Lambda}$ on Λ, whereas the third term arises from the implicit dependence of $\mathcal{G}_{c,\Lambda}$ on Λ through J_Λ. It is obvious that the first and third terms cancel, viz

$$\partial_\Lambda \mathcal{L}_\Lambda[\bar{\phi}'] = -\partial_\Lambda \mathcal{G}_{c,\Lambda}. \tag{11.279}$$

By using equations (11.270), (11.276) and (11.240) we obtain

$$\begin{aligned}\partial_\Lambda \Gamma_\Lambda[\bar{\phi}'] &= -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\left(\frac{\delta}{\delta J}\otimes\frac{\delta}{\delta J}\mathcal{G}_{c,\Lambda}\right) + \partial_\Lambda \ln Z_{0,\Lambda}\\ &= -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\left(\frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\mathcal{L}_\Lambda\right)^{-1} + \partial_\Lambda \ln Z_{0,\Lambda},\end{aligned} \tag{11.280}$$

or equivalently

$$\partial_\Lambda \Gamma_\Lambda[\bar{\phi}'] = -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\left(\frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda - G_{0,\Lambda}^{-1}\right)^{-1} + \partial_\Lambda \ln Z_{0,\Lambda}. \tag{11.281}$$

This is the Wetterich equation [3, 14]. The average effective action introduced by Wetterich is given by [3]

$$\Gamma_\Lambda^W[\bar{\phi}'] = \Gamma_\Lambda[\bar{\phi}'] - \frac{1}{2}(\bar{\phi}',\, G_0^{-1}\bar{\phi}') - \ln Z_{0,\Lambda}. \tag{11.282}$$

This satisfies the FRG equation [3]

$$\partial_\Lambda \Gamma_\Lambda^W[\bar{\phi}'] = -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\left(\frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda - G_{0,\Lambda}^{-1}\right)^{-1}. \tag{11.283}$$

11.4.2 The vertex expansion of the Wetterich FRG

One approximation method for solving the FRG equation (11.281) is the vertex expansion [14] which allows us to reduce this equation to an infinite number of coupled integro-differential equations for the irreducible vertices. Before we can explain this method we rewrite equation (11.281) in a suitable form as follows. First, we note

$$\begin{aligned}\partial_\Lambda Z_{0,\Lambda} &= \int d\phi \frac{1}{2}\left(\phi,\, \partial_\Lambda G_{0,\Lambda}^{-1}\phi\right)\exp(-S_0[\phi]) \\ &= \frac{Z_{0,\Lambda}}{2}\left(\frac{\delta}{\delta J},\, \partial_\Lambda G_{0,\Lambda}^{-1}\frac{\delta}{\delta J}\right)\exp\left(-\frac{1}{2}(J,\, G_{0,\Lambda}J)\right)|_{J=0}.\end{aligned} \tag{11.284}$$

Thus

$$\partial_\Lambda \ln Z_{0,\Lambda} = -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\cdot G_{0,\Lambda}. \tag{11.285}$$

Second, we have

$$\begin{aligned}\frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\mathcal{L}_\Lambda &= \frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\left(\Gamma_\Lambda - \frac{1}{2}\left(\bar{\phi}',\, G_{0,\Lambda}^{-1}\bar{\phi}'\right)\right) \\ &= \frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda - \frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda|_{\bar{\phi}'=0} + \Sigma_\Lambda - G_{0,\Lambda}^{-1} \\ &= U_\Lambda[\bar{\phi}'] - G_\Lambda^{-1}.\end{aligned} \tag{11.286}$$

In the above equation the second and the third terms are identically equal, viz

$$\Sigma_\Lambda = \frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda|_{\bar{\phi}'=0} = \Sigma_\Lambda = G_{0,\Lambda}^{-1} - G_\Lambda^{-1}. \tag{11.287}$$

This equation is equivalent to

$$G_\Lambda^{-1} = G_{0,\Lambda}^{-1}(1 - G_{0,\Lambda}\Sigma_\Lambda),\ G_\Lambda = (1 - G_{0,\Lambda}\Sigma_\Lambda)^{-1}G_{0,\Lambda}. \tag{11.288}$$

Furthermore, we have defined U_Λ as

$$U_\Lambda[\bar{\phi}'] = \frac{\delta}{\delta\bar{\phi}'} \otimes \frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda - \frac{\delta}{\delta\bar{\phi}'} \otimes \frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda|_{\bar{\phi}'=0}. \tag{11.289}$$

By using the above two results (11.285) and (11.286) we rewrite the FRG equation (11.281) in the form

$$\partial_\Lambda\Gamma_\Lambda[\bar{\phi}'] = -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\Big([U_\Lambda[\bar{\phi}'] - G_\Lambda^{-1}]^{-1} + G_{0,\Lambda}\Big). \tag{11.290}$$

We compute

$$\begin{aligned}
[U_\Lambda[\bar{\phi}'] - G_\Lambda^{-1}]^{-1} + G_{0,\Lambda} &= -[1 - G_\Lambda U_\Lambda]^{-1}G_\Lambda + G_{0,\Lambda} \\
&= -[1 + G_\Lambda U_\Lambda + G_\Lambda U_\Lambda \cdot G_\Lambda U_\Lambda + \cdots]G_\Lambda + G_{0,\Lambda} \\
&= -G_\Lambda - G_\Lambda U_\Lambda[1 + G_\Lambda U_\Lambda + G_\Lambda U_\Lambda \cdot G_\Lambda U_\Lambda + \cdots] \\
&\quad \times G_\Lambda + G_{0,\Lambda} \\
&= -G_\Lambda + G_{0,\Lambda} - G_\Lambda U_\Lambda[1 - G_\Lambda U_\Lambda]^{-1}G_\Lambda \\
&= -(1 - G_{0,\Lambda}\Sigma_\Lambda)^{-1}G_{0,\Lambda} + G_{0,\Lambda} - G_\Lambda U_\Lambda \\
&\quad \times [1 - G_\Lambda U_\Lambda]^{-1}G_\Lambda \\
&= -G_{0,\Lambda}\Sigma_\Lambda G_\Lambda - G_\Lambda U_\Lambda[1 - G_\Lambda U_\Lambda]^{-1}G_\Lambda.
\end{aligned} \tag{11.291}$$

In the fifth equality of the above equation we have used equation (11.288). By substitution we get

$$\partial_\Lambda\Gamma_\Lambda[\bar{\phi}'] = -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\Big(-G_{0,\Lambda}\Sigma_\Lambda G_\Lambda - G_\Lambda U_\Lambda[1 - G_\Lambda U_\Lambda]^{-1}G_\Lambda\Big). \tag{11.292}$$

We define the single-scale propagator $\dot{G}_\Lambda$ by

$$\begin{aligned}
\dot{G}_\Lambda &= -G_\Lambda\partial_\Lambda G_{0,\Lambda}^{-1}G_\Lambda \\
&= (1 - G_{0,\Lambda}\Sigma_\Lambda)^{-1}\dot{G}_{0,\Lambda}G_{0,\Lambda}^{-1}(1 - G_{0,\Lambda}\Sigma_\Lambda)^{-1}G_{0,\Lambda} \\
&= (1 - G_{0,\Lambda}\Sigma_\Lambda)^{-1}\dot{G}_{0,\Lambda}(1 - \Sigma_\Lambda G_{0,\Lambda})^{-1},
\end{aligned} \tag{11.293}$$

where the single-scale propagator of the free theory is given by

$$\begin{aligned}
\dot{G}_{0,\Lambda} &= -G_{0,\Lambda}\partial_\Lambda G_{0,\Lambda}^{-1}G_{0,\Lambda} \\
&= \partial_\Lambda G_{0,\Lambda}.
\end{aligned} \tag{11.294}$$

Using these definitions the above FRG equation becomes

$$\partial_\Lambda\Gamma_\Lambda[\bar{\phi}'] = -\frac{1}{2}\,\mathrm{Tr}\Big(\dot{G}_{0,\Lambda}\Sigma_\Lambda[1 - G_{0,\Lambda}\Sigma_\Lambda]^{-1} + \dot{G}_\Lambda U_\Lambda[\bar{\phi}'][1 - G_\Lambda U_\Lambda[\bar{\phi}']]^{-1}\Big). \tag{11.295}$$

For a free field theory $G_{0,c} = -(J,\, G_0 J)/2$ and $G_{0,ac} = 0$ and thus $\bar{\phi}' = \bar{\phi}$ and

$$\begin{aligned}\frac{\delta}{\delta\bar{\phi}'} \otimes \frac{\delta}{\delta\bar{\phi}'}\mathcal{L}_\Lambda &= -G_{0,\Lambda}^{-1} \\ &= \frac{\delta}{\delta\bar{\phi}'} \otimes \frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda - G_{0,\Lambda}^{-1} \\ &= U_{0,\Lambda}[\bar{\phi}'] - G_{0,\Lambda}^{-1}.\end{aligned} \tag{11.296}$$

In other words, $U_{0,\Lambda} = \Sigma_{0,\Lambda} = 0$.

We want now to derive the expansion of the functional U_Λ in terms of the irreducible vertices. Towards this end, we recall the expansion of the effective action in terms of irreducible vertices given by

$$\Gamma_\Lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha_1} \cdots \int_{\alpha_n} \Gamma^{(n)}_{\Lambda\alpha_1\ldots\alpha_n} \bar{\phi}_{\alpha_1} \ldots \bar{\phi}_{\alpha_n}. \tag{11.297}$$

We take the derivative with respect to $\bar{\phi}_\alpha$ to obtain

$$\frac{\delta\Gamma_\Lambda}{\delta\bar{\phi}_\alpha} = \Gamma^{(1)}_{\Lambda\alpha} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\alpha_1} \cdots \int_{\alpha_n} \Gamma^{(n+1)}_{\Lambda\alpha\alpha_1\ldots\alpha_n} \bar{\phi}_{\alpha_1} \ldots \bar{\phi}_{\alpha_n}. \tag{11.298}$$

We take the derivative a second time with respect to $\bar{\phi}_{\alpha'}$ to obtain

$$\frac{\delta\Gamma_\Lambda}{\delta\bar{\phi}_\alpha \delta\bar{\phi}_{\alpha'}} = \Gamma^{(1)}_{\Lambda\alpha'} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\alpha_1} \cdots \int_{\alpha_n} \Gamma^{(n+2)}_{\Lambda\alpha\alpha'\alpha_1\ldots\alpha_n} \bar{\phi}_{\alpha_1} \ldots \bar{\phi}_{\alpha_n}. \tag{11.299}$$

From this result we obtain the expansion of the functional U_Λ in terms of the irreducible vertices given by

$$\begin{aligned}(U_\Lambda)_{\alpha\alpha'} &= \frac{\delta^2\Gamma_\Lambda}{\delta\bar{\phi}_\alpha \delta\bar{\phi}_{\alpha'}} - \frac{\delta^2\Gamma_\Lambda}{\delta\bar{\phi}_\alpha \delta\bar{\phi}_{\alpha'}}|_{\bar{\phi}=0} \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\alpha_1} \cdots \int_{\alpha_n} \Gamma^{(n+2)}_{\Lambda\alpha\alpha'\alpha_1\ldots\alpha_n} \bar{\phi}_{\alpha_1} \ldots \bar{\phi}_{\alpha_n}.\end{aligned} \tag{11.300}$$

We will define the matrix

$$\left(\mathbf{\Gamma}^{(n+2)}_{\Lambda\alpha_1\ldots\alpha_n}\right)_{\alpha\alpha'} = \Gamma^{(n+2)}_{\Lambda\alpha\alpha'\alpha_1\ldots\alpha_n}. \tag{11.301}$$

We compute from the right-hand side of the FRG equation (11.295) the following expansion

$$\partial_\Lambda \Gamma^{(0)}_\Lambda + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\alpha_1} \cdots \int_{\alpha_n} \partial_\Lambda \Gamma^{(n)}_{\Lambda\alpha_1\ldots\alpha_n} \bar{\phi}_{\alpha_1} \ldots \bar{\phi}_{\alpha_n}. \tag{11.302}$$

From the left-hand side of the FRG equation (11.295) we obtain

$$-\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_{0,\Lambda}\Sigma_\Lambda[1-G_{0,\Lambda}\Sigma_\Lambda]^{-1}-\frac{1}{2}\sum_{\nu=1}^{\infty}\mathrm{Tr}\,\dot{G}_\Lambda U_\Lambda(G_\Lambda U_\Lambda)^{\nu-1}. \tag{11.303}$$

We deduce

$$\partial_\Lambda\Gamma_\Lambda^{(0)}=-\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_{0,\Lambda}\Sigma_\Lambda[1-G_{0,\Lambda}\Sigma_\Lambda]^{-1}. \tag{11.304}$$

In the expansion of the second term of (11.303) we will adopt the following notation. The fields appearing in the first U_Λ are denoted by the labels $\alpha_1,\ldots,\alpha_{n_1}$, those appearing in the second U_Λ are denoted by $\alpha_{n_1+1},\ldots,\alpha_{n_1+n_2},\ldots$, those appearing in the kth U_Λ are denoted by $\alpha_{n_1+\cdots+n_{k-1}+1},\ldots,\alpha_{n_1+\cdots+n_k}$, etc. We will also employ the notation $n=n_1+\cdots+n_\nu$. We get

$$\begin{aligned}-\frac{1}{2}\sum_{\nu=1}^{\infty}\mathrm{Tr}\,\dot{G}_\Lambda U_\Lambda(G_\Lambda U_\Lambda)^{\nu-1}=&-\frac{1}{2}\sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_\nu=1}^{\infty}\frac{\delta_{n,\,n_1+\cdots+n_\nu}}{n_1!\,\ldots\,n_\nu!}\\&\times\int_{\alpha_1}\cdots\int_{\alpha_{n_1}}\int_{\alpha_{n_1+1}}\cdots\int_{\alpha_{n_1+n_2}}\cdots\\&\times\int_{\alpha_{n-n_\nu-n_{\nu-1}+1}}\cdots\int_{\alpha_{n-n_\nu}}\int_{\alpha_{n-n_\nu+1}}\cdots\\&\times\int_{\alpha_n}\bar\phi_{\alpha_1}\cdots\bar\phi_{\alpha_{n_1}}\bar\phi_{\alpha_{n_1+1}}\cdots\bar\phi_{\alpha_{n_1+n_2}}\cdots\\&\times\bar\phi_{\alpha_{n-n_\nu-n_{\nu-1}+1}}\cdots\bar\phi_{\alpha_{n-n_\nu}}\bar\phi_{\alpha_{n-n_\nu+1}}\cdots\\&\times\bar\phi_{\alpha_n}\,\mathrm{Tr}\,\dot{G}_\Lambda\mathbf{\Gamma}^{(n_\nu+2)}_{\Lambda\alpha_{n-n_\nu+1}\ldots\alpha_n}\\&\times G_\Lambda\mathbf{\Gamma}^{(n_{\nu-1}+2)}_{\Lambda\alpha_{n-n_\nu-n_{\nu-1}+1}\ldots\alpha_{n-n_\nu}}\cdots\\&\times G_\Lambda\mathbf{\Gamma}^{(n_2+2)}_{\Lambda\alpha_{n_1+1}\ldots\alpha_{n_1+n_2}}G_\Lambda\mathbf{\Gamma}^{(n_1+1)}_{\Lambda\alpha_1\ldots\alpha_{n_1}}.\end{aligned} \tag{11.305}$$

By symmetrizing the above expression (11.305) in the fields and then comparing equations (11.302) and (11.305) we obtain

$$\begin{aligned}\partial_\Lambda\Gamma^{(n)}_{\Lambda,\alpha_1\ldots\alpha_n}=&-\frac{1}{2}\sum_{\nu=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_\nu=1}^{\infty}\delta_{n,n_1+\cdots+n_\nu}\\&\times\mathcal{S}_{\alpha_1\ldots\alpha_{n_1};\,\ldots\,;\alpha_{n-n_\nu+1}\ldots\alpha_n}\,\mathrm{Tr}\,\dot{G}_\Lambda\mathbf{\Gamma}^{(n_\nu+2)}_{\Lambda,\alpha_{n-n_\nu+1}\ldots\alpha_n}\\&\times G_\Lambda\mathbf{\Gamma}^{(n_{\nu-1}+2)}_{\Lambda,\alpha_{n-n_\nu-n_{\nu-1}+1}\ldots\alpha_{n-n_\nu}}\cdots\\&\times G_\Lambda\mathbf{\Gamma}^{(n_2+2)}_{\Lambda,\alpha_{n_1+1}\ldots\alpha_{n_1+n_2}}G_\Lambda\mathbf{\Gamma}^{(n_1+2)}_{\Lambda,\alpha_1\ldots\alpha_{n_1}}.\end{aligned} \tag{11.306}$$

The symmetrizer is obviously defined by

$$\mathcal{S}_{\alpha_1\ldots\alpha_{n_1};\,\ldots\,;\alpha_{n-n_\nu+1}\ldots\alpha_n} F_{\alpha_1\ldots\alpha_n} = \frac{1}{n_1!\,\ldots\,n_\nu!}\sum_P F_{\alpha_{P(1)}\,\ldots\,\alpha_{P(n)}}. \tag{11.307}$$

As a first example we consider the 2-point vertex $\Gamma^{(2)}_{\Lambda\alpha_1\alpha_2}$. We have

$$\begin{aligned}\partial_\Lambda \Gamma^{(2)}_{\Lambda\alpha_1\alpha_2} = -\frac{1}{2}\sum_{\nu=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_\nu=1}^{\infty}\delta_{2,n_1+\cdots+n_\nu}\\ \times \mathcal{S}_{\alpha_1\ldots\alpha_{n_1};\,\ldots\,;\alpha_{3-n_\nu}\,\ldots\,\alpha_2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(n_\nu+2)}_{\Lambda\alpha_{3-n_\nu}\,\ldots\,\alpha_2}\\ \times G_\Lambda \mathbf{\Gamma}^{(n_{\nu-1}+2)}_{\Lambda\alpha_{3-n_\nu-n_{\nu-1}}\,\ldots\,\alpha_{2-n_\nu}}\cdots\\ \times G_\Lambda \mathbf{\Gamma}^{(n_2+2)}_{\Lambda\alpha_{n_1+1}\ldots\alpha_{n_1+n_2}} G_\Lambda \mathbf{\Gamma}^{(n_1+2)}_{\Lambda\alpha_1\ldots\alpha_{n_1}}.\end{aligned} \tag{11.308}$$

The first term is directly given by

$$-\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(4)}_{\Lambda\alpha_1\alpha_2}. \tag{11.309}$$

The second term is also directly given by

$$\begin{aligned}&-\frac{1}{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\delta_{2,\,n_1+n_2}\mathcal{S}_{\alpha_1\ldots\alpha_{n_1};\alpha_{n_1+1}\ldots\alpha_2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(n_2+2)}_{\Lambda\alpha_{n_1+1}\ldots\alpha_2} G_\Lambda \mathbf{\Gamma}^{(n_1+2)}_{\Lambda\alpha_1\ldots\alpha_{n_1}}\\ &= -\frac{1}{2}\Big[\mathcal{S}_{\alpha_1\alpha_2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(2)}_\Lambda G_\Lambda \mathbf{\Gamma}^{(4)}_{\Lambda\alpha_1\alpha_2} + \mathcal{S}_{\alpha_1\alpha_2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(4)}_{\Lambda\alpha_1\alpha_2} G_\Lambda \mathbf{\Gamma}^{(2)}_\Lambda\\ &\quad + \mathcal{S}_{\alpha_1;\alpha_2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(3)}_{\Lambda\alpha_1} G_\Lambda \mathbf{\Gamma}^{(3)}_{\Lambda\alpha_2}\Big]\\ &= -\frac{1}{2}\mathcal{S}_{\alpha_1;\alpha_2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(3)}_{\Lambda\alpha_1} G_\Lambda \mathbf{\Gamma}^{(3)}_{\Lambda\alpha_2}.\end{aligned} \tag{11.310}$$

In the above equation we have used the fact that $\mathbf{\Gamma}^{(2)}_\Lambda = 0$ from equation (11.300). Higher-order corrections vanish for the same reason. We get then

$$\partial_\Lambda \Gamma^{(2)}_{\Lambda\alpha_1\alpha_2} = -\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(4)}_{\Lambda\alpha_1\alpha_2} - \frac{1}{2}\mathcal{S}_{\alpha_1;\alpha_2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(3)}_{\Lambda\alpha_1} G_\Lambda \mathbf{\Gamma}^{(3)}_{\Lambda\alpha_2}. \tag{11.311}$$

In the absence of external fields and symmetry breaking, all irreducible vertices with an odd numbers of legs vanish. In this case the above flow equation simplifies to

$$\partial_\Lambda \Gamma^{(2)}_{\Lambda\alpha_1\alpha_2} = -\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(4)}_{\Lambda\alpha_1\alpha_2}. \tag{11.312}$$

As a second example, we compute the FRG equation of the 4-point vertex. This turns out to involve eight terms corresponding to $\nu = 1, 2, 3, 4$. Explicitly, we have

$$
\begin{aligned}
\partial_\Lambda \Gamma^{(4)}_{\Lambda\alpha_1\ldots\alpha_4} = &-\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(6)}_{\Lambda\alpha_1\ldots\alpha_4} - \frac{1}{2}\Big(\mathcal{S}_{\alpha_1;\alpha_2\alpha_3\alpha_4}\,\mathrm{Tr}\,\dot{G}_\Lambda \Gamma^{(5)}_{\Lambda\alpha_2\alpha_3\alpha_4} G_\Lambda \Gamma^{(3)}_{\Lambda\alpha_1} \\
&+ \mathcal{S}_{\alpha_1\alpha_2\alpha_3;\alpha_4}\,\mathrm{Tr}\,\dot{G}_\Lambda \Gamma^{(3)}_{\Lambda\alpha_4} G_\Lambda \Gamma^{(5)}_{\Lambda\alpha_1\alpha_2\alpha_3} \\
&+ \mathcal{S}_{\alpha_1\alpha_2;\alpha_3\alpha_4}\,\mathrm{Tr}\,\dot{G}_\Lambda \Gamma^{(4)}_{\Lambda\alpha_3\alpha_4} G_\Lambda \Gamma^{(4)}_{\Lambda\alpha_1\alpha_2}\Big) \\
&- \frac{1}{2}\Big(\mathcal{S}_{\alpha_1\alpha_2;\alpha_3;\alpha_4}\,\mathrm{Tr}\,\dot{G}_\Lambda \Gamma^{(3)}_{\Lambda\alpha_4} G_\Lambda \Gamma^{(3)}_{\Lambda\alpha_3} G_\Lambda \Gamma^{(4)}_{\Lambda\alpha_1\alpha_2} \\
&+ \mathcal{S}_{\alpha_1;\alpha_2\alpha_3;\alpha_4}\,\mathrm{Tr}\,\dot{G}_\Lambda \Gamma^{(3)}_{\Lambda\alpha_4} G_\Lambda \Gamma^{(4)}_{\Lambda\alpha_2\alpha_3} G_\Lambda \Gamma^{(3)}_{\Lambda\alpha_1} \\
&+ \mathcal{S}_{\alpha_1;\alpha_2;\alpha_3\alpha_4}\,\mathrm{Tr}\,\dot{G}_\Lambda \Gamma^{(4)}_{\Lambda\alpha_3\alpha_4} G_\Lambda \Gamma^{(3)}_{\Lambda\alpha_2} G_\Lambda \Gamma^{(3)}_{\Lambda\alpha_1}\Big) \\
&- \frac{1}{2}\mathcal{S}_{\alpha_1;\alpha_2;\alpha_3;\alpha_4}\,\mathrm{Tr}\,\dot{G}_\Lambda \Gamma^{(3)}_{\Lambda\alpha_4} G_\Lambda \Gamma^{(3)}_{\Lambda\alpha_3} G_\Lambda \Gamma^{(3)}_{\Lambda\alpha_2} G_\Lambda \Gamma^{(3)}_{\Lambda\alpha_1}.
\end{aligned}
\tag{11.313}
$$

Again, in the case of the absence of external fields and symmetry breaking all irreducible vertices with an odd numbers of legs vanish and as a consequence the above expression simplifies to

$$
\begin{aligned}
\partial_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_1\ldots\alpha_4} &= -\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(6)}_{\Lambda,\alpha_1\ldots\alpha_4} - \frac{1}{2}\mathcal{S}_{\alpha_1\alpha_2;\alpha_3\alpha_4}\,\mathrm{Tr}\,\dot{G}_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_3\alpha_4} G_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_2} \\
&= -\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(6)}_{\Lambda,\alpha_1\ldots\alpha_4} - \frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda\Big[\Gamma^{(4)}_{\Lambda,\alpha_3\alpha_4} G_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_2} \\
&\quad + \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_2} G_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_3\alpha_4} + \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_3} G_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_2\alpha_4} + \Gamma^{(4)}_{\Lambda,\alpha_2\alpha_4} G_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_3} \\
&\quad + \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_4} G_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_2\alpha_3} + \Gamma^{(4)}_{\Lambda,\alpha_2\alpha_3} G_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_4}\Big].
\end{aligned}
\tag{11.314}
$$

11.5 The vertex expansion approximation method for ϕ^4

11.5.1 The disordered phase

We consider ϕ^4 in D dimensions. We make the replacements $\alpha \longrightarrow k$, $\phi_\alpha \longrightarrow \phi(k)$ and $\int_\alpha \longrightarrow \int_k = d^D k/(2\pi)^D$. The action is given by

$$
S_\Lambda[\phi] = S_{0,\Lambda}[\phi] + S_{\mathrm{I}}[\Phi]. \tag{11.315}
$$

We start by writing down the interaction part. This is given by

$$
\begin{aligned}
S_{\mathrm{I}}[\Phi] = Vf_0 + \frac{r_0}{2}\int_k \phi(-k)\phi(k) + \frac{u_0}{4!}\int_{k_1}\cdots \\
\times \int_{k_4} (2\pi)^D \delta^D(k_1 + \cdots + k_4)\phi(k_1)\ldots\phi(k_4).
\end{aligned}
\tag{11.316}
$$

The free part is

$$
S_{0,\Lambda}[\phi] = -\frac{1}{2}\big(\phi,\, G^{-1}_{0,\Lambda}\phi\big) = \frac{1}{2}\int_k G^{-1}_{0,\Lambda}(k)\phi(-k)\phi(k). \tag{11.317}
$$

In other words,

$$(G_{0,\Lambda})_{kk'} = -(2\pi)^D \delta^D(k + k') G_{0,\Lambda}(k). \tag{11.318}$$

By translational invariance, we have

$$(G_\Lambda)_{kk'} = -(2\pi)^D \delta^D(k + k') G_\Lambda(k). \tag{11.319}$$

We also have

$$G_\Lambda^{-1}(k) = G_{0,\Lambda}^{-1}(k) + \Sigma_\Lambda(k),\ G_\Lambda(k) = (1 + G_{0,\Lambda}(k)\Sigma_\Lambda(k))^{-1} G_{0,\Lambda}(k). \tag{11.320}$$

Also

$$(\Sigma_\Lambda)_{kk'} = (2\pi)^D \delta(k + k')\Sigma_\Lambda(k'). \tag{11.321}$$

In the disordered phase, the functional renormalization group equation for the effective action is given by equation (11.295) which becomes in this notation given by

$$\partial_\Lambda \Gamma_\Lambda = \frac{1}{2} \int_k \dot{G}_\Lambda(k)(U_\Lambda(1 - G_\Lambda U_\Lambda)^{-1})_{-k,k} + \frac{V}{2} \int_k \frac{\dot{G}_{0,\Lambda}(k)\Sigma_\Lambda(k)}{1 + G_{0,\Lambda}(k)\Sigma_\Lambda(k)}. \tag{11.322}$$

We have implicitly used the box convention $(2\pi)^D \delta^D(0) = V$. Obviously, the function U_Λ is given by

$$U_\Lambda(\bar{\phi})_{kk'} = \frac{\delta^2 \Gamma_\Lambda(\bar{\phi})}{\delta\bar{\phi}(k)\delta\bar{\phi}(k')} - (2\pi)^D \delta^D(k + k')\Sigma_\Lambda(k). \tag{11.323}$$

The single-scale propagator in this case is

$$\dot{G}_\Lambda(k) = -G_\Lambda(k)\partial_\Lambda G_{0,\Lambda}^{-1} G_\Lambda(k) = \frac{\dot{G}_{0,\Lambda}}{(1 + G_{0,\Lambda}(k)\Sigma_\Lambda(k))^2},\ \dot{G}_{0,\Lambda} = \partial_\Lambda G_{0,\Lambda}. \tag{11.324}$$

The FRG equation (11.295) has led to the FRG equation (11.304) for the free energy, and the FRG equation (11.312) for the self-energy, and the FRG equation (11.314) for the 4-point vertex. These equations become, in our notation, given respectively by

$$\partial_\Lambda \Gamma_\Lambda^{(0)} = \frac{V}{2} \int_k \frac{\dot{G}_{0,\Lambda}(k)\Sigma_\Lambda(k)}{1 + G_{0,\Lambda}(k)\Sigma_\Lambda(k)} \tag{11.325}$$

$$\partial_\Lambda \Sigma_\Lambda(p) = \frac{1}{2} \int_k \dot{G}_\Lambda(k)\Gamma_\Lambda^{(4)}(k, -k, p, -p) \tag{11.326}$$

$$\begin{aligned}\partial_\Lambda \Gamma^{(4)}_\Lambda(p_1, \dots, p_4) = \frac{1}{2}\int_k \dot{G}_\Lambda(k)\Gamma^{(6)}_\Lambda(k, -k, p_1, \dots, p_4) \\ - \int_k \Big[\dot{G}_\Lambda(k) G_\Lambda(k + p_1 + p_2) \\ \times \Gamma^{(4)}_\Lambda(-k - p_1 - p_2, k, p_1, p_2) \\ \times \Gamma^{(4)}_\Lambda(-k, k + p_1 + p_2, p_3, p_4) \\ + (p_2 \leftrightarrow p_3) + (p_2 \leftrightarrow p_4)\Big].\end{aligned} \tag{11.327}$$

In the above equations we have factored out the energy–momentum conserving delta functions contained in the irreducible vertices as follows

$$\Gamma^{(2)}_{\Lambda, p_1 p_2} = (2\pi)^D \delta^D(p_1 + p_2)\Gamma^{(2)}_\Lambda(p_1, -p_1),\ \Gamma^{(2)}_\Lambda(p_1, -p_1) = \Sigma_\Lambda(p_1) \tag{11.328}$$

$$\Gamma^{(4)}_{\Lambda, p_1 \dots p_4} = (2\pi)^D \delta^D(p_1 + \cdots + p_4)\Gamma^{(4)}_\Lambda(p_1, \dots, p_4) \tag{11.329}$$

$$\Gamma^{(6)}_{\Lambda, p_1 \dots p_6} = (2\pi)^D \delta^D(p_1 + \cdots + p_6)\Gamma^{(6)}_\Lambda(p_1, \dots, p_6). \tag{11.330}$$

Generalization is obvious.

We define the dimensionless n-point vertex $\tilde{\Gamma}^{(n)}_l$ by (use $[\phi] = M^{-(D+1)/2}$, $[\delta/\delta\phi] = M^{(1-D)/2}$, etc)

$$\tilde{\Gamma}^{(n)}_l(q_1, \dots, q_n) = \Lambda^{D\left(\frac{n}{2}-1\right)-n}\left(\frac{Z_l}{c_0}\right)^{n/2}\Gamma^{(n)}_\Lambda(\Lambda q_1, \dots, \Lambda q_n). \tag{11.331}$$

The logarithmic scale l is defined by

$$l = -\ln \Lambda/\Lambda_0. \tag{11.332}$$

The dimensionless rescaled momenta are

$$q_i = k_i/\Lambda. \tag{11.333}$$

From $G^{-1}_\Lambda(k) = G^{-1}_{0,\Lambda}(k) + \Sigma_\Lambda(k)$ we get

$$G_\Lambda(k) = \frac{1}{c_0 k^2 + \Sigma_\Lambda(k)}. \tag{11.334}$$

We expand as $\Sigma_\Lambda(k) = \Sigma_\Lambda(0) + k^2 \frac{\partial \Sigma_\Lambda}{\partial k^2}|_{k^2=0}$ to get

$$G_\Lambda(k) = \frac{Z_l}{c_0(k^2 + \zeta^{-2})}. \tag{11.335}$$

The wavefunction renormalization constant Z_l and the critical length ζ are defined by

$$Z_l = \frac{1}{1 + \frac{\partial \Sigma_\Lambda}{\partial k^2}|_{k^2=0}},\ c_0\zeta^{-2} = Z_l \Sigma_\Lambda(0). \tag{11.336}$$

The dimensionless 2-point vertex is

$$\tilde{\Gamma}_l^{(2)}(q,-q) = \frac{Z_l}{c_0\Lambda^2}\Gamma_\Lambda^{(2)}(\Lambda q,-\Lambda q) \Rightarrow \tilde{\Gamma}_l^{(2)}(q) = \frac{Z_l}{c_0\Lambda^2}\Sigma_\Lambda(\Lambda q). \tag{11.337}$$

By using this equation, it is interesting to show that the wavefunction renormalization constant can be rewritten as

$$Z_l = 1 - \frac{\partial\tilde{\Gamma}_\Lambda^{(2)}(q)}{\partial q^2}|_{q^2=0} \tag{11.338}$$

We can also see that

$$\tilde{G}_l(q) = \frac{c_0\Lambda^2}{Z_l}G_\Lambda(\Lambda q). \tag{11.339}$$

The dimensionless 4-point and 6-point vertices are given by

$$\tilde{\Gamma}_l^{(4)}(q_1,\ldots,q_4) = \Lambda^{D-4}\left(\frac{Z_l}{c_0}\right)^2\Gamma_\Lambda^{(4)}(\Lambda q_1,\ldots,\Lambda q_4) \tag{11.340}$$

$$\tilde{\Gamma}_l^{(6)}(q_1,\ldots,q_6) = \Lambda^{2D-6}\left(\frac{Z_l}{c_0}\right)^3\Gamma_\Lambda^{(6)}(\Lambda q_1,\ldots,\Lambda q_6). \tag{11.341}$$

Thus, we compute (with $k = \Lambda p$)

$$\begin{aligned}\partial_\Lambda\Sigma_\Lambda(k) &= \partial_\Lambda\Sigma_\Lambda + \partial_\Lambda k\partial_k\Sigma_\Lambda \\ &= \partial_\Lambda\Sigma_\Lambda + \frac{p}{\Lambda}\partial_p\Sigma_\Lambda.\end{aligned} \tag{11.342}$$

Thus (with $\partial_l = -\Lambda\partial_\Lambda$)

$$\begin{aligned}-\frac{Z_l}{c_0\Lambda}\partial_\Lambda\Sigma_\Lambda(k) &= -\frac{Z_l}{c_0\Lambda}\partial_\Lambda\Sigma_\Lambda - p\partial_p\tilde{\Gamma}_l^{(2)}(p) \\ (\eta_l - 2 + \partial_l)\tilde{\Gamma}_l^{(2)}(p) &= \dot{\Gamma}_l^{(2)}(p) - p\partial_p\tilde{\Gamma}_l^{(2)}(p).\end{aligned} \tag{11.343}$$

In the above we have defined the anomalous dimension by the usual formula in terms of the wavefunction renormalization constant given by

$$\eta_l = -\partial_l \ln Z_l. \tag{11.344}$$

We will write the above equation as

$$\begin{aligned}\dot{\Gamma}_l^{(2)}(p) &\equiv -\frac{Z_l}{c_0\Lambda}\partial_\Lambda\Sigma_\Lambda(k) \\ &= (\eta_l - 2 + \partial_l)\tilde{\Gamma}_l^{(2)}(p) + p\partial_p\tilde{\Gamma}_l^{(2)}(p).\end{aligned} \tag{11.345}$$

Equivalently,

$$\partial_l \tilde{\Gamma}_l^{(2)}(p) = (2 - \eta_l - p\partial_p)\tilde{\Gamma}_l^{(2)}(p) + \dot{\Gamma}_l^{(2)}(p). \tag{11.346}$$

The last term is given by the functional renormalization group equation (11.326). We find explicitly

$$\dot{\Gamma}_l^{(2)}(p) = \frac{1}{2}\int_q \dot{\tilde{G}}_l(q)\tilde{\Gamma}_l^{(4)}(q, -q, p, -p). \tag{11.347}$$

The dimensionless single-scale propagator is given by

$$\dot{\tilde{G}}_l(q) = -\frac{c_0\Lambda^3}{Z_l}\dot{G}_\Lambda(\Lambda q). \tag{11.348}$$

Furthermore, we compute

$$\begin{aligned}\partial_l \tilde{\Gamma}_l^{(4)}(q) &= (4 - D - 2\eta_l)\tilde{\Gamma}_l^{(4)}(q) + \Lambda^{D-4}\left(\frac{Z_l}{c_0}\right)^2 \partial_l \tilde{\Gamma}_l^{(4)}(q) \\ &= (4 - D - 2\eta_l)\tilde{\Gamma}_l^{(4)}(q) - q_i\partial_i\tilde{\Gamma}_l^{(4)}(q) + \dot{\Gamma}_l^{(4)}(q).\end{aligned} \tag{11.349}$$

The $\dot{\Gamma}_l^{(4)}(q)$ is given by

$$\dot{\Gamma}_l^{(4)}(q) = -\Lambda^{D-4}\left(\frac{Z_l}{c_0}\right)^2 \Lambda\partial_\Lambda \Gamma_\Lambda^{(4)}(k). \tag{11.350}$$

This is computed from the FRG equation (11.327). We get

$$\begin{aligned}\dot{\Gamma}_l^{(4)}(q_1, \ldots, q_4) = \frac{1}{2}\int_p \dot{\tilde{G}}_\Lambda(p)\tilde{\Gamma}_l^{(6)}(p, -p, q_1, \ldots, q_4) - \int_p \Big[\dot{\tilde{G}}_l(p)\tilde{G}_l(p + q_1 + q_2) \\ \times \tilde{\Gamma}_l^{(4)}(-p - q_1 - q_2, p, q_1, q_2)\tilde{\Gamma}_l^{(4)}(-p, p + q_1 + q_2, q_3, q_4) \\ + (q_2 \leftrightarrow q_3) + (q_2 \leftrightarrow q_4)\Big].\end{aligned} \tag{11.351}$$

11.5.2 The sharp momentum cutoff

In the disordered phase, we can use a multiplicative cutoff given by

$$G_{0,\Lambda}(k) = \theta_\epsilon(|k| - \Lambda)G_0(k),\ G_0(k) = \frac{1}{c_0 k^2}. \tag{11.352}$$

For a sharp momentum cutoff we have

$$\theta_\epsilon(|k| - \Lambda) = \theta(|k| - \Lambda). \tag{11.353}$$

We compute

$$\begin{aligned}\dot{G}_\Lambda(k) &= \frac{\partial_\Lambda G_{0,\Lambda}(k)}{[1 + G_{0,\Lambda}(k)\Sigma_\Lambda(k)]^2} \\ &= -\frac{1}{c_0 k^2}\frac{\delta(|k| - \Lambda)}{\left(1 + \frac{\theta(|k| - \Lambda)}{c_0 k^2}\Sigma_\Lambda(k)\right)^2}.\end{aligned} \tag{11.354}$$

This is an ambiguous expression of the form

$$I(x) = \delta(x) f(\theta(x)). \tag{11.355}$$

In the disordered phase there is no further dependence on the step function in the cutoff dependence of the self-energy $\Sigma_\Lambda(k)$. In other words, the function f is known here. We regularize the integral I as follows

$$I(x) = \lim \delta_\epsilon(x) f(\theta_\epsilon(x)); \epsilon \longrightarrow 0. \tag{11.356}$$

We remark that

$$\begin{aligned}\frac{d}{dx}\int_0^{\theta_\epsilon(x)} dt f(t) &= \lim \frac{1}{\delta x}\int_{\theta_\epsilon(x)}^{\theta_\epsilon(x)+\delta x \delta_\epsilon(x)} dt f(t),\ \delta x \longrightarrow 0 \\ &= \delta_\epsilon(x) f(\theta_\epsilon(x)).\end{aligned} \tag{11.357}$$

In other words,

$$\begin{aligned}I(x) &= \lim \frac{d}{dx}\int_0^{\theta_\epsilon(x)} dt f(t);\ \epsilon \longrightarrow 0 \\ &= \frac{d}{dx}\lim \int_0^{\theta_\epsilon(x)} dt f(t);\ \epsilon \longrightarrow 0 \\ &= \frac{d}{dx}\theta(x)\int_0^1 dt f(t) \\ &= \delta(x)\int_0^1 dt f(t).\end{aligned} \tag{11.358}$$

This is called the Morris lemma. By using this result we obtain

$$\begin{aligned}\dot{G}_\Lambda(k) &= -\frac{\delta(|k| - \Lambda)}{c_0 k^2}\int_0^1 \frac{dt}{\left(1 + t\frac{\Sigma_\Lambda(k)}{c_0 k^2}\right)^2} \\ &= -\frac{\delta(|k| - \Lambda)}{c_0\Lambda^2 + \Sigma_\Lambda(k)}.\end{aligned} \tag{11.359}$$

The single-scale propagator becomes

$$\begin{aligned}\dot{\tilde{G}}_l(q) &= -\frac{c_0\Lambda^3}{Z_l}\dot{G}_\Lambda(\Lambda q)\\ &= \frac{\delta(|q|-1)}{Z_l + \tilde{\Gamma}_l^{(2)}(q)}.\end{aligned} \tag{11.360}$$

The few first terms of the effective action are given by

$$\begin{aligned}\Gamma_\Lambda = \Gamma_\Lambda^{(0)} &+ \int_{k_1}\int_{k_2}\frac{1}{2!}(2\pi)^D\delta^D(k_1+k_2)\Gamma_\Lambda^{(2)}(k_1,k_2)\bar{\phi}(k_1)\bar{\phi}(k_2)\\ &+ \int_{k_1}\cdots\int_{k_4}\frac{1}{4!}(2\pi)^D\delta^D(k_1+\cdots+k_4)\\ &\times\Gamma_\Lambda^{(4)}(k_1,\ldots,k_4)\bar{\phi}(k_1)\ldots\bar{\phi}(k_4)+\cdots\end{aligned} \tag{11.361}$$

By comparing with the classical action (11.316) we obtain

$$\Gamma_\Lambda^{(2)}(0,0) = \Sigma_\Lambda(k=0) = r_\Lambda \tag{11.362}$$

$$\Gamma_\Lambda^{(4)}(0,0,0,0) = u_\Lambda. \tag{11.363}$$

From the formula (11.331) we can define the dimensionless quantities $\bar{r}_l$ and $\bar{u}_l$ by

$$\bar{r}_l = \frac{Z_l}{c_0\Lambda^2}r_\Lambda \equiv \tilde{\Gamma}_l^{(2)}(0) \tag{11.364}$$

$$\bar{u}_l = K_D\Lambda^{D-4}\left(\frac{Z_l}{c_0}\right)^2 u_\Lambda. \tag{11.365}$$

The constant K_D will be determined shortly. From the FRG equations (11.326) and (11.327) we have

$$\partial_\Lambda r_\Lambda = \frac{1}{2}\int_k \dot{G}_\Lambda(k)\Gamma_\Lambda^{(4)}(k,-k,0,0) \tag{11.366}$$

$$\begin{aligned}\partial_\Lambda u_\Lambda = &\frac{1}{2}\int_k \dot{G}_\Lambda(k)\Gamma_\Lambda^{(6)}(k,-k,0,0,0,0)\\ &-3\int_k \dot{G}_\Lambda(k)G_\Lambda(k)\Gamma_\Lambda^{(4)}(-k,k,0,0)\Gamma_\Lambda^{(4)}(0,0,k,-k).\end{aligned} \tag{11.367}$$

By substituting equation (11.359) into (11.366) we get

$$\partial_\Lambda r_\Lambda = -\frac{\Lambda^{D-1}}{2}\int\frac{d\Omega_D}{(2\pi)^D}\frac{1}{c_0\Lambda^2+\Sigma_\Lambda(k)}|_{|k|=\Lambda}\Gamma_\Lambda^{(4)}(k,-k,0,0)|_{|k|=\Lambda}. \tag{11.368}$$

By making the further approximations

$$\Sigma_\Lambda(k) \simeq r_\Lambda + \cdots \tag{11.369}$$

$$\Gamma^{(4)}_\Lambda(k, -k, 0, 0) \simeq u_\Lambda + \cdots, \tag{11.370}$$

we obtain (with $K_D = \int d\Omega_D/(2\pi)^D$)

$$\partial_\Lambda r_\Lambda = -\frac{K_D}{2}\frac{\Lambda^{D-1}}{c_0\Lambda^2 + r_\Lambda}u_\Lambda. \tag{11.371}$$

A very important approximation is to truncate the theory by neglecting the contribution coming from the 6-point vertex. We get then

$$\begin{aligned}\partial_\Lambda u_\Lambda &= -3\int_k \dot{G}_\Lambda(k)G_\Lambda(k)\Gamma^{(4)}_\Lambda(-k, k, 0, 0)\Gamma^{(4)}_\Lambda(0, 0, k, -k)\\ &= -3u_\Lambda^2\int_k \dot{G}_\Lambda(k)G_\Lambda(k)\\ &= -3u_\Lambda^2\int\frac{d^Dk}{(2\pi)^D}\dot{G}_\Lambda(k)\frac{\theta(|k| - \Lambda)}{c_0k^2 + \theta(|k| - \Lambda)r_\Lambda}.\end{aligned} \tag{11.372}$$

Because of the delta function in $\dot{G}_\Lambda(k)$, the integrand is of the form $\delta(x)f(\theta(x))$ so we need to employ the Morris lemma. In order to use this lemma just once, we need to go back to the formula (11.354) for the single-scale propagator $\dot{G}_\Lambda(k)$, and after substitution using the Morris lemma, we get

$$\begin{aligned}\partial_\Lambda u_\Lambda &= 3u_\Lambda^2c_0\Lambda^2\int\frac{d^Dk}{(2\pi)^D}\delta(|k| - \Lambda)\frac{\theta(|k| - \Lambda)}{(c_0k^2 + \theta(|k| - \Lambda)r_\Lambda)^3}\\ &= 3u_\Lambda^2\int\frac{d^Dk}{(2\pi)^D}\delta(|k| - \Lambda)\int_0^1 dt\frac{c_0\Lambda^2t}{(c_0\Lambda^2 + \mathrm{tr}_\Lambda)^3}\\ &= 3u_\Lambda^2\int\frac{d^Dk}{(2\pi)^D}\delta(|k| - \Lambda)\frac{1}{2(c_0\Lambda^2 + r_\Lambda)^2}\\ &= \frac{3K_D}{2}\frac{\Lambda^{D-1}u_\Lambda^2}{(c_0\Lambda^2 + r_\Lambda)^2}.\end{aligned} \tag{11.373}$$

In terms of the dimensionless quantities we obtain the Wilson momentum shell RG equations

$$\partial_l\bar{r}_l = 2\bar{r}_l + \frac{1}{2}\frac{\bar{u}_l}{1 + \bar{r}_l} \tag{11.374}$$

$$\partial_l\bar{u}_l = (4 - D)\bar{u}_l - \frac{3}{2}\frac{\bar{u}_l}{(1 + \bar{r}_l)^2}. \tag{11.375}$$

11.5.3 The ordered phase

The vacuum expectation value of the field (classical field) is defined by

$$\bar{\phi}' = \langle \phi_\alpha \rangle = \frac{\delta \mathcal{G}_{c,\Lambda}}{\delta J_\alpha}. \tag{11.376}$$

In the ordered phase, $\lim\langle\phi\rangle \longrightarrow 0$ when $J \longrightarrow 0$. In the ordered phase, we have instead

$$\lim \bar{\phi}' = \lim\langle\phi\rangle = \frac{\delta \mathcal{G}_{c,\Lambda}}{\delta J_\alpha}|_{J=0} = \bar{\phi}^0 \neq 0;\ J \longrightarrow 0. \tag{11.377}$$

This effect arises in general when spontaneous symmetry breaking occurs. We recall the results

$$\mathcal{L}_\Lambda[\bar{\phi}'] = (J_\Lambda[\bar{\phi}'],\ \bar{\phi}') - \mathcal{G}_{c,\Lambda}[J_\Lambda[\bar{\phi}']],\ J_\Lambda[\bar{\phi}'] \equiv J \tag{11.378}$$

$$\frac{\delta \mathcal{L}_\Lambda[\bar{\phi}']}{\delta \bar{\phi}'} = J_\Lambda[\bar{\phi}']. \tag{11.379}$$

Thus, we see that in the limit $J \longrightarrow 0$ the value $\bar{\phi}' = \bar{\phi}^0$ becomes an extremum of the functional $\mathcal{L}_\Lambda$. The fluctuation field will be defined by

$$\delta\bar{\phi} = \bar{\phi}' - \bar{\phi}^0. \tag{11.380}$$

We recall also the definition of the effective action in the disordered phase, which is the generating functional of the irreducible vertices, given by

$$\Gamma_\Lambda[\bar{\phi}'] = \mathcal{L}_\Lambda[\bar{\phi}'] + \frac{1}{2}\Big(\bar{\phi}',\ G_{0,\Lambda}^{-1}\bar{\phi}'\Big). \tag{11.381}$$

In the ordered phase, we will define this generating functional by [4]

$$\begin{aligned}\Gamma_\Lambda[\delta\bar{\phi},\ \bar{\phi}^0] &= \mathcal{L}_\Lambda[\bar{\phi}'] + \frac{1}{2}\Big(\delta\bar{\phi},\ G_{0,\Lambda}^{-1}\delta\bar{\phi}\Big) \\ &= (J_\Lambda[\bar{\phi}'],\ \bar{\phi}') - \mathcal{G}_{c,\Lambda}[J_\Lambda[\bar{\phi}']] + \frac{1}{2}\Big(\delta\bar{\phi},\ G_{0,\Lambda}^{-1}\delta\bar{\phi}\Big).\end{aligned} \tag{11.382}$$

The irreducible vertices are, therefore, given by

$$\Gamma_\Lambda[\delta\bar{\phi},\ \bar{\phi}^0] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha_1} \cdots \int_{\alpha_n} \Gamma^{(n)}_{\Lambda,\alpha_1\ldots\alpha_n}[\bar{\phi}^0]\delta\bar{\phi}_{\alpha_1} \ldots \delta\bar{\phi}_{\alpha_n}. \tag{11.383}$$

By following the same steps that led to the FRG equation (11.280) we now obtain the FRG equation instead

$$\begin{aligned}\partial_\Lambda \Gamma_\Lambda[\delta\bar{\phi},\,\bar{\phi}^0] = &-\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\left(\frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\mathcal{L}_\Lambda\right)^{-1} + \partial_\Lambda \ln Z_{0,\Lambda}\\ &-\left(\delta\bar{\phi},\,\partial_\Lambda\left[G_{0,\Lambda}^{-1}\bar{\phi}^0\right]\right) - \frac{1}{2}\left(\bar{\phi}^0,\,\partial_\Lambda\left[G_{0,\Lambda}^{-1}\right]\bar{\phi}^0\right).\end{aligned} \tag{11.384}$$

In deriving the above equation we have used the fact that the expectation value $\bar{\phi}^0$ depends on the cutoff and that the RGF equation (11.270) for $\mathcal{G}_{c,\Lambda}[J]$, derived in the disordered phase, is still valid in the ordered phase. Similarly to what we have done in the disordered phase, we define the self-energy Σ_Λ and the U_Λ matrix by

$$\Sigma_\Lambda[\bar{\phi}^0] = \frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda[\delta\bar{\phi},\,\bar{\phi}^0]|_{\delta\bar{\phi}=0} \tag{11.385}$$

$$U_\Lambda[\delta\bar{\phi},\,\bar{\phi}^0] = \frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda[\delta\bar{\phi},\,\bar{\phi}^0] - \frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda[\delta\bar{\phi},\,\bar{\phi}^0]|_{\delta\bar{\phi}=0}. \tag{11.386}$$

By going through the same steps which led to equation (11.295) we obtain now the modified FRG equation

$$\begin{aligned}\partial_\Lambda \Gamma_\Lambda[\bar{\phi}'] = &-\frac{1}{2}\,\mathrm{Tr}\,(\dot{G}_{0,\Lambda}\Sigma_\Lambda[1 - G_{0,\Lambda}\Sigma_\Lambda]^{-1} + \dot{G}_\Lambda U_\Lambda[\bar{\phi}'][1 - G_\Lambda U_\Lambda[\bar{\phi}']]^{-1})\\ &-\left(\delta\bar{\phi},\,\partial_\Lambda\left[G_{0,\Lambda}^{-1}\bar{\phi}^0\right]\right) - \frac{1}{2}\left(\bar{\phi}^0,\,\partial_\Lambda\left[G_{0,\Lambda}^{-1}\right]\bar{\phi}^0\right).\end{aligned} \tag{11.387}$$

The corresponding FRG equations of the irreducible vertices, which generalize equations (11.304) and (11.306), are given now, respectively, by

$$\begin{aligned}\partial_\Lambda \Gamma_\Lambda^{(0)} = &-\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_{0,\Lambda}\Sigma_\Lambda[1 - G_{0,\Lambda}\Sigma_\Lambda]^{-1}\\ &+\int_\alpha \Gamma_{\Lambda,\alpha}^{(1)}[\bar{\phi}^0]\partial_\Lambda\bar{\phi}_\alpha^0 - \frac{1}{2}\left(\bar{\phi}^0,\,\partial_\Lambda G_{0,\Lambda}^{-1}\bar{\phi}^0\right)\end{aligned} \tag{11.388}$$

$$\begin{aligned}\partial_\Lambda \Gamma_{\Lambda,\alpha_1\ldots\alpha_n}^{(n)} = &-\frac{1}{2}\sum_{\nu=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_\nu=1}^{\infty}\delta_{n,n_1+\cdots+n_\nu}\\ &\times \mathcal{S}_{\alpha_1\ldots\alpha_{n_1};\,\ldots\,;\alpha_{n-n_\nu+1}\ldots\alpha_n}\,\mathrm{Tr}\,\dot{G}_\Lambda \boldsymbol{\Gamma}_{\Lambda\alpha_{n-n_\nu+1}\ldots\alpha_n}^{(n_\nu+2)}\\ &\times G_\Lambda \boldsymbol{\Gamma}_{\Lambda\alpha_{n-n_\nu-n_{\nu-1}+1}\ldots\alpha_{n-n_\nu}}^{(n_{\nu-1}+2)}\cdots\\ &\times G_\Lambda \boldsymbol{\Gamma}_{\Lambda\alpha_{n_1+1}\ldots\alpha_{n_1+n_2}}^{(n_2+2)} G_\Lambda \boldsymbol{\Gamma}_{\Lambda\alpha_1\ldots\alpha_{n_1}}^{(n_1+2)}\\ &+\int_\alpha \Gamma_{\Lambda,\alpha\alpha_1\ldots\alpha_n}^{(n+1)}[\bar{\phi}^0]\partial_\Lambda\bar{\phi}_\alpha^0.\end{aligned} \tag{11.389}$$

As opposed to the case of the disordered phase, in the ordered phase the irreducible vertices with odd numbers of external legs do not vanish. In particular, $\Gamma^{(1)}_{\Lambda,\alpha_1}[\bar{\phi}^0]$ does not vanish. The corresponding flow equation is found to be given by

$$\begin{aligned}\partial_\Lambda \Gamma^{(1)}_{\Lambda\alpha_1}[\bar{\phi}^0] = &-\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(3)}_{\Lambda,\alpha_1}[\bar{\phi}^0] + \int_\alpha \Gamma^{(2)}_{\Lambda,\alpha\alpha_1}[\bar{\phi}^0]\partial_\Lambda \bar{\phi}^0_\alpha \\ &- \int_\alpha \Big[G^{-1}_{0,\Lambda}\Big]_{\alpha_1\alpha}\partial_\Lambda \bar{\phi}^0_\alpha - \int_\alpha \Big[\partial_\Lambda G^{-1}_{0,\Lambda}\Big]_{\alpha_1\alpha}\bar{\phi}^0_\alpha .\end{aligned} \tag{11.390}$$

As it turns out, it is useful to set the 1-point vertex to zero even in the ordered phase. To this end, we choose the regularized inverse free propagator, by adding appropriate counter terms, such that

$$G^{-1}_{0,\Lambda}\partial_\Lambda \bar{\phi}^0 = 0,\ \partial_\Lambda G^{-1}_{0,\Lambda}\bar{\phi}^0 = 0. \tag{11.391}$$

The FRG equations for the 0-point and the 1-point vertices become then

$$\partial_\Lambda \Gamma^{(0)}_\Lambda = -\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_{0,\Lambda}\Sigma_\Lambda[1 - G_{0,\Lambda}\Sigma_\Lambda]^{-1} + \int_\alpha \Gamma^{(1)}_{\Lambda,\alpha}[\bar{\phi}^0]\partial_\Lambda \bar{\phi}^0_\alpha \tag{11.392}$$

$$\partial_\Lambda \Gamma^{(1)}_{\Lambda\alpha_1}[\bar{\phi}^0] = -\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(3)}_{\Lambda,\,\alpha_1}[\bar{\phi}^0] + \int_\alpha \Gamma^{(2)}_{\Lambda,\alpha\alpha_1}[\bar{\phi}^0]\partial_\Lambda \bar{\phi}^0_\alpha . \tag{11.393}$$

The flow of the vacuum expectation value $\bar{\phi}^0$ is fixed such that the 1-point vertex vanishes identically. In other words, we have

$$\Gamma^{(1)}_{\Lambda\alpha_1}[\bar{\phi}^0] = 0. \tag{11.394}$$

Equivalently,

$$\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_\Lambda \mathbf{\Gamma}^{(3)}_{\Lambda,\alpha_1}[\bar{\phi}^0] = \int_\alpha \Gamma^{(2)}_{\Lambda,\alpha\alpha_1}[\bar{\phi}^0]\partial_\Lambda \bar{\phi}^0_\alpha . \tag{11.395}$$

This equation determines the dependence of the vacuum expectation value on the cutoff, viz $\bar{\phi}^0 = \bar{\phi}^0(\Lambda)$. The flow equation of the free energy is now given by the same formula as in the disordered phase, i.e.

$$\partial_\Lambda \Gamma^{(0)}_\Lambda = -\frac{1}{2}\,\mathrm{Tr}\,\dot{G}_{0,\Lambda}\Sigma_\Lambda[1 - G_{0,\Lambda}\Sigma_\Lambda]^{-1}. \tag{11.396}$$

We consider now as before ϕ^4 in D dimension. In other words, we will make the replacements $\alpha \longrightarrow k$, $\phi_\alpha \longrightarrow \phi(k)$ and $\int_\alpha \longrightarrow \int_k = d^Dk/(2\pi)^D$. We will also replace $\bar{\phi}^0_\alpha \longrightarrow \bar{\phi}^0(k)$. We will assume further that the expectation value $\bar{\phi}^0$ is uniform, i.e.

$$\bar{\phi}^0(k) = (2\pi)^D \delta^D(k)\bar{\phi}^0_\Lambda . \tag{11.397}$$

By using also $(G_\Lambda)_{k_1k_2} = -(2\pi)^D\delta^D(k_1 + k_2)G_\Lambda(k_1)$, $\Gamma^{(3)}_{\Lambda,k_1k_2p} = (2\pi)^D\delta^D(k_1 + k_2 + p)\,\Gamma^{(3)}_\Lambda(k_1, k_2, p)$, etc, we can show that the condition (11.396) takes the form

$$\Gamma^{(2)}_{\Lambda}(-p,\, p)(2\pi)^D\delta^D(p)\partial_\Lambda\bar{\phi}^{\,0}_{\Lambda} = -\frac{1}{2}\int_k \dot{G}_\Lambda(k)(2\pi)^D\delta^D(p)\Gamma^{(3)}_{\Lambda}(k,\, -k,\, p). \tag{11.398}$$

Equivalently,

$$\Sigma_\Lambda(0)\partial_\Lambda\bar{\phi}^{\,0}_{\Lambda} = -\frac{1}{2}\int_k \dot{G}_\Lambda(k)\Gamma^{(3)}_{\Lambda}(k,\, -k,\, 0). \tag{11.399}$$

The flow equation of the 2-point vertex (self-energy) can be derived from the FRG equation (11.389). This is a generalization of the FRG equation (11.312) or equivalently the FRG equation (11.326) (see also the FRG equation (11.311)). We get

$$\begin{aligned}\partial_\Lambda\Sigma_\Lambda(p) = {} & \frac{1}{2}\int_k \dot{G}_\Lambda(k)\Gamma^{(4)}_{\Lambda}(k,\, -k,\, p,\, -p) + \partial_\Lambda\bar{\phi}^{\,0}_{\Lambda}\Gamma^{(3)}_{\Lambda}(k-k,\, 0) \\ & - \int_{k'} \dot{G}_\Lambda(k')G_\Lambda(k'+k)\Gamma^{(3)}_{\Lambda}(k,\, -k,\, -k',\, k')\Gamma^{(3)}_{\Lambda}(-k',\, k+k',\, -k).\end{aligned} \tag{11.400}$$

The rescaled vacuum expectation value M_l is defined by

$$\Lambda\partial_\Lambda\bar{\phi}^{\,0}_{\Lambda} = \Lambda^{\frac{D}{2}-1}\left(\frac{Z_l}{c_0}\right)^{1/2}\left[\frac{D-2+\eta_l}{2} - \partial_l\right]M_l. \tag{11.401}$$

By now using equation (11.331) we have the rescaled objects

$$\tilde{\Gamma}^{(2)}_{l}(q) = \frac{Z_l}{c_0\Lambda^2}\Sigma_\Lambda(\Lambda q) \tag{11.402}$$

$$\tilde{\Gamma}^{(3)}_{l}(q_1,\, \ldots,\, q_3) = \Lambda^{\frac{D}{2}-3}\left(\frac{Z_l}{c_0}\right)^{3/2}\Gamma^{(3)}_{\Lambda}(\Lambda q_1,\, \ldots,\, \Lambda q_3). \tag{11.403}$$

Also we use equation (11.348) defining the single-scale propagator. The condition (11.399) becomes

$$\Lambda\partial_\Lambda\bar{\phi}^{\,0}_{\Lambda} = \Lambda^{\frac{D}{2}-1}\left(\frac{Z_l}{c_0}\right)^{1/2}\frac{1}{2}\int_q \dot{\tilde{G}}_l(q)\frac{\tilde{\Gamma}^{(3)}_{l}(q,\, -q,\, 0)}{\tilde{\Gamma}^{(2)}_{l}(0)}. \tag{11.404}$$

By comparing equations (11.401) and (11.404) we obtain the final result

$$\partial_l M_l = \left[\frac{D-2+\eta_l}{2}\right]M_l - \frac{1}{2}\int_q \dot{\tilde{G}}_l(q)\frac{\tilde{\Gamma}^{(3)}_{l}(q,\, -q,\, 0)}{\tilde{\Gamma}^{(2)}_{l}(0)}. \tag{11.405}$$

As in the disordered phase, the renormalization group flow equation of the dimensionless 2-point vertex $\tilde{\Gamma}^{(2)}_{l}(q)$ is given by the equation

$$\partial_l \tilde{\Gamma}_l^{(2)}(q) = (2 - \eta_l - q\partial_q)\tilde{\Gamma}_l^{(2)}(q) + \dot{\Gamma}_l^{(2)}(q), \tag{11.406}$$

where

$$\dot{\Gamma}_l^{(2)}(q) \equiv -\frac{Z_l}{c_0 \Lambda}\partial_\Lambda \Sigma_\Lambda(k). \tag{11.407}$$

Now this last quantity is given explicitly from the FRG equation (11.400) by the result

$$\begin{aligned}\dot{\Gamma}_l^{(2)}(q) \;=\; & \frac{1}{2}\int_p \dot{\tilde{G}}_l(p)\tilde{\Gamma}_l^{(4)}(p, -p, q, -q) - \tilde{\Gamma}_l^{(3)}(q, -q, 0) \\ & \times\left[\partial_l - \frac{D - 2 + \eta_l}{2}\right]M_l \\ & - \int_p \dot{\tilde{G}}_l(p)\tilde{G}_l(p + q)\tilde{\Gamma}_l^{(3)}(q, -q - p, p)\tilde{\Gamma}_l^{(3)}(-p, q + p, -q).\end{aligned} \tag{11.408}$$

This is a generalization of the FRG equation (11.347). Obviously, we can use equation (11.405) to eliminate M_l from equation (11.408).

11.6 The gradient expansion approximation method

11.6.1 Effective potential

We will consider the $O(N)$ sigma model

$$S[\phi] = \int d^D x\left[\frac{r_0}{2}\phi_a^2 + \frac{c_0}{2}(\partial_\mu \phi_a)^2 + \frac{u_0}{4!}\left(\phi_a^2\right)^2\right]. \tag{11.409}$$

The average effective action or the Wetterich Legendre effective action is defined by

$$\Gamma_\Lambda^W[\bar{\phi}'] = \Gamma_\Lambda[\bar{\phi}'] - \frac{1}{2}(\bar{\phi}',\, G_0^{-1}\bar{\phi}') - \ln Z_{0,\Lambda}. \tag{11.410}$$

We will employ an additive regularization R_Λ given by

$$G_{0,\Lambda}^{-1} = G_0^{-1} - R_\Lambda. \tag{11.411}$$

In order that $G_{0,\Lambda} \longrightarrow G_0$ when $\Lambda \longrightarrow 0$, and $G_{0,\Lambda} \longrightarrow 0$ when $\Lambda \longrightarrow \infty$, we must demand that

$$|R_\Lambda| \sim 0,\ \Lambda \longrightarrow 0;\ |R_\Lambda| \sim \infty,\ \Lambda \longrightarrow \infty. \tag{11.412}$$

By comparing with the multiplicative regularization θ_Λ given by $G_{0,\Lambda} = \theta_\Lambda G_0$ we see that $\theta_\Lambda = (1 - R_\Lambda G_0)^{-1}$. By using this additive regularization we can rewrite the Wetterich Legendre effective action as

$$\Gamma_\Lambda^W[\bar{\phi}'] = \mathcal{L}_\Lambda[\bar{\phi}'] - \frac{1}{2}(\bar{\phi}',\, R_\Lambda\bar{\phi}') - \ln Z_{0,\Lambda}. \tag{11.413}$$

Thus, in the limit $\Lambda \longrightarrow 0$ we have the behavior $\Gamma_\Lambda^W \longrightarrow \mathcal{L}$ and thus the name Legendre effective action. Furthermore, from the limiting behavior $\Gamma_\Lambda \longrightarrow S_1$ when $\Lambda \longrightarrow \Lambda_0 = \infty$, we obtain the initial condition

$$\Gamma_\Lambda^W = S,\ \Lambda \longrightarrow \Lambda_0, \tag{11.414}$$

where S is the above bare action. The Wetterich Legendre effective action satisfies the FRG equation

$$\begin{aligned}\partial_\Lambda \Gamma_\Lambda^W[\bar{\phi}'] &= -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda G_{0,\Lambda}^{-1}\left(\frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda - G_{0,\Lambda}^{-1}\right)^{-1}\\ &= -\frac{1}{2}\,\mathrm{Tr}\,\partial_\Lambda R_\Lambda\left(\frac{\delta}{\delta\bar{\phi}'}\otimes\frac{\delta}{\delta\bar{\phi}'}\Gamma_\Lambda^W + R_\Lambda\right)^{-1}\\ &= -\frac{1}{2}\,\mathrm{Tr}\,\frac{\partial_\Lambda R_\Lambda}{\Gamma_\Lambda^{W(2)} + R_\Lambda}.\end{aligned} \tag{11.415}$$

The classical field is defined by $\langle\phi\rangle = \bar{\phi}'$ and it does not depend on Λ. In the limit of vanishing external source $J \equiv J_\Lambda \longrightarrow 0$ it may acquire a non-zero expectation value due to spontaneous symmetry breaking, i.e. $\lim \bar{\phi}' \longrightarrow \bar{\phi}_0$, $J \longrightarrow 0$, and it becomes given by the equation

$$\frac{\delta\mathcal{L}_\Lambda[\bar{\phi}']}{\delta\bar{\phi}'} = 0 = \frac{\delta\Gamma_\Lambda^W[\bar{\phi}']}{\delta\bar{\phi}'} + R_\Lambda\bar{\phi}'. \tag{11.416}$$

$\bar{\phi}' = \bar{\phi}_0$ is a solution of this equation. Since we have the limit $R_\Lambda \longrightarrow 0$ when $\Lambda \longrightarrow 0$ we see that $\bar{\phi}_0$ is a minimum of the Wetterich Legendre effective action Γ_Λ^W. The minimum of the Wetterich action for any Λ will be denoted by $\bar{\phi}_{0,\Lambda}$. Since in the limit $\Lambda \longrightarrow \Lambda_0$ in which $\Gamma_\Lambda^W \longrightarrow S$ we have $\bar{\phi}_{0,\Lambda} \longrightarrow \bar{\phi}_0^{MF}$ where $\bar{\phi}_0^{MF}$ is the mean-field value. Hence, $\bar{\phi}_{0,\Lambda}$ interpolates between the classical value $\bar{\phi}_0^{MF}$ and the quantum value $\bar{\phi}_0$ which contains all fluctuations.

In the derivative or gradient expansion method, we expand the Wetterich Legendre effective action Γ_Λ^W in terms of the derivatives of the field. Clearly, this should consist of a good approximation in the low energy limit. In this expansion, the first term will obviously contain no derivatives. We write it as (in the following we will omit for simplicity the prime on the field)

$$\Gamma_\Lambda^W[\bar{\phi}] = \int d^D x U_\Lambda(\rho(x)) + \cdots \tag{11.417}$$

In the above equation ρ is the field density defined by

$$\rho = \frac{1}{2}\bar{\phi}_i^{\,2}. \tag{11.418}$$

Obviously, for a uniform configuration we obtain exactly

$$\Gamma^W_\Lambda[\bar{\phi}] = V U_\Lambda(\rho). \tag{11.419}$$

In other words, $U_\Lambda[\bar{\phi}]$ is the effective potential. Thus, $\bar{\phi}_{0,\Lambda}$ is the minimum of the effective potential $U_\Lambda[\bar{\phi}]$.

In terms of $U_\Lambda[\bar{\phi}]$ the above FRG equation (11.415) becomes

$$V\partial_\Lambda U_\Lambda[\bar{\phi}] = -\frac{1}{2}\,\mathrm{Tr}\,\frac{\partial_\Lambda R_\Lambda}{\Gamma^{W(2)}_\Lambda + R_\Lambda}. \tag{11.420}$$

We use $\langle k'|R_\Lambda|k\rangle = R_\Lambda(k)(2\pi)^D\delta^D(k+k')$ and $V = (2\pi)^D\delta^D(0)$ and $\left(G^{-1}_{0,\Lambda}\right)_{kk'} = -(2\pi)^D\delta^D(k+k')G^{-1}_{0,\Lambda}(k)$. Also, we use $G^{-1}_\Lambda = G^{-1}_{0,\Lambda} - \Gamma^{(2)}_\Lambda$ to derive $G^{-1}_\Lambda(k) = \Gamma^{W(2)}_\Lambda(k) + R_\Lambda(k)$. We get

$$\partial_\Lambda U_\Lambda[\bar{\phi}] = \frac{1}{2}\sum_{i=1}^{N}\int_k \partial_\Lambda R_\Lambda(k) G_{i,\Lambda}(\bar{\phi}, k), \tag{11.421}$$

where

$$G_{i,\Lambda}(\bar{\phi}, k) = \left(\frac{1}{V}\frac{\delta^2}{\delta\bar{\phi}_i(k)\delta\bar{\phi}_i(-k)}\Gamma^W_\Lambda|_{\bar{\phi}(x)=\bar{\phi}} + R_\Lambda(k)\right)^{-1}. \tag{11.422}$$

The most general expression of the Wetterich Legendre effective action Γ^W_Λ, which includes up to the quartic term in the derivatives of the field, is of the form

$$\begin{aligned}\Gamma^W_\Lambda[\bar{\phi}] = &\int d^D x \\ &\times\left[U_\Lambda(\rho(x)) + \frac{c_0}{2}Z^{-1}_\Lambda(\rho(x))(\partial_\mu\phi_i)^2 + \frac{c_0}{4}Y_\Lambda(\rho(x))(\partial_\mu\rho(x))^2 + \cdots\right].\end{aligned} \tag{11.423}$$

We can compute the functional derivative

$$\frac{\delta^2}{\delta\bar{\phi}_i(k)\delta\bar{\phi}_i(-k)}\Gamma^W_\Lambda|_{\bar{\phi}(x)=\bar{\phi}} = V\left(U'_\Lambda + \bar{\phi}_i{}^2 U''_\Lambda + c_0 k^2 Z^{-1}_\Lambda + \frac{c_0}{2}k^2\bar{\phi}_i{}^2 Y_\Lambda\right). \tag{11.424}$$

By choosing $O(N)$ invariance we can always choose $\bar{\phi}$ in such a way that the only component is non-zero, say $\bar{\phi}_1$, and thus we have

$$\bar{\phi} = \begin{pmatrix}\bar{\phi}_1\\ 0\\ \vdots\\ 0\end{pmatrix}. \tag{11.425}$$

In other words, $\rho = \bar{\phi}_1{}^2/2$. The longitudinal and transverse propagators become, therefore, given by

$$G_{l,\Lambda}^{-1}(\rho, k^2) \equiv G_{1,\Lambda}^{-1} = U'_\Lambda + 2\rho U''_\Lambda + c_0(Z_\Lambda^{-1} + \rho^2 Y_\Lambda)k^2 + R_\Lambda \tag{11.426}$$

$$G_{t,\Lambda}^{-1}(\rho, k^2) \equiv G_{i,\Lambda}^{-1} = U'_\Lambda + c_0 Z_\Lambda^{-1} k^2 + R_\Lambda,\ i = 2, \ldots, N. \tag{11.427}$$

Thus, the above FRG equation becomes

$$\begin{aligned}\partial_\Lambda U_\Lambda[\bar{\phi}] &= \frac{1}{2}\int_k \partial_\Lambda R_\Lambda(k)(G_{l,\Lambda}(\rho, k^2) + (N-1)G_{t,\Lambda}(\rho, k^2)) \\ &= \frac{K_D}{2}\int_0^\infty k^{D-1} dk \partial_\Lambda R_\Lambda(k)(G_{l,\Lambda}(\rho, k^2) + (N-1)G_{t,\Lambda}(\rho, k^2)).\end{aligned} \tag{11.428}$$

Clearly, in order to solve the above equation, one needs also to determine the flow equations of the wavefunction renormalization Z_Λ and of Y_Λ. We will employ the local potential approximation to solve this equation in which we set these factors equal to their initial classical values $Z_{\Lambda_0} = 1$ and $Y_{\Lambda_0} = 0$. This allows us then to close the hierarchy of flow equations. Indeed, in this approximation we will only need to deal with the flow equation of the effective potential which depends only on the effective potential itself and nothing else. In this approximation the Wetterich Legendre effective action becomes

$$\Gamma_\Lambda^W[\bar{\phi}] = \int d^D x \left[U_\Lambda(\rho(x)) + \frac{c_0}{2}(\partial_\mu \phi_i)^2 + \cdots \right]. \tag{11.429}$$

The initial classical value of the effective potential is, therefore, given by the classical potential, viz

$$\begin{aligned}U_{\Lambda_0}(\rho) &= f_0 + \frac{r_0}{2}\phi_i^2 + \frac{u_0}{4!}\left(\phi_i^2\right) \\ &= \frac{u_0}{6}(\rho - \rho_0)^2.\end{aligned} \tag{11.430}$$

In the above equation we have chosen $f_0 = 3r_0^2/2u_0$ and thus $\rho_0 = -3r_0/u_0$. In the following we will use the Litim regulator given by [15]

$$R_\Lambda(k) = c_0(\Lambda^2 - k^2)\theta(\Lambda^2 - k^2). \tag{11.431}$$

Obviously, for $k \leqslant \Lambda$ we have $R_\Lambda(k) = c_0(\Lambda^2 - k^2)$, and hence the longitudinal and transverse propagators become k-independent given by

$$G_{l,\Lambda}^{-1}(\rho, k^2) = U'_\Lambda + 2\rho U''_\Lambda + c_0\Lambda^2 \tag{11.432}$$

$$G_{t,\Lambda}^{-1}(\rho, k^2) = U'_\Lambda + c_0\Lambda^2. \tag{11.433}$$

We also compute $\partial_\Lambda R_\Lambda(k) = 2c_0\Lambda\theta(\Lambda^2 - k^2)$ and as a consequence the integral over k reduces to $\int_0^\infty k^{D-1}dk(2c_0\Lambda\theta(\Lambda^2 - k^2)(\ldots) = 2c_0\Lambda^{D+1}(\ldots)/D$. The flow equation becomes

$$\partial_\Lambda U_\Lambda(\rho) = \frac{c_0 K_D \Lambda^{D+1}}{D}\left(\frac{1}{U'_\Lambda + 2\rho U''_\Lambda + c_0\Lambda^2} + \frac{N-1}{U'_\Lambda + c_0\Lambda^2}\right). \tag{11.434}$$

This can be solved numerically for fixed values of the dimensionless parameters $\bar{u}_0 = u_0/c_0^2\Lambda$, $\bar{\rho}_0 = c_0\rho_0/\Lambda$ for various values of the cutoff Λ. It is useful to shift the potential by its running minimum, viz $U_\Lambda(\rho) \longrightarrow U_\Lambda(\rho) - U_\Lambda(\rho_0)$, measure it in units of f_0, and plot it as a function of $\bar{\phi}/\bar{\phi}^{MF}$. We will be interested in the behavior of the minimum $\bar{\phi}_{0,\Lambda}$ in the continuum limit $\Lambda \longrightarrow 0$ and whether or not spontaneous symmetry breaking occurs. In the ordered phase we observe that $\bar{\phi}_{0,\Lambda}$ approaches a non-zero value in the limit $\Lambda \longrightarrow 0$, whereas in the disordered phase it will approach a zero value.

11.6.2 Fixed points and critical exponents

In order to discuss fixed points and critical exponents we introduce the running cutoff $l = \Lambda_0 \exp(-l)$ and the following dimensionless quantities

$$q = k/\Lambda,\ \tilde{x} = \Lambda x,\ \tilde{\phi}(\tilde{x}) = \sqrt{c_0}\Lambda^{(2-D)/2}\bar{\phi}(x),\ \tilde{\rho}(x) = c_0\Lambda^{2-D}\rho(x). \tag{11.435}$$

The dimensionless potential is therefore

$$\tilde{U}_l(\tilde{\phi}) = \frac{U_\Lambda(\bar{\phi})}{\Lambda^D} = \tilde{f}_0 + \frac{\tilde{r}_l}{2}\tilde{\phi}_i^{\,2} + \frac{\tilde{u}_l}{4!}\left(\tilde{\phi}_i^{\,2}\right)^2 + \cdots \tag{11.436}$$

The definition of the dimensionless parameters $\tilde{f}_l$, $\tilde{r}_l$ and $\tilde{u}_l$ is obvious. By substituting in the above FRG equation we obtain

$$\partial_l \tilde{U}_l(\tilde{\rho}) = D\tilde{U}_l(\tilde{\rho}) - (D-2)\tilde{\rho}\,\tilde{U}'_l(\tilde{\rho}) - \frac{K_D}{D}\left(\frac{1}{1 + \tilde{U}'_l + 2\tilde{\rho}\,\tilde{U}''_l} + \frac{N-1}{1+\tilde{U}'_l}\right). \tag{11.437}$$

We introduce the function

$$\tilde{W}_l(\tilde{\rho}) = \tilde{U}'_l(\tilde{\rho}). \tag{11.438}$$

In terms of this new function, the above FRG equation becomes

$$\begin{aligned}\partial_l \tilde{W}_l(\tilde{\rho}) &= 2\tilde{W}_l(\tilde{\rho}) - (D-2)\tilde{\rho}\,\tilde{W}'_l(\tilde{\rho}) \\ &+ \frac{K_D}{D}\left(\frac{3\tilde{W}'_l + 2\tilde{\rho}\,\tilde{W}''_l}{[1 + \tilde{W}_l + 2\tilde{\rho}\,\tilde{W}'_l]^2} + \frac{(N-1)\tilde{W}'_l}{[1+\tilde{W}_l]^2}\right).\end{aligned} \tag{11.439}$$

The fixed points are solutions of the equation

$$\partial_l \tilde{U}_\Lambda(\tilde{\rho}_*) = 0 \Rightarrow \partial_l \tilde{W}_l(\tilde{\rho}_*) = 0. \tag{11.440}$$

It is well established that there are three fixed points in the $O(N)$ sigma model in D dimensions which are

- The ferromagnetic low-temperature fixed point $\mathcal{F}$ in which the $O(N)$ symmetry is spontaneously broken. In this case, we have a finite vacuum expectation value $\bar{\phi}_0$, and as a consequence, the rescaled vacuum expectation value $\tilde{\phi}_0$ diverges with the running cutoff exponentially, viz

$$\tilde{\phi}_0 = \sqrt{c_0}\bar{\phi}_0\Lambda_0^{-\frac{D-2}{2}}e^{\frac{D-2}{2}l} \longrightarrow \infty,\ l \longrightarrow \infty. \tag{11.441}$$

  $\bar{\phi}_0$ is the spontaneous magnetization.
- The paramagnetic high-temperature fixed point $\mathcal{P}$ in which the $O(N)$ symmetry is not broken. In this case, both the vacuum expectation value $\bar{\phi}_0$ and the rescaled vacuum expectation value $\tilde{\phi}_0$ vanish. The vanishing of the vacuum expectation value $\bar{\phi}_0$ is driven by fluctuations and it occurs at a value of the cutoff $\Lambda_* > 0$. This gives a measure of the critical length $1/\xi$ of the system.
- The Wilson–Fisher fixed critical point $\mathcal{C}$. This is given by the solution of the equation $\partial_l \tilde{W}_l(\tilde{\rho}_*) = 0$ such that $\tilde{W}_l(\tilde{\rho}_*) \neq 0$ which should be contrasted with the Gaussian trivial fixed point for which the solution is such that $\tilde{W}_l(\tilde{\rho}_*) = 0$. In this case the rescaled vacuum expectation value $\tilde{\phi}_0$ is finite in the limit $\Lambda \longrightarrow 0$, whereas the vacuum expectation value $\bar{\phi}_0$ goes to zero as

$$\bar{\phi}_0 = \frac{1}{\sqrt{c_0}}\tilde{\phi}_0\Lambda^{\frac{D-2}{2}} \longrightarrow 0,\ \Lambda \longrightarrow 0. \tag{11.442}$$

There are six critical exponents related by the so-called scaling relations and thus only two of them are linearly independent. We take them to be the anomalous dimension η and the mass critical exponent ν. In the local potential approximation, $\eta = 0$, since the wavefunction renormalization is set equal to 1. It remains, therefore, to calculate the mass critical exponent ν. This can be extracted from the behavior of the critical length as a function of the mass parameter (temperature) given by

$$\Lambda_0\xi \sim |\tilde{r} - \tilde{r}_0^*|^{-\nu}. \tag{11.443}$$

We consider the matrix ϕ^4 action

$$S[\phi] = \int d^D x\, \mathrm{Tr}\left[\frac{r_0}{2}\phi^2 + \frac{c_0}{2}(\partial_\mu\phi)^2 + \frac{u_0}{4!}(\phi^2)^2\right]. \tag{11.444}$$

The path integral is

$$Z = \int [d\phi]\exp(-S(\phi)). \tag{11.445}$$

By diagonalization $\phi = U\psi U^{-1}$ we obtain the path integral

$$Z = \int [d\psi]\exp(-\tilde{S}(\psi)). \tag{11.446}$$

The effective action is given by

$$\tilde{S}(\psi) = S(\psi) - \int d^D x \sum_{i<j} \ln(\psi_i - \psi_j)^2 + F(\psi), \tag{11.447}$$

where the second term comes from the Vandermonde determinant, and $F(\psi)$ comes from the integration over the $SU(N)$ group, viz

$$\exp(-F(\psi)) = \int [dU] \exp\left(-\frac{c_0}{2} \int d^D x \, \mathrm{Tr}\, ([U^{-1}\partial_\mu U, \psi]^2)\right). \tag{11.448}$$

The free action and the interaction part are given by

$$\tilde{S}_0(\psi) = \int d^D x \, \mathrm{Tr}\, \frac{c_0}{2} (\partial_\mu \psi)^2 = -\frac{1}{2}(\psi, G_0^{-1}\psi) \tag{11.449}$$

$$(G_0^{-1})_{kk'} = -c_0 k^2 (2\pi)^D \delta^D(k + k')\delta_{ij}. \tag{11.450}$$

The Wetterich equation in this case is given by

$$\frac{1}{V}\partial_\Lambda \Gamma_\Lambda^W[\bar{\psi}] = \frac{1}{2}\sum_{i=1}^{N} \int_k \partial_\Lambda R_\Lambda(k) G_{i,\Lambda}(\bar{\psi}, k), \tag{11.451}$$

where

$$G_{i,\Lambda}(\bar{\psi}, k) = \left(\frac{1}{V}\frac{\delta^2}{\delta\bar{\psi}_i(k)\delta\bar{\psi}_i(-k)}\Gamma_\Lambda^W|_{\bar{\psi}(x)=\bar{\psi}} + R_\Lambda(k)\right)^{-1}. \tag{11.452}$$

The initial condition is

$$\Gamma_\Lambda^W \longrightarrow \tilde{S}, \, \Lambda \longrightarrow \Lambda_0. \tag{11.453}$$

We assume the Wetterich effective action to be of the form

$$\Gamma_\Lambda^W[\bar{\psi}] = \int d^D x \, \mathrm{Tr}\left[U_\Lambda(\bar{\psi}(x)) + \frac{c_0}{2} Z_\Lambda^{-1}(\bar{\psi})(\partial_\mu \bar{\psi})^2 + \cdots\right]. \tag{11.454}$$

The initial condition evaluated on $\bar{\psi}(x) = \bar{\psi}$ becomes

$$U_\Lambda(\bar{\psi}) \longrightarrow \mathrm{Tr}\left(\frac{r_0}{2}\bar{\psi}^2 + \frac{u_0}{4!}(\bar{\psi}^2)^2\right) - \sum_{i<j} \ln(\bar{\psi}_i - \bar{\psi}_j) + \frac{F}{V}, \, \Lambda \longrightarrow \Lambda_0. \tag{11.455}$$

We compute

$$\begin{aligned} G^{-1}_{ij,ij,\Lambda}(\bar{\phi}, k) = {} & U'_{\Lambda,ij}\delta_{ij} + \frac{1}{4}\bar{\phi}_{ji}\{\bar{\phi}, U''_\Lambda\}_{ji} + \frac{1}{2}\rho_{ji} U''_{\Lambda,ji} \\ & + \frac{1}{4}\left(\phi U''_\Lambda \phi\right)_{ji}\delta_{ij} + c_0 k^2 Z^{-1}_{\Lambda,ii}\delta_{ij} + R_\Lambda(k). \end{aligned} \tag{11.456}$$

The expectation value $\bar{\phi} = \langle \phi \rangle$ was assumed to be uniform in space. The analogous assumption in matrix space is to take $\bar{\phi}$ as a diagonal matrix given by

$$\bar{\phi}_i = \lambda_i \delta_{ij}. \tag{11.457}$$

In this case we obtain

$$G^{-1}_{ij,ij,\Lambda}(\bar{\phi}, k) = \left(U'_{\Lambda,i} + 2\rho_i U''_{\Lambda,i} + c_0 k^2 Z^{-1}_{\Lambda,i} \right) \delta_{ij} + R_\Lambda(k). \tag{11.458}$$

On general ground, we know that $G^{-1}_{ij,kl} \sim \delta_{ij,kl}$, and thus

$$G_{ij,ij,\Lambda}(\bar{\phi}, k) = \frac{1}{\left(U'_{\Lambda,i} + 2\rho_i U''_{\Lambda,i} + c_0 k^2 Z^{-1}_{\Lambda,i} \right) \delta_{ij} + R_\Lambda(k)}. \tag{11.459}$$

11.7 Polchinski's renormalization of ϕ^4 theory

One of the most fundamental applications of the functional renormalization group equation approach is the efficient proof of the renormalizability of phi-four theory in four dimensions given by Polchinski in [2]. Instead of reviewing this result here we will review in the next chapter the proof of the renormalizability of a more general phi-four theory in four dimensions which is a theory defined on noncommutative Moyal–Weyl space. This will serve the purpose of our main application of the functional renormalization group equation approach in this book as well as allowing us the opportunity to introduce noncommutative field theory.

11.8 Exercise

Exercise 1:

Consider the Landau–Wilson energy functional in a constant magnetic field

$$E[m] = \int d^d x \left[\frac{1}{2} (\vec{\partial} m(x))^2 + \frac{1}{2} r_0 m^2(x) + u_0 m^4(x) \right] + \int d^d x h m(x). \tag{11.460}$$

We will integrate over a small momentum shell characterized by $b \simeq 1$, i.e. over the spherical shell of radius Λ and thickness $\delta k = \tau \cdot \Lambda$ where $\tau = \ln b$. We expand the field as

$$m(x) = \bar{m}(x) + \delta m(x), \; \delta m(x) = \sum_{i=1}^{N} c_i \phi_i(x). \tag{11.461}$$

The N wave packets (plane waves!) $\phi_i(x)$ are of spatial size $\delta x = 1/\delta k$ and they are localized in the momentum shell δk. They are orthonormal and they also satisfy

$$\int d^d x (\phi_i(x))^{2n+1} = 0. \tag{11.462}$$

The background $\bar{m}$ is assumed to be constant over wave packets.

- Show that the RG-transformed energy functional (by integrating out the background at one-loop) is given by

$$E'[\bar{m}] = E[\bar{m}] + \frac{1}{2}\sum_{i=1}^{N} \log\left(1 + \frac{r_0}{\Lambda^2} + \frac{12u_0}{\Lambda^2}\bar{m}^2(x_i)\right), \tag{11.463}$$

  where x_i is the center of mass of the wave packet i.
- We approximate the sum over x_i by an integral $\int dx$ and then we expand the logarithm. Show that we obtain

$$E'[\bar{m}] = \int d^d x\left[\frac{1}{2}(\vec{\partial}\bar{m}(x))^2 + \frac{1}{2}\tilde{r}_0\bar{m}^2(x) + \tilde{u}_0\bar{m}^4(x)\right] + \int d^d x h\bar{m}(x). \tag{11.464}$$

  Determine the values of $\tilde{r}_0$ and $\tilde{u}_0$ in terms of r_0 and u_0.
- By performing the second step (rescaling) and the third step (normalization) of the renormalization group transformation show that we can express the RG transformation as

$$r_0' - r_0 = [2r_0 + 12C_d(\Lambda^{d-2}u_0 - \Lambda^{d-4}r_0u_0)]\ln b \tag{11.465}$$

$$u_0' - u_0 = [(4-d)u_0 - 36C_d\Lambda^{d-4}u_0^2)]\ln b \tag{11.466}$$

$$h' = b^{\frac{d+2}{2}}h. \tag{11.467}$$

- Determine the magnetic field critical exponent D_h.
- Show that the linearized renormalized group transformation can be put into the differential form

$$\frac{dr}{d\tau} = 2r + 12C_d(\Lambda^{d-2}u - \Lambda^{d-4}ru), \quad \frac{du}{d\tau} = (4-d)u - 36C_d\Lambda^{d-4}u^2. \tag{11.468}$$

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Chapter 12

Noncommutative scalar field theory and its renormalizability

In this chapter, and after an efficient introduction to noncommutative scalar field theory, we will apply the Wilson–Polchinski renormalization group equation, discussed in the previous chapter, to the problem of renormalizing noncommutative phi-four theory in two and four dimensions with and without the harmonic oscillator term. Noncommutative field theory is discussed in great detail in our book [1], whereas we follow closely the original programme of Grosse and Wulkenhaar [7–9] in the very difficult problem of renormalization of noncommutative phi-four theory on Moyal–Weyl spaces.

12.1 Noncommutative Moyal–Weyl spaces

This section is taken from [1].

12.1.1 The Weyl map

A Groenewold–Moyal–Weyl space $\mathbf{R}^d_\theta$ is a deformation of ordinary d-dimensional Euclidean space $\mathbf{R}^d$ in which the coordinates x_i are replaced with Hermitian operators $\hat{x}_i$ satisfying the Heisenberg–Weyl commutation relations

$$\left[\hat{x}_i, \hat{x}_j\right] = i\theta_{ij}. \tag{12.1}$$

The space $\mathbf{R}^d_\theta$ in general can be only partially noncommutative, i.e. the Poisson tensor θ_{ij} is of rank $2r \leqslant d$. This means in particular that we have only $2r$ noncommuting coordinates. The Poisson tensor, also known as the noncommutativity parameter, can thus be brought by means of an appropriate linear transformation of the coordinate operators to the canonical form

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$$\theta = \begin{pmatrix} 0 & \theta_1 & 0 & \cdot & \cdot & 0 & \cdot \\ -\theta_1 & 0 & 0 & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \theta_r & \cdot \\ 0 & 0 & \cdot & \cdot & -\theta_r & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{12.2}$$

In the above equation $\theta_r = \theta_{2r-12r}$. In the spirit of Connes' noncommutative geometry [2], we will describe the Groenewold–Moyal–Weyl spacetime $\mathbf{R}^d_\theta$ in terms of the algebra of functions on $\mathbf{R}^d$, endowed with an associative noncommutative product between elements f and g denoted by $f * g$. This star product is, precisely, the Groenewold–Moyal–Weyl star product [3–5] which we will derive shortly.

The algebra corresponding to the space $\mathbf{R}^d_\theta$ will be denoted $\mathcal{A}_\theta$. The algebra $\mathcal{A}_0$ corresponding to the commutative space $\mathbf{R}^d$ is clearly the algebra of functions on $\mathbf{R}^d$ with the usual pointwise multiplication of functions. Specification of the algebra will determine only topological properties of the space $\mathbf{R}^d_\theta$. In order to specify the metric aspects we must also define proper derivation operations on the algebra $\mathcal{A}_\theta$. The Weyl map will allow us to map the algebra $\mathcal{A}_\theta$ to the correct operator algebra generated by the coordinate operators $\hat{x}_i$.

The algebra of functions on $\mathbf{R}^d$, of interest to us here, is the algebra of Schwartz functions of sufficiently rapid decrease at infinity. These are functions with all their derivatives vanishing at infinity. Equivalently Schwartz functions are functions $f(x)$ which admit well defined Fourier transforms $\tilde{f}(k)$, viz

$$\tilde{f}(k) = \int d^d x f(x) e^{-ikx}. \tag{12.3}$$

The Fourier transforms $\tilde{f}(k)$ are also Schwartz functions, i.e. their derivatives to any order vanish at infinity in momentum space. The functions $f(x)$ are given by the inverse Fourier transforms, viz

$$f(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{f}(k) e^{ikx}. \tag{12.4}$$

The Weyl operator $\hat{f}$ acting in some, infinite-dimensional separable, Hilbert space $\mathcal{H}$ which corresponds to the function $f(x)$ is obtained by requiring that $f(x)$ is the Weyl symbol of $\hat{f}$, i.e.

$$\hat{f} = \int \frac{d^d k}{(2\pi)^d} \tilde{f}(k) e^{ik\hat{x}}. \tag{12.5}$$

We have only replaced the coordinates x_i by the coordinate operators $\hat{x}_i$. This is a bounded operator which is also compact.

For simplicity we will assume maximal noncommutativity, i.e. $d = 2r$. The Weyl map, or quantizer, is given by

$$\Delta(\hat{x}_i, x_i) = \int \frac{d^d k}{(2\pi)^d} e^{ik_i \hat{x}_i} e^{-ik_i x_i}. \tag{12.6}$$

This corresponds to the symmetric ordering of the operator. We have explicitly

$$\hat{f} = \int d^d x f(x_i)\Delta(\hat{x}_i, x_i). \tag{12.7}$$

It is obvious that if $f = f_k = \exp(ikx)$ then $\hat{f} = \hat{f}_k = \exp(ik\hat{x})$, viz

$$\exp(ik\hat{x}) = \int d^d x \exp(ikx)\Delta(\hat{x}_i, x_i). \tag{12.8}$$

The derivative operators on the noncommutative space $\mathbf{R}^d_\theta$ can be given by the inner derivations

$$\hat{\partial}_i = \frac{1}{i}(\theta^{-1})_{ij}\left(\hat{x}_j - \hat{x}_j^R\right). \tag{12.9}$$

The coordinate operators $\hat{x}_i^R$ act on the right of the algebra, viz $\hat{x}_i^R\hat{f} = \hat{f}\,\hat{x}_i$. These derivative operators satisfy the conditions

$$\left[\hat{\partial}_i, \hat{\partial}_j\right] = 0, \quad \left[\hat{\partial}_i, \hat{x}_j\right] = \delta_{ij}. \tag{12.10}$$

The derivative operators on $\mathbf{R}^d_\theta$ can also be given by any outer derivations satisfying the above two requirements.

We have the basic identity

$$\begin{aligned}\left[\hat{\partial}_i, e^{ik\hat{x}}\right] &= \sum_{n=1}^{\infty}\frac{i^n}{n!}\left[\hat{\partial}_i, (k\hat{x})^n\right] \\ &= \sum_{n=1}^{\infty}\frac{i^n}{n!}nk_i(k\hat{x})^{n-1} \\ &= ik_i e^{ik\hat{x}}.\end{aligned} \tag{12.11}$$

In the second line we have used the second equation of (12.10) which is valid for all derivations inner or outer. We have, therefore, the result

$$\left[\hat{\partial}_i, \hat{f}\right] = \int \frac{d^d k}{(2\pi)^d} ik_i \tilde{f}(k) e^{ik\hat{x}}. \tag{12.12}$$

This suggests that we associate the operator $[\hat{\partial}_i, \hat{f}]$ with the function $\partial_i f(x_i)$. The proof goes as follows. First we have

$$\begin{aligned}\left[\hat{\partial}_i, \Delta(\hat{x}_i, x_i)\right] &= \int \frac{d^d k}{(2\pi)^d} ik_i e^{ik_i\hat{x}_i} e^{-ik_i x_i} \\ &= -\partial_i \Delta(\hat{x}_i, x_i).\end{aligned} \tag{12.13}$$

By using the above result we have

$$\begin{aligned}\left[\hat{\partial}_i, \hat{f}\right] &= \int d^d x f(x_i)\left[\hat{\partial}_i, \Delta(\hat{x}_i, x_i)\right] \\ &= -\int d^d x f(x_i)\partial_i \Delta(\hat{x}_i, x_i) \\ &= \int d^d x \partial_i f(x_i)\Delta(\hat{x}_i, x_i).\end{aligned} \tag{12.14}$$

In other words, the operator $[\hat{\partial}_i, \hat{f}\,]$ corresponds to the function $\partial_i f(x_i)$ as it should.

In the commutative limit $\theta \longrightarrow 0$ the operator $\Delta(\hat{x}_i, x_i)$ reduces in an obvious way to the delta function $\delta^2(\hat{x} - x)$. This is, in fact, obvious from equation (12.8). For $\alpha_i \in \mathbf{R}$ we compute

$$\begin{aligned}e^{\alpha\hat{\partial}} e^{ik\hat{x}} e^{-\alpha\hat{\partial}} &= e^{\frac{i}{2}\alpha k} e^{\alpha\hat{\partial} + ik\hat{x}} e^{-\alpha\hat{\partial}} \\ &= e^{i\alpha k} e^{ik\hat{x}}.\end{aligned} \tag{12.15}$$

The unitary operator $\exp(\alpha\hat{\partial})$ corresponds to a translation operator in space by a vector $\vec{\alpha}$. By using the above result we obtain

$$e^{\alpha\hat{\partial}}\Delta(\hat{x}_i, x_i)e^{-\alpha\hat{\partial}} = \Delta(\hat{x}_i, x_i - \alpha_i). \tag{12.16}$$

We can then conclude that $\mathrm{Tr}_{\mathcal{H}}\Delta(\hat{x}_i, x_i)$ is independent of x for any trace $\mathrm{Tr}_{\mathcal{H}}$ on $\mathcal{H}$ since $\mathrm{Tr}_{\mathcal{H}}\Delta(\hat{x}_i, x_i) = \mathrm{Tr}_{\mathcal{H}}\Delta(\hat{x}_i, x_i - \alpha_i)$. In other words $\mathrm{Tr}_{\mathcal{H}}\Delta(\hat{x}_i, x_i)$ is simply an overall normalization which we can choose appropriately. We choose

$$\mathrm{Tr}_{\mathcal{H}}\Delta(\hat{x}_i, x_i) = \frac{1}{\sqrt{\det(2\pi\theta)}}. \tag{12.17}$$

In some sense $\sqrt{\det(2\pi\theta)}$ is the volume of an elementary cell in noncommutative spacetime if we think of $\mathbf{R}^d_\theta$ as a phase space. This can also be understood from the result

$$\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}} e^{ik\hat{x}} = \int d^d x e^{ikx} = (2\pi)^d \delta^d(k). \tag{12.18}$$

Similarly, we can compute

$$\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\hat{f} = \int d^d x f(x_i). \tag{12.19}$$

The analogue of the identity (12.17) is the identity

$$\int d^d x \Delta(\hat{x}_i, x_i) = 1. \tag{12.20}$$

We now want to show that the Weyl map is indeed one-to-one. The proof goes as follows. First we compute

$$\begin{aligned}\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}} e^{ik\hat{x}} e^{ip\hat{x}} &= \sqrt{\det(2\pi\theta)}\,\mathrm{Tr}\, e^{-\frac{i}{2}\theta_{ij}k_i p_j} e^{i(k+p)\hat{x}} \\ &= (2\pi)^d \delta^d(k + p).\end{aligned} \tag{12.21}$$

Hence

$$\begin{aligned}\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\Delta(\hat{x}_i,\,x_i)\Delta(\hat{x}_i,\,y_i) &= \int \frac{d^dk}{(2\pi)^d}e^{-ikx} \\ &\quad\times\int\frac{d^dp}{(2\pi)^d}e^{-ipy}\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}e^{ik\hat{x}}e^{ip\hat{x}} \\ &= \delta^2(x-y).\end{aligned} \tag{12.22}$$

Using this last formula one can deduce

$$f(x_i) = \sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\hat{f}\;\Delta(\hat{x}_i,\,x_i). \tag{12.23}$$

This shows explicitly that the Weyl map Δ provides indeed a one-to-one correspondence between fields and operators.

12.1.2 Star product and scalar action

The most natural problem now is to determine the image under the Weyl map of the pointwise product $\hat{f}\,\hat{g}$ of the two operators $\hat{f}$ and $\hat{g}$.

First, we compute the generalization of equation (12.21) given by

$$\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}e^{ik\hat{x}}e^{ip\hat{x}}e^{iq\hat{x}} = e^{-\frac{i}{2}\theta_{ij}k_ip_j}(2\pi)^d\delta^d(k+p+q). \tag{12.24}$$

This leads to

$$\begin{aligned}\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\Delta(\hat{x},\,y)\Delta(\hat{x},\,z)\Delta(\hat{x},\,x) &= \int\frac{d^dk}{(2\pi)^d} \\ &\quad\times\frac{d^dp}{(2\pi)^d}e^{ik(x-y)}e^{ip(x-z)}e^{-\frac{i}{2}\theta_{ij}k_ip_j}.\end{aligned} \tag{12.25}$$

Hence

$$\begin{aligned}\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\hat{f}\,\hat{g}\Delta(\hat{x}_i,\,x_i) &= \int\frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\tilde{f}(k)\tilde{g}(p)e^{-\frac{i}{2}\theta_{ij}k_ip_j}e^{i(k+p)x} \\ &\equiv f*g(x).\end{aligned} \tag{12.26}$$

This star product can also be given by

$$f*g(x) = e^{\frac{i}{2}\theta_{ij}\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\eta_j}}f(x+\xi)g(x+\eta)|_{\xi=\eta=0}. \tag{12.27}$$

The above result can also be put in the form

$$\hat{f}\,\hat{g} = \int d^dx f*g(x_i)\Delta(\hat{x}_i,\,x_i). \tag{12.28}$$

This leads to the identity

$$\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\hat{f}\,\hat{g} \equiv \int d^d x\, f * g(x) = \int d^d x f(x) g(x). \tag{12.29}$$

On the other hand we know that the operator $[\hat{\partial}_i, \hat{f}\,]$ corresponds to the function $\partial_i f(x_i)$. We can then also write

$$\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\Big[\hat{\partial}_i, \hat{f}\,\Big]^2 \equiv \int d^d x\, \partial_i f * \partial_i f(x) = \int d^d x (\partial_i f)^2. \tag{12.30}$$

We are now in a position to propose a free, i.e. quadratic, scalar action. This will be given simply by

$$\begin{aligned} S_{\text{free}} &= \int d^d x\, \Phi\left(-\frac{1}{2}\partial_i^2 + \frac{\mu^2}{2}\right)\Phi \\ &\equiv \sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\hat{\Phi}\left(-\frac{1}{2}[\hat{\partial}_i, [\hat{\partial}_i, \ldots]] + \frac{\mu^2}{2}\right)\hat{\Phi}. \end{aligned} \tag{12.31}$$

Next we add a phi-four interaction as a typical example of noncommutative interacting field theory. First, we note that the operators $\hat{\Phi}$ and $\hat{\Phi}^2$ correspond to the fields Φ and Φ^2 respectively. Indeed we have

$$\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\hat{\Phi}\Delta(\hat{x}_i, x_i) = \Phi(x) \tag{12.32}$$

$$\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\hat{\Phi}^2\Delta(\hat{x}_i, x_i) = \Phi * \Phi(x). \tag{12.33}$$

Hence, we must have

$$\sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\hat{\Phi}^4\Delta(\hat{x}_i, x_i) = \Phi * \Phi * \Phi * \Phi(x). \tag{12.34}$$

The phi-four interaction term must, therefore, be of the form

$$\begin{aligned} S_{\text{interaction}} &= \sqrt{\det(2\pi\theta)}\,\frac{\lambda}{4!}\,\mathrm{Tr}_{\mathcal{H}}\hat{\Phi}^4 \\ &= \frac{\lambda}{4!}\int d^d x\, \Phi * \Phi * \Phi * \Phi(x). \end{aligned} \tag{12.35}$$

12.1.3 The Langmann–Szabo–Zarembo models

In this section, we will write down the most general action, with a phi-four interaction, in a noncommutative $\mathbf{R}^d_\theta$, under the effect of a magnetic field which induces noncommutivity also in momentum space. Then, we will regularize the partition function of the theory by replacing the field operator $\hat{\Phi}$ by an $N \times N$ matrix M, and also replacing the infinite-dimensional trace $\mathrm{Tr}_{\mathcal{H}}$ by a finite-dimensional trace Tr_N. The resulting theory is a single-trace matrix model, with a matrix phi-four interaction, and a modified propagator. We will mostly follow [6].

For simplicity, we start by considering a two-dimensional Euclidean space, with noncommutativity given by θ_{ij}. Generalization to higher dimensions will be sketched in section 12.1.6. We also introduce noncommutativity in momentum space, by introducing a minimal coupling to a constant background magnetic field B_{ij}. The derivation operators become

$$\hat{D}_i = \hat{\partial}_i - iB_{ij}X_j, \;\; \hat{C}_i = \hat{\partial}_i + iB_{ij}X_j. \tag{12.36}$$

In the above, X_i is defined by

$$X_i = \frac{\hat{x}_i + \hat{x}_i^R}{2}. \tag{12.37}$$

Hence

$$\begin{aligned} \hat{D}_i &= -i\left(\theta^{-1} + \frac{B}{2}\right)_{ij} \hat{x}_j + i\left(\theta^{-1} - \frac{B}{2}\right)_{ij} \hat{x}_j^R \\ \hat{C}_i &= -i\left(\theta^{-1} - \frac{B}{2}\right)_{ij} \hat{x}_j + i\left(\theta^{-1} + \frac{B}{2}\right)_{ij} \hat{x}_j^R. \end{aligned} \tag{12.38}$$

We also note

$$[X_i, \, X_j] = 0 \tag{12.39}$$

$$\left[\hat{D}_i, \, \hat{D}_j\right] = 2iB_{ij}, \;\; \left[\hat{C}_i, \, \hat{C}_j\right] = -2iB_{ij} \tag{12.40}$$

$$\left[\hat{D}_i, \, X_j\right] = \left[\hat{C}_i, \, X_j\right] = \left[\hat{\partial}_i, \, X_j\right] = \delta_{ij}. \tag{12.41}$$

Instead of the conventional Laplacian $\Delta = (-\hat{\partial}_i^2 + \mu^2)/2$ we will consider the generalized Laplacians

$$\begin{aligned} \Delta &= -\sigma \hat{D}_i^{\,2} - \tilde{\sigma} \hat{C}_i^{\,2} + \frac{\mu^2}{2} \\ &= -(\sigma + \tilde{\sigma})\hat{\partial}_i^2 + (\sigma - \tilde{\sigma})iB_{ij}\{X_j, \, \hat{\partial}_i\} - (\sigma + \tilde{\sigma})(B^2)_{ij}X_iX_j + \frac{\mu^2}{2}. \end{aligned} \tag{12.42}$$

The case $\sigma = \tilde{\sigma}$ corresponds to the Grosse–Wulkenhaar model [7–9], while the model $\sigma = 1$, $\tilde{\sigma} = 0$ corresponds to Langmann–Szabo–Zarembo model considered in [10].

Let us introduce the operators $Z = X_1 + iX_2$, $\bar{Z} = Z^+ = X_1 - iX_2$, $\hat{\partial} = \hat{\partial}_1 - i\hat{\partial}_2$ and $\bar{\hat{\partial}} = -\hat{\partial}^+ = \hat{\partial}_1 + i\hat{\partial}_2$. Also, we introduce the creation and annihilation operators (with $\theta_0 = \theta/2$, $\theta = \theta_{12}$)

$$\hat{a} = \frac{1}{2}\left(\sqrt{\theta_0}\hat{\partial} + \frac{1}{\sqrt{\theta_0}}\bar{Z}\right), \;\; \hat{a}^+ = \frac{1}{2}\left(-\sqrt{\theta_0}\bar{\hat{\partial}} + \frac{1}{\sqrt{\theta_0}}Z\right) \tag{12.43}$$

$$\hat{b} = \frac{1}{2}\left(\sqrt{\theta_0}\bar{\hat{\partial}} + \frac{1}{\sqrt{\theta_0}}Z\right),\ \ \hat{b}^+ = \frac{1}{2}\left(-\sqrt{\theta_0}\hat{\partial} + \frac{1}{\sqrt{\theta_0}}\bar{Z}\right). \tag{12.44}$$

We have

$$\begin{aligned} X_1 &= \frac{\sqrt{\theta_0}}{2}\left(\hat{a} + \hat{b}^+ + \hat{a}^+ + \hat{b}\right),\ \ X_2 = \frac{i\sqrt{\theta_0}}{2}\left(\hat{a} + \hat{b}^+ - \hat{a}^+ - \hat{b}\right) \\ \hat{\partial}_1 &= \frac{1}{2\sqrt{\theta_0}}\left(\hat{a} - \hat{b}^+ - \hat{a}^+ + \hat{b}\right),\ \ \hat{\partial}_2 = \frac{i}{2\sqrt{\theta_0}}\left(\hat{a} - \hat{b}^+ + \hat{a}^+ - \hat{b}\right). \end{aligned} \tag{12.45}$$

We compute by using $[Z, \bar{Z}] = 0$, $[\hat{\partial}, Z] = [\bar{\hat{\partial}}, \bar{Z}] = 2$, $[\hat{\partial}, \bar{Z}] = [\bar{\hat{\partial}}, Z] = 0$, and $[\hat{\partial}, \bar{\hat{\partial}}] = 0$ the commutation relations

$$[\hat{a}, \hat{a}^+] = 1,\ \ [\hat{b}, \hat{b}^+] = 1. \tag{12.46}$$

The rest are zero.

We consider, now, the rank-one Fock space operators

$$\hat{\phi}_{l,m} = |l\rangle\langle m|. \tag{12.47}$$

We have

$$\hat{\phi}^{+}_{l,m} = \hat{\phi}_{m,l} \tag{12.48}$$

$$\hat{\phi}_{l,m}\hat{\phi}_{l',m'} = \delta_{m,l'}\hat{\phi}_{l,m'} \tag{12.49}$$

$$\mathrm{Tr}_{\mathcal{H}}\hat{\phi}_{l,m} = \delta_{l,m} \tag{12.50}$$

$$\mathrm{Tr}_{\mathcal{H}}\hat{\phi}^{+}_{l,m}\hat{\phi}_{l',m'} = \delta_{l,l'}\delta_{m,m'}. \tag{12.51}$$

We are, therefore, led to consider expanding the arbitrary scalar operators $\hat{\Phi}$, and $\hat{\Phi}^+$, in terms of $\hat{\phi}_{l,m}$, as follows

$$\hat{\Phi} = \sum_{l,m=1}^{\infty} M_{lm}\hat{\phi}_{l,m},\ \ \hat{\Phi}^+ = \sum_{l,m=1}^{\infty} M^*_{lm}\hat{\phi}^{+}_{l,m}. \tag{12.52}$$

The infinite-dimensional matrix M should be thought of as a compact operator, acting on some separable Hilbert space $\mathbf{H}_1$ of Schwartz sequences, with sufficiently rapid decrease [6]. This, in particular, will guarantee the convergence of the expansions of the scalar operators $\hat{\Phi}$, and $\hat{\Phi}^+$.

Next, we compute

$$\hat{\Phi}\hat{\Phi}' = \sum_{l,m=1}^{\infty} (MM')_{lm}\hat{\phi}_{l,m}. \tag{12.53}$$

The representation of this operator product, in terms of the star product of functions, can be obtained as follows. In the operators $\hat{\phi}_{l,m} = |l\rangle\langle m|$, we can identify the kets $|l\rangle$ with the states of the harmonic oscillator operators $\hat{a}$, and $\hat{a}^+$, whereas the bras $\langle m|$ can be identified with the states of the harmonic oscillator operators $\hat{b}$, and $\hat{b}^+$. More precisely, the operators $\hat{\phi}_{l,m}$ are in one-to-one correspondence with the wavefunctions $\phi_{l,m}(x) = \langle x|l, m\rangle$, known as Landau states [11, 12].

Landau states are defined by

$$\hat{a}\hat{\phi}_{l,m} = \sqrt{l-1}\,\hat{\phi}_{l-1,m},\ \ \hat{a}^+\hat{\phi}_{l,m} = \sqrt{l}\,\hat{\phi}_{l+1,m} \tag{12.54}$$

$$\hat{b}\hat{\phi}_{l,m} = \sqrt{m-1}\,\hat{\phi}_{l,m-1},\ \ \hat{b}\hat{\phi}_{l,m} = \sqrt{m}\,\hat{\phi}_{l,m+1}. \tag{12.55}$$

These states, as we will show, satisfy, among other things, the following properties

$$\phi^*_{l,m}(x) = \phi_{m,l}(x) \tag{12.56}$$

$$\phi_{l_1,m_1} * \phi_{l_2,m_2}(x) = \frac{1}{\sqrt{4\pi\theta_0}}\delta_{m_1,l_2}\phi_{l_1,m_2}(x) \tag{12.57}$$

$$\int d^2x\ \phi_{l,m}(x) = \sqrt{4\pi\theta_0}\ \delta_{l,m} \tag{12.58}$$

$$\int d^2x\ \phi^*_{l_1,m_1} * \phi_{l_2,m_2}(x) = \delta_{l_1,l_2}\delta_{m_1,m_2}. \tag{12.59}$$

By comparing with equations (12.48), (12.49), (12.50), and (12.51) we conclude that the field/operator (Weyl) map is given by

$$\sqrt{2\pi\theta}\ \phi_{l_1,m_1} \leftrightarrow \hat{\phi}_{l_1,m_1} \tag{12.60}$$

$$\int d^2x \leftrightarrow \sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}. \tag{12.61}$$

We are, therefore, led to consider scalar functions Φ and Φ^+ corresponding to the scalar operators $\hat{\Phi}$, $\hat{\Phi}^+$ given explicitly by

$$\Phi = \sqrt{2\pi\theta}\sum_{l,m=1}^{\infty} M_{lm}\phi_{l,m},\ \ \Phi^+ = \sqrt{2\pi\theta}\sum_{l,m=1}^{\infty} M^*_{lm}\phi^*_{l,m} \longrightarrow \hat{\Phi},\ \ \hat{\Phi}^+. \tag{12.62}$$

Indeed the star product is given by

$$\Phi * \Phi' = \sqrt{2\pi\theta}\sum_{l,m=1}^{\infty}(MM')_{lm}\phi_{l,m}. \tag{12.63}$$

In other words, the star product is mapped to the operator product as it should be, viz

$$\Phi * \Phi' \leftrightarrow \hat{\Phi}\hat{\Phi}'. \tag{12.64}$$

The operator $X_i\hat{\Phi} = \frac{1}{2}\hat{x}_i\hat{\Phi} + \frac{1}{2}\hat{\Phi}\hat{x}_i$ corresponds to the function $\frac{1}{2}x_i * \Phi + \frac{1}{2}\Phi * x_i = x_i\Phi$. As a consequence, the differential operators $\hat{D}_i^{\,2}$, and $\hat{C}_i^{\,2}$ will be represented in the star picture by the differential operators

$$D_i = \partial_i - iB_{ij}x_j, \;\; C_i = \partial_i + iB_{ij}x_j. \tag{12.65}$$

The Landau states are actually eigenstates of the Laplacians D_i^2, and C_i^2 at the special point

$$B^2\theta_0^2 \equiv \frac{B^2\theta^2}{4} = 1. \tag{12.66}$$

Indeed, we can compute (with $\alpha = 1 + B\theta_0$, $\beta = 1 - B\theta_0$, and $\alpha\beta = 1 - B^2\theta_0^2$) the following

$$4\theta_0\hat{D}_i^{\,2} = -4\alpha^2\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) - 4\beta^2\left(\hat{b}^+\hat{b} + \frac{1}{2}\right) + 4\alpha\beta\left(\hat{a}\hat{b} + \hat{a}^+\hat{b}^+\right) \tag{12.67}$$

$$4\theta_0\hat{C}_i^{\,2} = -4\beta^2\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) - 4\alpha^2\left(\hat{b}^+\hat{b} + \frac{1}{2}\right) + 4\alpha\beta\left(\hat{a}\hat{b} + \hat{a}^+\hat{b}^+\right). \tag{12.68}$$

For $\beta = 0$, we observe that D_i^2, and C_i^2 depend only on the number operators $\hat{a}^+\hat{a}$, and $\hat{b}^+\hat{b}$ respectively. For $\alpha = 0$, the roles of D_i^2, and C_i^2, are reversed.

Next, we write the most general single-trace action with a phi-four interaction in a noncommutative $\mathbf{R}_\theta^d$ under the effect of a magnetic field, as follows

$$\begin{aligned} S = \sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\Bigg[\hat{\Phi}^+\left(-\sigma\hat{D}_i^{\,2} - \tilde{\sigma}\hat{C}_i^{\,2} + \frac{\mu^2}{2}\right)\hat{\Phi} \\ + \frac{\lambda}{4!}\hat{\Phi}^+\hat{\Phi}\,\hat{\Phi}^+\hat{\Phi} + \frac{\lambda'}{4!}\hat{\Phi}^+\hat{\Phi}^+\,\hat{\Phi}\hat{\Phi}\Bigg]. \end{aligned} \tag{12.69}$$

By using the above results, as well as the results of the previous section, we can rewrite this action in terms of the star product as follows

$$\begin{aligned} S = \int d^2x\Bigg[\Phi^+\left(-\sigma D_i^2 - \tilde{\sigma}C_i^2 + \frac{\mu^2}{2}\right)\Phi \\ + \frac{\lambda}{4!}\Phi^+ * \Phi * \Phi^+ * \Phi + \frac{\lambda'}{4!}\Phi^+ * \Phi^+ * \Phi * \Phi\Bigg]. \end{aligned} \tag{12.70}$$

In most of the following we will assume $\lambda' = 0$.

12.1.4 Duality transformations

The action, of interest, reads

$$\begin{aligned} S &= \sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\left[\hat{\Phi}^{+}\left(-\sigma\hat{D}_i^{\,2} - \tilde{\sigma}\hat{C}_i^{\,2} + \frac{\mu^2}{2}\right)\hat{\Phi} + \frac{\lambda}{4!}\hat{\Phi}^{+}\hat{\Phi}\,\hat{\Phi}^{+}\hat{\Phi}\right] \\ &= \int d^2x\left[\Phi^{+}\left(-\sigma D_i^{\,2} - \tilde{\sigma}C_i^{\,2} + \frac{\mu^2}{2}\right)\Phi + \frac{\lambda}{4!}\Phi^{+} * \Phi \,*\, \Phi^{+} * \Phi\right]. \end{aligned} \tag{12.71}$$

This action enjoys a remarkable symmetry under certain duality transformations which exchange, among other things, positions and momenta. See [6, 10, 13] for the original derivation. This property can be shown as follows. We start with the quadratic action given by

$$S_2[\Phi, B] = \int d^2x\left[\Phi^{+}\left(-\sigma D_i^{\,2} - \tilde{\sigma}C_i^{\,2} + \frac{\mu^2}{2}\right)\Phi\right]. \tag{12.72}$$

We define $\tilde{k}_i = B_{ij}^{-1}k_j$. The Fourier transform of $\Phi(x)$, and $D_i\Phi(x)$, are $\tilde{\Phi}(k)$, and $-\tilde{D}_i\tilde{\Phi}(k)$, where

$$\tilde{\Phi}(k) = \int d^2x\Phi(x)\, e^{-ik_ix_i} \tag{12.73}$$

$$\begin{aligned} -\tilde{D}_i\tilde{\Phi}(k) &= \int d^2x D_i\Phi(x)\, e^{-ik_ix_i} \\ &= -\left(\frac{\partial}{\partial\tilde{k}_i} - iB_{ij}\tilde{k}_j\right)\tilde{\Phi}(k). \end{aligned} \tag{12.74}$$

Then, we can compute that

$$\int d^2x\,(D_i\Phi)^{+}(x)(D_i\Phi)(x) = \int d^2\tilde{k}\,(\tilde{D}_i\bar{\Phi})^{+}(\tilde{k})(\tilde{D}_i\bar{\Phi})(\tilde{k}). \tag{12.75}$$

The new field, $\bar{\Phi}$, is defined by

$$\bar{\Phi}(\tilde{k}) = \sqrt{\left|\det\frac{B}{2\pi}\right|}\,\tilde{\Phi}(B\tilde{k}). \tag{12.76}$$

A similar result holds for the other quadratic terms. By renaming the variable as $\tilde{k} = x$, we can see that the resulting quadratic action has, therefore, the same form as the original quadratic action, viz

$$S_2[\Phi, B] = S_2[\bar{\Phi}, B]. \tag{12.77}$$

Next, we consider the interaction term

$$\begin{aligned} S_{\text{int}}[\Phi, B] &= \int d^2x \Phi^+ * \Phi \ * \ \Phi^+ * \Phi \\ &= \int \frac{d^2k_1}{(2\pi)^2} \cdots \\ &\quad \times \int \frac{d^2k_1}{(2\pi)^2} \tilde{\Phi}^+(k_1)\tilde{\Phi}(k_2)\tilde{\Phi}^+(k_3)\tilde{\Phi}^+(k_4)\tilde{V}(k_1, k_2, k_3, k_4). \end{aligned} \tag{12.78}$$

The vertex in momentum space is given by

$$\tilde{V}(k_1, k_2, k_3, k_4) = (2\pi)^2\delta^2(k_1 - k_2 + k_3 - k_4)e^{-i\theta^{\mu\nu}\left(k_{1\mu}k_{2\nu} + k_{3\mu}k_{4\nu}\right)}. \tag{12.79}$$

By substituting, $k = B\tilde{k}$, we obtain

$$S_{\text{int}}[\Phi, B] = \int d^2\tilde{k}_1 \ldots \int d^2\tilde{k}_4 \bar{\Phi}^+(\tilde{k}_1)\bar{\Phi}(\tilde{k}_2)\bar{\Phi}^+(\tilde{k}_3)\bar{\Phi}^+(\tilde{k}_4)\bar{V}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4). \tag{12.80}$$

The new vertex is given by

$$\bar{V}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4) = \frac{\det B}{(2\pi)^2}\delta^2(\tilde{k}_1 - \tilde{k}_2 + \tilde{k}_3 - \tilde{k}_4)e^{i(B\theta B)^{\mu\nu}\left(\tilde{k}_{1\mu}\tilde{k}_{2\nu} + \tilde{k}_{3\mu}\tilde{k}_{4\nu}\right)}. \tag{12.81}$$

The interaction term can also be rewritten as

$$S_{\text{int}}[\Phi, B] = \int d^2x_1 \ldots \int d^2x_4 \Phi^+(x_1)\Phi(x_2)\Phi^+(x_3)\Phi^+(x_4)V(x_1, x_2, x_3, x_4). \tag{12.82}$$

The vertex in position space is given by

$$\begin{aligned} V(x_1, x_2, x_3, x_4) &= \int \frac{d^2k_1}{(2\pi)^2} \cdots \\ &\quad \times \int \frac{d^2k_4}{(2\pi)^2} \tilde{V}(k_1, k_2, k_3, k_4)e^{ik_1x_1 - ik_2x_2 + ik_3x_3 - ik_4x_4} \\ &= \frac{1}{(2\pi)^2|\det\theta|}\delta^2(x_1 - x_2 + x_3 - x_4)e^{-i(\theta^{-1})_{\mu\nu}\left(x_1^\mu x_2^\mu + x_3^\mu x_4^\mu\right)}. \end{aligned} \tag{12.83}$$

We can see immediately from comparing equations (12.80), and (12.81), to equations (12.82), and (12.83), that the interaction term in momentum space, has the same form as the interaction term in position space, provided that the new noncommutativity parameter, and the new coupling constant, are given by

$$\bar{\theta} = -B^{-1}\theta^{-1}B^{-1} \tag{12.84}$$

$$\lambda\frac{\det B}{(2\pi)^2} = \frac{\bar{\lambda}}{(2\pi)^2}\frac{1}{\det\bar{\theta}} \Leftrightarrow \bar{\lambda} = \frac{\lambda}{|\det B\theta|}. \tag{12.85}$$

In summary, the duality transformations under which the action retains the same form, are given by

$$x_i \longrightarrow \tilde{k}_i = B_{ij}^{-1} k_j \tag{12.86}$$

$$\Phi(x) \longrightarrow \bar{\Phi}(\tilde{k}) = \sqrt{\left| \det \frac{B}{2\pi} \right|}\, \tilde{\Phi}(B\tilde{k}) \tag{12.87}$$

$$\theta \longrightarrow \bar{\theta} = -B^{-1}\theta^{-1}B^{-1} \tag{12.88}$$

$$\lambda \longrightarrow \bar{\lambda} = \frac{\lambda}{|\det B\theta|}. \tag{12.89}$$

12.1.5 Matrix regularization

Now, we want to express the above action, which is given by equation (12.71), in terms of the compact operators M, and M^+. First, we compute

$$\mathrm{Tr}_{\mathcal{H}} \Phi^+ \Phi = \mathrm{Tr}_{\mathbf{H}_1} M^+ M \tag{12.90}$$

$$\mathrm{Tr}_{\mathcal{H}} \Phi^+ (ab + a^+ b^+) \Phi = \mathrm{Tr}_{\mathbf{H}_1} (\Gamma^+ M^+ \Gamma M + M^+ \Gamma^+ M \Gamma) \tag{12.91}$$

$$\mathrm{Tr}_{\mathcal{H}} \Phi^+ \left(a^+ a + \frac{1}{2} \right) \Phi = \mathrm{Tr}_{\mathbf{H}_1} M^+ E M \tag{12.92}$$

$$\mathrm{Tr}_{\mathcal{H}} \Phi^+ \left(b^+ b + \frac{1}{2} \right) \Phi = \mathrm{Tr}_{\mathbf{H}_1} M E M^+ \tag{12.93}$$

$$\mathrm{Tr}_{\mathcal{H}} \Phi^+ \Phi\, \Phi^+ \Phi = \mathrm{Tr}_{\mathbf{H}_1} M^+ M M^+ M. \tag{12.94}$$

The infinite-dimensional matrices Γ, and E are defined by

$$(\Gamma)_{lm} = \sqrt{m-1}\, \delta_{lm-1},\ (E)_{lm} = \left(l - \frac{1}{2} \right) \delta_{lm}. \tag{12.95}$$

The action becomes

$$\begin{aligned} S = \frac{\sqrt{\det(2\pi\theta)}}{\theta_0} \Big[& -(\sigma + \tilde{\sigma})\alpha\beta\, \mathrm{Tr}_{\mathbf{H}_1} (\Gamma^+ M^+ \Gamma M + M^+ \Gamma^+ M \Gamma) \\ & + \left(\sigma\alpha^2 + \tilde{\sigma}\beta^2 \right) \mathrm{Tr}_{\mathbf{H}_1} M^+ E M + \left(\sigma\beta^2 + \tilde{\sigma}\alpha^2 \right) \mathrm{Tr}_{\mathbf{H}_1} M E M^+ \\ & + \frac{\mu^2 \theta_0}{2}\, \mathrm{Tr}_{\mathbf{H}_1} M^+ M + \frac{\lambda \theta_0}{4!}\, \mathrm{Tr}_{\mathbf{H}_1} M^+ M M^+ M \Big]. \end{aligned} \tag{12.96}$$

We regularize the theory by taking M to be an $N \times N$ matrix. The states $\phi_{l,m}(x)$, with $l, m < N$, where N is some large integer, corresponding to a cutoff in position, and momentum spaces [9]. The infrared cutoff is found to be proportional to $R = \sqrt{2\theta N}$, while the UV cutoff is found to be proportional to $\Lambda = \sqrt{8N/\theta}$. In [6], a double scaling strong noncommutativity limit, in which N/θ (and thus Λ) is kept fixed, was considered.

12.1.6 The Grosse–Wulkenhaar model

In two dimensions

We will be mostly interested in the so-called Grosse–Wulkenhaar model. This contains, compared with the usual case, a harmonic oscillator term in the Laplacian, which modifies, and thus allows us to control, the IR behavior of the theory. This model is perturbatively renormalizable, which makes it more interesting. The Grosse–Wulkenhaar model corresponds to the values $\sigma = \tilde{\sigma} \neq 0$, so that the mixing term, in equation (12.42), cancels. We consider, without any loss of generality, $\sigma = \tilde{\sigma} = 1/4$. We obtain, therefore, the action

$$\begin{aligned} S &= \sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\left[\hat{\Phi}^{+}\left(-\frac{1}{2}\hat{\partial}_i^2 + \frac{1}{2}(B_{ij}X_j)^2 + \frac{\mu^2}{2}\right)\hat{\Phi} + \frac{\lambda}{4!}\hat{\Phi}^{+}\hat{\Phi}\,\hat{\Phi}^{+}\hat{\Phi}\right] \\ &= \int d^2x\left[\Phi^{+}\left(-\frac{1}{2}\partial_i^2 + \frac{1}{2}(B_{ij}x_j)^2 + \frac{\mu^2}{2}\right)\Phi + \frac{\lambda}{4!}\Phi^{+} * \Phi * \Phi^{+} * \Phi(x)\right]. \end{aligned} \tag{12.97}$$

In two dimensions, we can show that $(B_{ij}x_j)^2 = \Omega^2\tilde{x}_i^2$, where $\tilde{x}_i = 2(\theta^{-1})_{ij}x_j$, and Ω is defined by

$$B\theta = 2\Omega. \tag{12.98}$$

We get, therefore, the action

$$\begin{aligned} S &= \sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\left[\hat{\Phi}^{+}\left(-\frac{1}{2}\hat{\partial}_i^2 + \frac{1}{2}\Omega^2\tilde{X}_i^2 + \frac{\mu^2}{2}\right)\hat{\Phi} + \frac{\lambda}{4!}\hat{\Phi}^{+}\hat{\Phi}\,\hat{\Phi}^{+}\hat{\Phi}\right] \\ &= \int d^2x\left[\Phi^{+}\left(-\frac{1}{2}\partial_i^2 + \frac{1}{2}\Omega^2\tilde{x}_i^2 + \frac{\mu^2}{2}\right)\Phi + \frac{\lambda}{4!}\Phi^{+} * \Phi * \Phi^{+} * \Phi\right]. \end{aligned} \tag{12.99}$$

Similar to $\tilde{x}_i = 2(\theta^{-1})_{ij}x_j$, we have defined $\tilde{X}_i = 2(\theta^{-1})_{ij}X_j$. This action is also found to be covariant under a duality transformation which exchanges, among other things, positions and momenta as $x_i \leftrightarrow \tilde{p}_i = B_{ij}^{-1}p_j$. Let us note here, that because of the properties of the star product, the phi-four interaction, is actually invariant under this duality transformation. The value $\Omega^2 = 1$, gives an action which is invariant under this duality transformation, i.e. the kinetic term becomes invariant under this duality transformation for $\Omega^2 = 1$.

In the Landau basis, the above action, reads

$$\begin{aligned} S = \frac{\nu_2}{\theta}\Big[& (\Omega^2 - 1)\mathrm{Tr}_H(\Gamma^+ M^+ \Gamma M + M^+ \Gamma^+ M \Gamma) \\ & + (\Omega^2 + 1)\mathrm{Tr}_H(M^+ E M + M E M^+) \\ & + \frac{\mu^2\theta}{2}\,\mathrm{Tr}_H M^+ M + \frac{\lambda\theta}{4!}\,\mathrm{Tr}_H M^+ M M^+ M\Big]. \end{aligned} \tag{12.100}$$

This is a special case of equation (12.96). Equivalently

$$S = \nu_2 \sum_{m,n,k,l}\left(\frac{1}{2}(M^+)_{mn} G_{mn,kl} M_{kl} + \frac{\lambda}{4!}(M^+)_{mn} M_{nk} (M^+)_{kl} M_{lm}\right) \tag{12.101}$$

$$\begin{aligned} G_{mn,kl} = & \Big(\mu^2 + {\mu_1}^2(m + n - 1)\Big)\delta_{n,k}\delta_{m,l} \\ & - {\mu_1}^2\sqrt{\omega(m-1)(n-1)}\;\delta_{n-1,k}\delta_{m-1,l} \\ & - {\mu_1}^2\sqrt{\omega mn}\;\delta_{n+1,k}\delta_{m+1,l}. \end{aligned} \tag{12.102}$$

The parameters of the model are μ^2, λ, and

$$\nu_2 = \sqrt{\det(2\pi\theta)}\,,\;\; {\mu_1}^2 = 2(\Omega^2 + 1)/\theta,\;\; \sqrt{\omega} = (\Omega^2 - 1)/(\Omega^2 + 1). \tag{12.103}$$

There are only three independent coupling constants in this theory, which we can take to be μ^2, λ, and Ω^2.

Generalization

Generalization of the above results, to higher dimensions $d = 2n$, assuming maximal noncommutativity for simplicity, is straightforward. The action reads

$$\begin{aligned} S &= \sqrt{\det(2\pi\theta)}\,\mathrm{Tr}_{\mathcal{H}}\left[\hat{\Phi}^+\left(-\sigma\hat{D}_i^{\,2} - \tilde{\sigma}\hat{C}_i^{\,2} + \frac{\mu^2}{2}\right)\hat{\Phi} + \frac{\lambda}{4!}\hat{\Phi}^+\hat{\Phi}\,\hat{\Phi}^+\hat{\Phi}\right] \\ &= \int d^d x\left[\Phi^+\left(-\sigma D_i^2 - \tilde{\sigma}\tilde{C}_i^{\,2} + \frac{\mu^2}{2}\right)\Phi + \frac{\lambda}{4!}\Phi^+ * \Phi\; * \;\Phi^+ * \Phi\right]. \end{aligned} \tag{12.104}$$

In order to be able to proceed, we will assume that the noncommutativity tensor θ, and the magnetic tensor B, are simultaneously diagonalizable. In other words, θ and B, can be brought together, to the canonical form (12.2). For example, in four dimensions, we will have

$$\theta = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix},\;\; B = \begin{pmatrix} 0 & B_{12} & 0 & 0 \\ -B_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} \\ 0 & 0 & -B_{34} & 0 \end{pmatrix}. \tag{12.105}$$

The d-dimensional problem will, thus, split into a direct sum, of n independent, and identical, two-dimensional problems.

The expansion of the scalar field operator is now given by

$$\hat{\Phi} = \sum_{\vec{l},\vec{m}}^{\infty} M_{\vec{l}\vec{m}} \hat{\phi}_{\vec{l},\vec{m}}, \ \vec{l} = (l_1, \dots, l_n), \ \vec{m} = (m_1, \dots, m_n). \tag{12.106}$$

Obviously,

$$\hat{\phi}_{\vec{l},\vec{m}} = \prod_{i=1}^{n} \hat{\phi}_{l_i,m_i}, \tag{12.107}$$

and

$$\hat{\phi}_{\vec{l},\vec{m}} \leftrightarrow \det(2\pi\theta)^{1/4} \, \phi_{\vec{l},\vec{m}}. \tag{12.108}$$

The quantum numbers l_i, and m_i correspond to the plane $x_{2i-1} - x_{2i}$. They correspond to the operators $\hat{x}_{2i-1}$, $\hat{x}_{2i}$, $\hat{\partial}_{2i-1}$, and $\hat{\partial}_{2i}$, or equivalently, to the creation, and annihilation operators $\hat{a}^{(i)}$, $\hat{a}^{(i)+}$, $\hat{b}^{(i)}$, $\hat{b}^{(i)+}$. Indeed, the full Hilbert space $\mathbf{H}_n$, in this case, is a direct sum of the individual Hilbert spaces $\mathbf{H}_1^{(i)}$, associated, with the individual planes.

The above action can be given, in terms of the compact operators M and M^+, essentially by equation (12.96).

The Grosse–Wulkenhaar model in higher dimensions, corresponds, as before, to the values $\sigma = \tilde{\sigma} = 1/4$. However, in higher dimensions, we need also to choose the magnetic field B, such that

$$B\theta = 2\Omega 1. \tag{12.109}$$

The action reduces, then, to

$$\begin{aligned} S &= \sqrt{\det(2\pi\theta)}\, \mathrm{Tr}_{\mathcal{H}} \left[\hat{\Phi}^+ \left(-\frac{1}{2}\hat{\partial}_i^2 + \frac{1}{2}\Omega^2 \tilde{X}_i^2 + \frac{\mu^2}{2} \right) \hat{\Phi} + \frac{\lambda}{4!} \hat{\Phi}^+ \hat{\Phi} \, \hat{\Phi}^+ \hat{\Phi} \right] \\ &= \int d^d x \left[\Phi^+ \left(-\frac{1}{2}\partial_i^2 + \frac{1}{2}\Omega^2 \tilde{x}_i^2 + \frac{\mu^2}{2} \right) \Phi + \frac{\lambda}{4!} \Phi^+ * \Phi \ * \ \Phi^+ * \Phi \right]. \end{aligned} \tag{12.110}$$

Again this action will be given, in terms of the compact operators M and M^+, by essentially the same equations (12.100), and (12.101).

For example, we replace in equation (12.102) mn by $m_1n_1m_2n_2$, kl by $k_1l_1k_2l_2$, nk by $n_1k_1n_2k_2$, kl by $k_1l_1k_2l_2$ and lm by $l_1m_1l_2m_2$ to obtain on $\mathbf{R}_\theta^2 \times \mathbf{R}_\theta^2$ the result

$$\begin{aligned} G^{m_2n_2,k_2l_2}_{m_1n_1,k_1l_1} = &\left(\mu_0^{\ 2} + \mu_1^{\ 2}(m_1 + n_1 + m_2 + n_2 - 2)\right)\delta_{n_1,k_1}\delta_{m_1,l_1}\delta_{n_2,k_2}\delta_{m_2,l_2} \\ &- \mu_1^{\ 2}\sqrt{\omega}\Big(\sqrt{(m_1 - 1)(n_1 - 1)}\ \delta_{n_1-1,k_1}\delta_{m_1-1,l_1} \\ &+ \sqrt{m_1 n_1}\ \delta_{n_1+1,k_1}\delta_{m_1+1,l_1})\delta_{n_2,k_2}\delta_{m_2,l_2} \\ &- \mu_1^{\ 2}\sqrt{\omega}\Big(\sqrt{(m_2 - 1)(n_2 - 1)}\ \delta_{n_2-1,k_2}\delta_{m_2-1,l_2} \\ &+ \sqrt{m_2 n_2}\ \delta_{n_2+1,k_2}\delta_{m_2+1,l_2})\delta_{n_1,k_1}\delta_{m_1,l_1}. \end{aligned} \tag{12.111}$$

12.1.7 Noncommutative phi-four on $\mathbf{R}^D \times \mathbf{R}^2_\theta$

Let us consider a phi-four theory on a noncommutative Moyal–Weyl space with only two noncommuting coordinates, viz $\mathbf{R}^d_\theta = \mathbf{R}^D \times \mathbf{R}^2_\theta$ where $D = d - 2$. We also introduce noncommutativity in momentum space by introducing a minimal coupling to a constant background magnetic field B_{ij} as was done in [6]. This corresponds to the addition of a harmonic oscillator potential to the kinetic action which modifies, and thus allows us to control, the IR behavior of the theory. A particular version of this theory was shown to be renormalizable by Grosse and Wulkenhaar in [8].

The action of interest is given by

$$S = \int d^d x \left[\Phi \left(-\frac{1}{2}\partial_i^2 + \frac{1}{2}(B_{ij}x_j)^2 - \frac{1}{2}\partial_\mu^2 + \frac{\mu^2}{2} \right) \Phi + \frac{\lambda}{4!} \Phi * \Phi * \Phi * \Phi \right]. \quad (12.112)$$

The index i runs over the noncommuting directions while the index μ runs over the commuting directions. We rewrite this action as

$$S = \int d^d x \left[\Phi \left(-\frac{1}{2}\partial_i^2 + \frac{1}{2}\Omega^2 \tilde{x}_i^2 - \frac{1}{2}\partial_\mu^2 + \frac{\mu^2}{2} \right) \Phi + \frac{\lambda}{4!} \Phi * \Phi * \Phi * \Phi \right]. \quad (12.113)$$

The harmonic oscillator coupling constant Ω is defined by $\Omega^2 = B^2\theta^2/4$, whereas the coordinate $\tilde{x}_i$ is defined by $\tilde{x}_i = 2(\theta^{-1})_{ij}x_j$. It was shown in [10] that this action is covariant under a duality transformation which exchanges among other things positions and momenta as $p_i \leftrightarrow \tilde{x}_i$. The value $\Omega^2 = 1$ in particular gives an action which is invariant under this duality transformation. Under the field/operator Weyl map we can rewrite the action as

$$S = \nu_2 \int d^D x \, \mathrm{Tr}_{\mathcal{H}} \left[\hat{\Phi} \left(-\frac{1}{2}\hat{\partial}_i^2 + \frac{1}{2}\Omega^2 \tilde{X}_i^2 - \frac{1}{2}\partial_\mu^2 + \frac{\mu^2}{2} \right) \hat{\Phi} + \frac{\lambda}{4!}\hat{\Phi}^4 \right]. \quad (12.114)$$

The Planck volume ν_2 is defined by $\nu_2 = \sqrt{\det 2\pi\theta}$. We can expand the scalar fields in the Landau basis $\{\hat{\phi}_{m,n}\}$ as (with x standing for commuting coordinates)

$$\hat{\Phi} = \frac{1}{\sqrt{\nu_2}} \sum_{m,n=1}^{\infty} M_{mn}(x)\hat{\phi}_{m,n}. \quad (12.115)$$

The infinite-dimensional matrix M should be thought of as a compact operator acting on the separable Hilbert space $H = \mathcal{S}(\mathbf{N})$ of Schwartz sequences $(a_m)_{m\geqslant 1}$ with sufficiently rapid decrease as $m \longrightarrow \infty$. In the Landau basis the action becomes

$$\begin{aligned} S = &\int d^D x \, \mathrm{Tr}_H \\ &\times \left[\frac{1}{2}(\partial_\mu M)^2 + \frac{1}{2}\mu^2 M^2 + \frac{1}{2}r^2 E M^2 + \frac{u}{N}M^4 + \text{remainder} \right]. \end{aligned} \quad (12.116)$$

The coupling constants r^2 and u are defined by

$$r^2 = \frac{4\pi(\Omega^2+1)}{\nu_2},\ u = \frac{\lambda}{4!}\frac{N}{\nu_2}. \tag{12.117}$$

The remainder is given by

$$\text{remainder} = \frac{1}{2}r^2\sqrt{\omega}(\Gamma^+M^+\Gamma M + M^+\Gamma^+M\Gamma),\ \sqrt{\omega} = \frac{1-\Omega^2}{1+\Omega^2}. \tag{12.118}$$

The matrices Γ and E are given by

$$(\Gamma)_{lm} = \sqrt{m-1}\,\delta_{lm-1},\ (E)_{lm} = \left(l-\frac{1}{2}\right)\delta_{lm}. \tag{12.119}$$

The above action can also be put in the form

$$S = \int d^Dx \sum_{m,n,k,l}\left(\frac{1}{2}M_{mn}G_{mn,kl}M_{kl} + \frac{\lambda}{4!}M_{mn}M_{nk}M_{kl}M_{lm}\right), \tag{12.120}$$

where

$$\begin{aligned} G_{mn,kl} = &\left(-\partial_\mu^2 + \mu^2 + {\mu_1}^2(m+n-1)\right)\delta_{n,k}\delta_{m,l} \\ &- {\mu_1}^2\sqrt{\omega(m-1)(n-1)}\ \delta_{n-1,k}\delta_{m-1,l} \\ &- {\mu_1}^2\sqrt{\omega mn}\ \delta_{n+1,k}\delta_{m+1,l}. \end{aligned} \tag{12.121}$$

In momentum space we have

$$\begin{aligned} S = &\int \frac{d^Dp}{(2\pi)^D}\sum_{m,n,k,l}\frac{1}{2}\tilde{M}_{mn}(-p)G_{mn,kl}(p)\tilde{M}_{kl}(p) \\ &+ \int\frac{d^Dp_1}{(2\pi)^D}\int\frac{d^Dp_2}{(2\pi)^D}\int\frac{d^Dp_3}{(2\pi)^D} \\ &\times\sum_{m,n,k,l}\frac{\lambda}{4!}\tilde{M}_{mn}(p_1)\tilde{M}_{nk}(p_2)\tilde{M}_{kl}(p_3)\tilde{M}_{lm}(-p_1-p_2-p_3), \end{aligned} \tag{12.122}$$

where

$$\begin{aligned} G_{mn,kl}(p) = &\left(p_\mu^2 + \mu^2 + {\mu_1}^2(m+n-1)\right)\delta_{n,k}\delta_{m,l} \\ &- {\mu_1}^2\sqrt{\omega(m-1)(n-1)}\ \delta_{n-1,k}\delta_{m-1,l} \\ &- {\mu_1}^2\sqrt{\omega mn}\ \delta_{n+1,k}\delta_{m+1,l}. \end{aligned} \tag{12.123}$$

12.2 Wilson–Polchinski renormalization group equation on $\mathbf{R}^D \times \mathbf{R}^2_\theta$

We are interested in the renormalizability of noncommutative phi-four theory with a harmonic oscillator term. Towards this end we need to develop the Wilson–Polchinski renormalization group equation for noncommutative scalar field theory

in the matrix basis [7]. As an example, we consider in this section the theory on $\mathbf{R}^D \times \mathbf{R}^2_\theta$.

We consider the path integral

$$Z[J] = \int \prod_{m,n,p} d\tilde{M}_{mn}(p) \exp\Bigg[-S[M] - \sqrt{\det(2\pi\tilde{\theta})} \int \frac{d^D p}{(2\pi)^D} \sum_{m,n} \tilde{M}_{mn}(p) J_{nm}(-p) \Bigg]. \tag{12.124}$$

The total action S and the interaction term L are given by

$$S[M] = \sqrt{\det(2\pi\tilde{\theta})} \int \frac{d^D p}{(2\pi)^D} \times \sum_{m,n,k,l} \frac{1}{2} \tilde{M}_{mn}(-p) G_{mn,kl}(p) \tilde{M}_{kl}(p) + L[M] \tag{12.125}$$

$$L[M] = \sqrt{\det(2\pi\tilde{\theta})} \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \times \int \frac{d^D p_3}{(2\pi)^D} \sum_{m,n,k,l} \frac{\lambda}{4!} \tilde{M}_{mn}(p_1) \tilde{M}_{nk}(p_2) \tilde{M}_{kl}(p_3) \times \tilde{M}_{lm}(-p_1 - p_2 - p_3). \tag{12.126}$$

The volume of an elementary cell is $\nu_2 = \sqrt{\det 2\pi\tilde{\theta}} = \pi\theta = 2\pi(1 + \Omega^2)/{\mu_1}^2$ where we have used $\tilde{\theta} = \theta/2$ and $\theta = 2(1 + \Omega^2)/{\mu_1}^2$. We introduce a momentum space cutoff Λ_1 and a cutoff Λ_2 on discrete indices. The dimensionless noncommutativity is $\bar{\theta} = \theta\Lambda_2^2$. The regularized inverse propagator is

$$G_{mn,kl}(p, \Lambda) = G_{mn,kl}(p) K_1^{-1}\left(\frac{p^2}{\Lambda_1^2}\right) \prod_{i \in \{m,n,k,l\}} K_2^{-1}\left(\frac{{\mu_1}^2 i}{\Lambda_2^2}\right). \tag{12.127}$$

The ultraviolet cutoff function K_1 is such that it is equal to 1 for $p^2 \leqslant \Lambda_1^2$ and it vanishes rapidly for $p^2 > \Lambda_1^2$. Similarly, the ultraviolet cutoff function $K_2(x)$ is such that it is equal to 1 for $0 \leqslant x \leqslant 1$ and it vanishes rapidly for $x > 1$. We will assume that $\Lambda_1 = \Lambda_2 = \Lambda$. We consider now the path integral

$$Z[J, \Lambda] = \int \prod_{m,n,p} d\tilde{M}_{mn}(p) \exp\Big[-S[M,\Lambda] - \sqrt{\det(2\pi\tilde{\theta})} \int \frac{d^D p}{(2\pi)^D} \sum_{m,n,k,l} \tilde{M}_{mn}(p) F_{mn,kl}[p, \Lambda] J_{kl}(-p) - S'[M, \Lambda]\Big]. \tag{12.128}$$

The action $S[M, \Lambda]$ is given in terms of the inverse propagator $G_{mn,k,l}(p, \Lambda)$ and a general interaction action $L[M, \Lambda]$ as follows

$$S[M, \Lambda] = \sqrt{\det(2\pi\tilde{\theta})} \int \frac{d^D p}{(2\pi)^D} \times \sum_{m,n,k,l} \frac{1}{2} \tilde{M}_{mn}(-p) G_{mn,kl}(p, \Lambda) \tilde{M}_{kl}(p) + L[M, \Lambda]. \tag{12.129}$$

Clearly we must have

$$S[M, \infty] = S[M], \;\; L[M, \infty] = L[M] \tag{12.130}$$

$$G_{mn,kl}(p, \infty) = G_{mn,kl}(p), \;\; F_{mn,kl}[p, \infty] = \delta_{ml}\delta_{nk}. \tag{12.131}$$

The extra piece $S'[M, \Lambda]$ is absent in the commutative theory. It is given by

$$S'[M, \Lambda] = C[\Lambda] + \frac{\nu_2}{2} \int \frac{d^D p}{(2\pi)^D} \sum_{m,n,k,l} J_{mn}(p) E_{mn,kl}[p, \Lambda] J_{kl}(-p). \tag{12.132}$$

We must have

$$C[\infty] = 0, \;\; E_{mn,kl}[p, \infty] = 0. \tag{12.133}$$

The Polchinski exact renormalization group equation is derived from the requirement that the path integral $Z[J, \Lambda]$ is independent of the cutoff Λ. We compute

$$\begin{aligned} -\Lambda \frac{\partial Z[J, \Lambda]}{\partial \Lambda} \;=\; & \int \prod\nolimits_{m,n,p} d\tilde{M}_{mn}(p) \Bigg[\Lambda \frac{\partial L[M, \Lambda]}{\partial \Lambda} \\ & + \nu_2 \int_p \sum_{m,n,k,l} \frac{1}{2} \tilde{M}_{mn}(-p) \Lambda \frac{\partial G_{mn,kl}(p, \Lambda)}{\partial \Lambda} \tilde{M}_{kl}(p) \\ & + \nu_2 \int_p \sum_{m,n,k,l} \tilde{M}_{mn}(p) \Lambda \frac{\partial F_{mn,kl}[p, \Lambda]}{\partial \Lambda} J_{kl}(-p) + \Lambda \frac{\partial C[\Lambda]}{\partial \Lambda} \\ & + \frac{\nu_2}{2} \int_p \sum_{m,n,k,l} J_{mn}(p) \Lambda \frac{\partial E_{mn,kl}[p, \Lambda]}{\partial \Lambda} J_{kl}(-p) \Bigg] \\ & \times \exp \Bigg[-S[M,\Lambda] \\ & - \nu_2 \int_p \sum_{m,n} \tilde{M}_{mn}(p) F_{mn,kl}[\Lambda] J_{kl}(-p) - S'[M, \Lambda] \Bigg]. \end{aligned} \tag{12.134}$$

By using $\partial \tilde{M}_{mn}(p)/\partial \tilde{M}_{kl}(q) = \delta_{mk}\delta_{nl}(2\pi)^D \delta^D(p - q)$ and $G_{mn,kl}(p, \Lambda) = G_{kl,mn}(-p, \Lambda)$ we find the results

$$\tilde{M}_{mn}(-p) = \frac{1}{\nu_2}\Delta_{nm,lk}(-p,\,\Lambda)\frac{\partial}{\partial\tilde{M}_{kl}(p)}(S[M,\,\Lambda] - L[M,\,\Lambda]) \tag{12.135}$$

$$\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{mn}(p)} = \nu_2\sum_{k,l}G_{mn,kl}(-p,\,\Lambda)\tilde{M}_{kl}(-p) + \frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(p)} \tag{12.136}$$

$$\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{mn}(p)} = \nu_2\sum_{k,l}F_{mn,kl}[p,\,\Lambda]J_{kl}(-p). \tag{12.137}$$

The propagator is as usual defined by

$$\Delta_{nm,lk}(p,\,\Lambda)G_{kl,st}(p,\,\Lambda) = G_{nm,lk}(p,\,\Lambda)\Delta_{kl,st}(p,\,\Lambda) = \delta_{nt}\delta_{ms}. \tag{12.138}$$

We have the result

$$\begin{aligned}\mathcal{R} &= \nu_2\int_p\sum_{m,n,k,l}\frac{1}{2}\tilde{M}_{mn}(-p)\Lambda\frac{\partial G_{mn,kl}(p,\,\Lambda)}{\partial\Lambda}\tilde{M}_{kl}(p)\\ &= -\frac{1}{2\nu_2}\int_p\sum_{m,n,k,l}\Lambda\frac{\partial\Delta_{nm,lk}(-p,\,\Lambda)}{\partial\Lambda}\\ &\quad\times\left[\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)} + \frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\right.\\ &\quad\left. - 2\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\right],\end{aligned} \tag{12.139}$$

where

$$\begin{aligned}\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\,e^{\cdots} &= \left[\frac{\partial^2}{\partial\tilde{M}_{kl}(p)\partial\tilde{M}_{mn}(-p)} + \frac{\partial^2 S[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)\partial\tilde{M}_{mn}(-p)}\right.\\ &\quad - \frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\\ &\quad - \frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\\ &\quad - \frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\nu_2 G_{mn,st}(p,\,\Lambda)\tilde{M}_{st}(p)\\ &\quad - \nu_2 G_{kl,st}(-p,\,\Lambda)\tilde{M}_{st}(-p)\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\\ &\quad\left. - \frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\right]e^{\cdots},\end{aligned} \tag{12.140}$$

and

$$\frac{\partial^2 S[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)\partial\tilde{M}_{mn}(-p)} = \nu_2 G_{mn,kl}(p,\,\Lambda) + \frac{\partial^2 L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)\partial\tilde{M}_{mn}(-p)}. \tag{12.141}$$

The first term in the above equation yields a total derivative in the path integral. The third and fourth terms lead to equal contributions. We have then

$$\begin{aligned}
\mathcal{R} &= \nu_2 \int_p \sum_{m,n,k,l} \frac{1}{2}\tilde{M}_{mn}(-p)\Lambda\frac{\partial G_{mn,kl}(p,\,\Lambda)}{\partial\Lambda}\tilde{M}_{kl}(p)\; e^{\cdots} \\
&= -\frac{1}{2\nu_2}\int_p \sum_{m,n,k,l} \Lambda\frac{\partial\Delta_{nm,lk}(-p,\,\Lambda)}{\partial\Lambda} \\
&\quad\times\left[\nu_2 G_{mn,kl}(p,\,\Lambda) + \frac{\partial^2 L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)\partial\tilde{M}_{mn}(-p)}\right. \\
&\quad - 2\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)} + \frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)} \\
&\quad\left. - 2\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\right] e^{\cdots} \\
&= -\frac{1}{2\nu_2}\int_p \sum_{m,n,k,l} \Lambda\frac{\partial\Delta_{nm,lk}(-p,\,\Lambda)}{\partial\Lambda} \\
&\quad\times\left[\nu_2 G_{mn,kl}(p,\,\Lambda) - \frac{\partial^2 L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)\partial\tilde{M}_{mn}(-p)}\right. \\
&\quad\left. + \frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\right] e^{\cdots} + \cdots.
\end{aligned} \tag{12.142}$$

In the above we have used $\Delta_{nm,lk}(-p,\,\Lambda) = \Delta_{lk,nm}(p,\,\Lambda)$ then we have neglected a term which will lead to a total derivative in the path integral. The last two terms can be cancelled if we choose $L[M,\,\Lambda]$ such that

$$\begin{aligned}
\Lambda\frac{\partial L[M,\,\Lambda]}{\partial\Lambda} &= \frac{1}{2\nu_2}\int_p \sum_{m,n,k,l} \Lambda\frac{\partial\Delta_{nm,lk}(-p,\,\Lambda)}{\partial\Lambda} \\
&\quad\times\left[\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)} - \frac{\partial^2 L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)\partial\tilde{M}_{mn}(-p)}\right].
\end{aligned} \tag{12.143}$$

The first term can be cancelled if we choose $C[\Lambda]$ (the vacuum energy) such that

$$\begin{aligned}\Lambda\frac{\partial C[\Lambda]}{\partial\Lambda} &= \frac{1}{2}\int_p \sum_{m,n,k,l} \Lambda\frac{\partial\Delta_{lk,nm}(p,\,\Lambda)}{\partial\Lambda}G_{mn,kl}(p,\,\Lambda)\\ &= \frac{1}{2}\int_p \sum_{m,n,k,l}\left(\Lambda\frac{\partial}{\partial\Lambda}\ln K_1\left(\frac{p^2}{\Lambda^2}\right)\right.\\ &\quad\left.+\sum_i \Lambda\frac{\partial}{\partial\Lambda}\ln K_2\left(\frac{{\mu_1}^2 i}{\Lambda^2}\right)\right)\Delta_{lk,nm}(p,\,\Lambda)G_{mn,kl}(p,\,\Lambda)\\ &= \frac{1}{2}\int_p \sum_{m,n}\Lambda\frac{\partial}{\partial\Lambda}\ln K_1\left(\frac{p^2}{\Lambda^2}\right)\\ &\quad+\int_p \sum_{m,n}\Lambda\frac{\partial}{\partial\Lambda}\ln K_2\left(\frac{{\mu_1}^2 m}{\Lambda^2}\right)K_2\left(\frac{{\mu_1}^2 n}{\Lambda^2}\right).\end{aligned} \tag{12.144}$$

We integrate this equation with the initial condition $C[\infty] = 0$ replaced with $C[\Lambda_0] = 0$. We obtain

$$\begin{aligned}C[\Lambda] &= \frac{1}{2}\int_p \sum_{m,n}\ln K_1\left(\frac{p^2}{\Lambda^2}\right)K_1^{-1}\left(\frac{p^2}{\Lambda_0^2}\right)\\ &\quad+2\int_p \sum_n \ln\prod_m K_2\left(\frac{{\mu_1}^2 m}{\Lambda^2}\right)K_2^{-1}\left(\frac{{\mu_1}^2 m}{\Lambda_0^2}\right).\end{aligned} \tag{12.145}$$

There remain the following terms

$$\begin{aligned}\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\,e^{\cdots} &= \left[-\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\nu_2 G_{mn,st}(p,\,\Lambda)\tilde{M}_{st}(p)\right.\\ &\quad-\nu_2 G_{kl,st}(-p,\,\Lambda)\tilde{M}_{st}(-p)\\ &\quad\left.\times\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}-\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\right]e^{\cdots}.\end{aligned} \tag{12.146}$$

The first two terms lead to equal contributions. They give

$$\begin{aligned}&\int_p \sum_{m,n,k,l}\sum_{s,t}\Lambda\frac{\partial\Delta_{nm,lk}(-p,\,\Lambda)}{\partial\Lambda}G_{kl,st}(-p,\,\Lambda)\tilde{M}_{st}(-p)\frac{\partial S_J[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}\\ &\quad=\nu_2\int_p\sum_{m,n,k,l}\tilde{M}_{mn}(p)\Delta F_{mn,kl}(p)J_{kl}(-p),\end{aligned} \tag{12.147}$$

where

$$\Delta F_{mn,kl}(p) = \sum_{s,t,s',t'} G_{mn,s't'}(-p,\,\Lambda)\Lambda\frac{\partial\Delta_{t's',ts}(-p,\,\Lambda)}{\partial\Lambda}F_{st,kl}[p,\,\Lambda]. \tag{12.148}$$

To cancel this term we choose $F_{mn,kl}[p, \Lambda]$ such that

$$\begin{aligned}\Lambda\frac{\partial F_{mn,kl}[p, \Lambda]}{\partial\Lambda} &= -\Delta F_{mn,kl}(p)\\ &= -\sum_{s,t,s',t'} G_{mn,s't'}(-p, \Lambda)\Lambda\frac{\partial\Delta_{t's',ts}(-p, \Lambda)}{\partial\Lambda}F_{st,kl}[p, \Lambda].\end{aligned} \tag{12.149}$$

The last term is quadratic in the source and it leads to the contribution

$$\begin{aligned}&\frac{1}{2\nu_2}\int_p \sum_{m,n,k,l}\Lambda\frac{\partial\Delta_{nm,lk}(-p, \Lambda)}{\partial\Lambda}\frac{\partial S_J[M, \Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial S_J[M, \Lambda]}{\partial\tilde{M}_{mn}(-p)}\\ &\quad= \frac{\nu_2}{2}\int_p \sum_{m,n,k,l} J_{mn}(p)\Delta E_{mn,kl}(p)J_{kl}(-p),\end{aligned} \tag{12.150}$$

where

$$\Delta E_{mn,kl}(p) = \sum_{s,t,s',t'} F_{st,mn}[-p, \Lambda]\Lambda\frac{\partial\Delta_{ts,t's'}(-p, \Lambda)}{\partial\Lambda}F_{s't',kl}[p, \Lambda]. \tag{12.151}$$

We choose $E_{mn,kl}[p, \Lambda]$ such that

$$\begin{aligned}\Lambda\frac{\partial E_{mn,kl}[p, \Lambda]}{\partial\Lambda} &= -\Delta E_{mn,kl}(p)\\ &= -\sum_{s,t,s',t'} F_{st,mn}[-p, \Lambda]\Lambda\frac{\partial\Delta_{ts,t's'}(-p, \Lambda)}{\partial\Lambda}F_{s't',kl}[p, \Lambda].\end{aligned} \tag{12.152}$$

We integrate equation (12.149) with the initial condition $F_{mn,kl}[p, \infty] = \delta_{ml}\delta_{nk}$ replaced with $F_{mn,kl}[p, \Lambda_0] = \delta_{ml}\delta_{nk}$. We obtain the solution

$$F_{mn,kl}[p, \Lambda] = \sum_{s',t'} G_{mn,s't'}(-p, \Lambda)\Delta_{t's',kl}(-p, \Lambda_0). \tag{12.153}$$

Thus, equation (12.152) becomes

$$\Lambda\frac{\partial E_{mn,kl}[p, \Lambda]}{\partial\Lambda} = \sum_{s,t,s',t'}\Delta_{mn,st}[-p, \Lambda_0]\Lambda\frac{\partial G_{ts,t's'}(-p, \Lambda)}{\partial\Lambda}\Delta_{s't',kl}[-p, \Lambda_0]. \tag{12.154}$$

The initial condition $E_{mn,kl}[p, \infty] = 0$ will be replaced with $E_{mn,kl}[p, \Lambda_0] = 0$ and then we obtain the solution

$$\begin{aligned}E_{mn,kl}[p, \Lambda] = \sum_{s,t,s',t'}&\Delta_{mn,st}[-p, \Lambda_0](G_{ts,t's'}(-p, \Lambda)\\ &- G_{ts,t's'}(-p, \Lambda_0))\Delta_{s't',kl}[p, \Lambda_0].\end{aligned} \tag{12.155}$$

It remains finally to solve the Wilson–Polchinski equation

$$\Lambda\frac{\partial L[M,\,\Lambda]}{\partial\Lambda}=\frac{1}{2\nu_2}\int_p\sum_{m,n,k,l}\Lambda\frac{\partial\Delta_{nm,lk}(-p,\,\Lambda)}{\partial\Lambda}\times\left[\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)}\frac{\partial L[M,\,\Lambda]}{\partial\tilde{M}_{mn}(-p)}-\frac{\partial^2 L[M,\,\Lambda]}{\partial\tilde{M}_{kl}(p)\partial\tilde{M}_{mn}(-p)}\right]. \tag{12.156}$$

The initial condition is

$$\begin{aligned}L[M,\,\Lambda_0]&=L[M]\\&=\nu_2\int_{p_1}\int_{p_2}\int_{p_3}\sum_{m,n,k,l}\frac{\lambda}{4!}\tilde{M}_{mn}(p_1)\tilde{M}_{nk}(p_2)\\&\quad\times\tilde{M}_{kl}(p_3)\tilde{M}_{lm}(-p_1-p_2-p_3).\end{aligned} \tag{12.157}$$

The vertices are given by

$$L_{m_1n_1\ldots m_Nn_N}[p_1,\,\ldots,\,p_N,\,\Lambda]=\frac{\partial^N L[M,\,\Lambda]}{\partial\tilde{M}_{m_1n_1}(p_1)\ldots\partial\tilde{M}_{m_Nn_N}(p_N)}|_{M=0}. \tag{12.158}$$

In the solution of the theory given by Feynman graphs the vertices are connected to each other by internal lines $\Delta_{mn,kl}(p,\,\Lambda)$ while they are connected to sources $J_{mn}(p)$ by external lines $\Delta_{mn,kl}(p,\,\Lambda_0)$. For finite Λ loops the summations are finite. Renormalizability is the requirement that the Wilson–Polchinski equation admits a regular solution which depends on a finite number of initial conditions.

12.3 Renormalization of scalar ϕ^4 in two dimensions

12.3.1 Wilson–Polchinski renormalization group equation

We start with the model

$$S[M]=\nu_2\sum_{m,n,k,l}\frac{1}{2}\tilde{M}_{mn}G_{mn,kl}\tilde{M}_{kl}+L[M]. \tag{12.159}$$

The Laplacian is

$$\begin{aligned}G_{mn,kl}&=\left(\mu^2+\mu_1^{\,2}(m+n-1)\right)\delta_{n,k}\delta_{m,l}\\&\quad-\mu_1^{\,2}\sqrt{\omega(m-1)(n-1)}\;\delta_{n-1,k}\delta_{m-1,l}\\&\quad-\mu_1^{\,2}\sqrt{\omega mn}\;\delta_{n+1,k}\delta_{m+1,l}.\end{aligned} \tag{12.160}$$

And the interaction term is

$$L[M]=\nu_2\frac{\lambda}{4!}\sum_{m,n,k,l}\tilde{M}_{mn}\tilde{M}_{nk}\tilde{M}_{kl}\tilde{M}_{lm}. \tag{12.161}$$

The parameters of the model are (with $\nu_2 = \sqrt{\det(2\pi\theta)} = \pi\theta$, $\sqrt{\omega} = (1 - \Omega^2)/(1 + \Omega^2)$)

$$\nu_2\mu^2 = \pi\theta\mu^2, \ \nu_2{\mu_1}^2\sqrt{\omega} = 2\pi(1 - \Omega^2), \ \nu_2\lambda = \pi\theta\lambda. \tag{12.162}$$

We introduce a momentum space cutoff Λ which will play the role of the running cutoff. The dimensionless noncommutativity is $\bar{\theta} = \theta\Lambda^2$. The regularized inverse propagator is

$$G_{mn,kl}(\Lambda) = G_{mn,kl} \prod_{i\in\{m,n,k,l\}} K^{-1}\left(\frac{{\mu_1}^2 i}{\Lambda^2}\right). \tag{12.163}$$

The ultraviolet cutoff function $K(x)$ is such that it is equal to 1 for $0 \leqslant x \leqslant 1$ and it vanishes rapidly for $x > 1$. The propagator is as usual defined by

$$\Delta_{nm,lk}(\Lambda)G_{kl,st}(\Lambda) = G_{nm,lk}(\Lambda)\Delta_{kl,st}(\Lambda) = \delta_{nl}\delta_{ms}. \tag{12.164}$$

We consider now the path integral

$$Z[J, \Lambda] = \int \prod_{m,n} d\tilde{M}_{mn} \, e^{-S[M,\Lambda]-\nu_2 \sum\limits_{m,n,k,l} \tilde{M}_{mn}F_{mn,kl}[\Lambda]J_{kl}-S'[M,\,\Lambda]}. \tag{12.165}$$

The action $S[M, \Lambda]$ is given in terms of the inverse propagator $G_{mn,k,l}(p, \Lambda)$ and a general interaction action $L[M, \Lambda]$ as follows

$$S[M, \Lambda] = \nu_2 \sum_{m,n,k,l} \frac{1}{2}\tilde{M}_{mn}G_{mn,kl}(\Lambda)\tilde{M}_{kl} + L[M, \Lambda]. \tag{12.166}$$

Clearly, we must have

$$S[M, \Lambda_0] = S[M], \ L[M, \Lambda_0] = L[M] \tag{12.167}$$

$$G_{mn,kl}(\Lambda_0) = G_{mn,kl}, \ F_{mn,kl}[\Lambda_0] = \delta_{ml}\delta_{nk}. \tag{12.168}$$

In the above equation Λ_0 is the ultraviolet cutoff or initial cutoff which must be sent to ∞ at the end. The extra piece $S'[M, \Lambda]$ is absent in the commutative theory. It is given by

$$S'[M, \Lambda] = C[\Lambda] + \frac{\nu_2}{2} \sum_{m,n,k,l} J_{mn}E_{mn,kl}[\Lambda]J_{kl}. \tag{12.169}$$

We must have

$$C[\Lambda_0] = 0, \ E_{mn,kl}[\Lambda_0] = 0. \tag{12.170}$$

The Polchinski exact renormalization group equation is derived from the requirement that the path integral $Z[J, \Lambda]$ is independent of the cutoff Λ, viz

$$-\Lambda\frac{\partial Z[J, \Lambda]}{\partial\Lambda} = 0. \tag{12.171}$$

We must then choose $L[M, \Lambda]$ such that

$$\Lambda\frac{\partial L[M,\Lambda]}{\partial\Lambda} = \frac{1}{2\nu_2}\sum_{m,n,k,l}\Lambda\frac{\partial\Delta_{nm,lk}(\Lambda)}{\partial\Lambda} \times\left[\frac{\partial L[M,\Lambda]}{\partial\tilde{M}_{kl}}\frac{\partial L[M,\Lambda]}{\partial\tilde{M}_{mn}} - \frac{\partial^2 L[M,\Lambda]}{\partial\tilde{M}_{kl}\partial\tilde{M}_{mn}}\right]. \tag{12.172}$$

We must choose the vacuum energy such that

$$C[\Lambda] = 2\sum_n \ln\prod\nolimits_m K_2\left(\frac{\mu_1^2 m}{\Lambda^2}\right)K_2^{-1}\left(\frac{\mu_1^2 m}{\Lambda_0^2}\right). \tag{12.173}$$

We must choose the functions F and E such that

$$F_{mn,kl}[\Lambda] = \sum_{s',t'} G_{mn,s't'}(\Lambda)\Delta_{t's',kl}(\Lambda_0) \tag{12.174}$$

$$E_{mn,kl}[\Lambda] = \sum_{s,t,s',t'}\Delta_{mn,st}(\Lambda_0)(G_{ts,t's'}(\Lambda) - G_{ts,t's'}(\Lambda_0))\Delta_{s't',kl}(\Lambda_0). \tag{12.175}$$

It remains to solve the Wilson–Polchinski equation (12.172).

12.3.2 Free propagator

We are dealing with the Laplacian

$$\begin{aligned} G_{mn,kl} = &\left(\mu^2 + \mu_1^2(m+n-1)\right)\delta_{n,k}\delta_{m,l} \\ &- \mu_1^2\sqrt{\omega(m-1)(n-1)}\;\delta_{n-1,k}\delta_{m-1,l} \\ &- \mu_1^2\sqrt{\omega mn}\;\delta_{n+1,k}\delta_{m+1,l}. \end{aligned} \tag{12.176}$$

The propagator is defined by

$$\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\Delta_{nm,lk}G_{kl,rs} = \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}G_{mn,kl}\Delta_{lk,sr} = \delta_{ns}\delta_{mr}. \tag{12.177}$$

We consider the massless case $\mu = 0$. The indices m,n,k,l of each term contributing to the Laplacian $G_{mn,kl}$ are such that $m + k = n + l$. Thus the indices m, n, k, l of each term contributing to the propagator $\Delta_{lk,nm}$ are such that $m + k = n + l$. By using $m + k = n + l$ we get with $n = m + \alpha$ and $s = r + \beta$ the identity

$$\sum_{l=1}^{\infty}G_{mm+\alpha,l+\alpha l}\Delta_{ll+\alpha,r+\beta r} = \delta_{\alpha\beta}\delta_{mr}. \tag{12.178}$$

For $\alpha \neq \beta$ we get $\sum_l G_{mm+\alpha,l+\alpha l}\Delta_{ll+\alpha,r+\beta r} = 0$ from which we conclude that

$$\Delta_{ll+\alpha,r+\beta r} = 0, \;\; \alpha \neq \beta. \tag{12.179}$$

For $\alpha = \beta$ we get the independent equations

$$\sum_{l=1}^{\infty} G_{mm+\alpha,l+\alpha l}\Delta_{ll+\alpha,r+\alpha r} = \delta_{mr}. \tag{12.180}$$

By defining $G_{mm+\alpha,l+\alpha l} = G^{(\alpha)}_{m,l}$, $\Delta_{ll+\alpha,r+\alpha r} = \Delta^{(\alpha)}_{l,\,r}$ we get for every α the ordinary matrix equation

$$\sum_{l=1}^{\infty} G^{(\alpha)}_{m,l}\Delta^{(\alpha)}_{l,r} = \delta_{mr}. \tag{12.181}$$

We introduce a cutoff $\mathcal{N}$. We have then to solve

$$\sum_{l=1}^{\mathcal{N}} G^{(\alpha)}_{m,l}\Delta^{(\alpha)}_{l,\,r} = \delta_{mr}. \tag{12.182}$$

As an example, let us choose $\mathcal{N} = 4$. We compute

$$\begin{aligned} G^{(\alpha)}_{m,l} - {\mu_1}^2 v\delta_{m,l} &= {\mu_1}^2(2m + \alpha - 1 - v)\delta_{m,l} - {\mu_1}^2\sqrt{\omega l(l+\alpha)} \\ &\times \delta_{m-1,l} - {\mu_1}^2\sqrt{\omega m(m+\alpha)}\;\delta_{m+1,l} \\ &= {\mu_1}^2\begin{pmatrix} \alpha+1-v & -\sqrt{\omega(1+\alpha)} & 0 & 0 \\ -\sqrt{\omega(1+\alpha)} & \alpha+3-v & -\sqrt{2\omega(2+\alpha)} & 0 \\ 0 & -\sqrt{2\omega(2+\alpha)} & \alpha+5-v & -\sqrt{3\omega(3+\alpha)} \\ 0 & 0 & -\sqrt{3\omega(3+\alpha)} & \alpha+7-v \end{pmatrix}. \end{aligned} \tag{12.183}$$

This can be rewritten as

$$G^{(\alpha)}_{m,l} - {\mu_1}^2 v\delta_{m,l} = {\mu_1}^2 TT^{+} \tag{12.184}$$

$$T = \begin{pmatrix} \sqrt{\alpha+1}\sqrt{A_1^{\alpha,\omega}(v)} & 0 & 0 & 0 \\ -\sqrt{\dfrac{\omega}{A_1^{\alpha,\omega}(v)}} & \sqrt{\alpha+2}\sqrt{A_2^{\alpha,\omega}(v)} & 0 & 0 \\ 0 & -\sqrt{\dfrac{2\omega}{A_2^{\alpha,\omega}(v)}} & \sqrt{\alpha+3}\sqrt{A_3^{\alpha,\omega}(v)} & 0 \\ 0 & 0 & -\sqrt{\dfrac{3\omega}{A_3^{\alpha,\omega}(v)}} & \sqrt{\alpha+4}\sqrt{A_4^{\alpha,\omega}(v)} \end{pmatrix} \tag{12.185}$$

$$A_n^{\alpha,\omega}(v) = \frac{1}{\alpha + n}(\alpha + 2n - 1 - v - \frac{(n-1)\omega}{A_{n-1}^{\alpha,\omega}(v)}),\ \ n \geqslant 1. \tag{12.186}$$

This last equation can be put in the form

$$nL_n^{\alpha,\omega}(v) - (\alpha + 2n - 1 - v)L_{n-1}^{\alpha,\omega}(v) + \omega(\alpha + n - 1)L_{n-2}^{\alpha,\omega}(v) = 0, \tag{12.187}$$

where

$$A_n^{\alpha,\omega}(v) = \frac{n}{\alpha + n}\frac{L_n^{\alpha,\omega}(v)}{L_{n-1}^{\alpha,\omega}(v)},\ \ L_0^{\alpha,\omega} \equiv 1. \tag{12.188}$$

The functions $L_n^{\alpha,\omega}(v)$ for $\omega = 1$ are the Laguerre polynomials. Thus, the functions $L_n^{\alpha,\omega}(v)$ are the deformed Laguerre polynomials.

The eigenvalues of the Laplacian are determined by the condition $\det\left(G_{m,l}^{(\alpha)} - \mu_1^2 v\delta_{m,l}\right) = 0$ which is equivalent to the condition $\det T = 0$. This leads to the condition $A_1A_2A_3A_4 = 0$ or equivalently $L_4^{\alpha,\omega}(v) = 0$. In general we will get the condition $L_{\mathcal{N}}^{\alpha,\omega}(v) = 0$. In other words, the eigenvalues are the zeroes of the deformed Laguerre polynomial $L_{\mathcal{N}}^{\alpha,\omega}(v)$, viz

$$L_{\mathcal{N}}^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right) = 0,\ \ i = 1, \ldots, \mathcal{N}. \tag{12.189}$$

Let $U_i^{(\alpha,\omega)}$ be the eigenvector of $G_{m,l}^{(\alpha)}$ with eigenvalue $\mu_1^2 v_i^{(\alpha,\omega)}$. We can write

$$G_{m,l}^{(\alpha)} = \sum_{i=1}^{\mathcal{N}} \mu_1^2 v_i^{(\alpha,\omega)}\left(U_i^{(\alpha,\omega)}\right)_{m+1}\left(U_i^{(\alpha,\omega)+}\right)_{l+1}. \tag{12.190}$$

We must have the completeness relation

$$\delta_{m,l} = \sum_{i=1}^{\mathcal{N}}\left(U_i^{(\alpha,\omega)}\right)_m\left(U_i^{(\alpha,\omega)+}\right)_l. \tag{12.191}$$

We must also have the orthogonality relation

$$\delta_{i,j} = \sum_{l=1}^{\mathcal{N}}\left(U_j^{(\alpha,\omega)+}\right)_l\left(U_i^{(\alpha,\omega)}\right)_l. \tag{12.192}$$

The propagator is, therefore, given by

$$\Delta_{m,l}^{(\alpha)} = \sum_{i=1}^{\mathcal{N}} \frac{1}{\mu_1^2 v_i^{(\alpha,\omega)}}\left(U_i^{(\alpha,\omega)}\right)_{m+1}\left(U_i^{(\alpha,\omega)+}\right)_{l+1}. \tag{12.193}$$

By inserting the mass we obtain

$$\Delta_{m,l}^{(\alpha)} = \sum_{i=1}^{\mathcal{N}} \frac{1}{\mu^2 + \mu_1^2 v_i^{(\alpha,\omega)}}\left(U_i^{(\alpha,\omega)}\right)_{m+1}\left(U_i^{(\alpha,\omega)+}\right)_{l+1}. \tag{12.194}$$

It remains to find the eigenvectors $(U_i^{(\alpha,\omega)})_j$. Let us do this for $\mathcal{N}=4$. We find by using $A_4^{\alpha,\omega}(v_i^{(\alpha,\omega)})=0$ the result

$$\begin{aligned}\left(U_i^{(\alpha,\omega)}\right)_3 &= \frac{\alpha+7-v_i^{(\alpha,\omega)}}{\sqrt{3\omega(3+\alpha)}}\left(U_i^{(\alpha,\omega)}\right)_4 \\ &= \sqrt{\frac{\omega(\alpha+3)}{3}}\,\frac{L_2^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right)}{L_3^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right)}\left(U_i^{(\alpha,\omega)}\right)_4.\end{aligned} \tag{12.195}$$

We also find

$$\begin{aligned}\left(U_i^{(\alpha,\omega)}\right)_2 &= \frac{(\alpha+7-v_i^{(\alpha,\omega)})(\alpha+5-v_i^{(\alpha,\omega)})-3\omega(3+\alpha)}{\sqrt{6\omega^2(2+\alpha)(3+\alpha)}}\left(U_i^{(\alpha,\omega)}\right)_4 \\ &= \sqrt{\frac{\omega^2(\alpha+2)(\alpha+3)}{6}}\,\frac{L_1^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right)}{L_3^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right)}\left(U_i^{(\alpha,\omega)}\right)_4\end{aligned} \tag{12.196}$$

$$\begin{aligned}\left(U_i^{(\alpha,\omega)}\right)_1 &= \frac{\sqrt{\omega(1+\alpha)}}{\alpha+1-v_i^{(\alpha,\omega)}}\left(U_i^{(\alpha,\omega)}\right)_2 \\ &= \sqrt{\frac{\omega^3(\alpha+1)(\alpha+2)(\alpha+3)}{6}}\,\frac{1}{L_3^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right)}\left(U_i^{(\alpha,\omega)}\right)_4.\end{aligned} \tag{12.197}$$

Generalization of this result is given by

$$\left(U_i^{(\alpha,\omega)}\right)_j = \sqrt{\frac{\omega^{\mathcal{N}}\Gamma(\alpha+\mathcal{N})\Gamma(j)}{\omega^j\Gamma(\alpha+j)\Gamma(\mathcal{N})}}\,\frac{L_{j-1}^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right)}{L_{\mathcal{N}-1}^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right)}\left(U_i^{(\alpha,\omega)}\right)_{\mathcal{N}}. \tag{12.198}$$

Normalizing this vector to 1 we get

$$\left(U_i^{(\alpha,\omega)}\right)_j = \sqrt{\frac{\dfrac{\Gamma(j)}{\omega^j\Gamma(\alpha+j)}\left(L_{j-1}^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right)\right)^2}{\displaystyle\sum_{h=1}^{\mathcal{N}}\frac{\Gamma(h)}{\omega^h\Gamma(\alpha+h)}\left(L_{h-1}^{\alpha,\omega}\left(v_i^{(\alpha,\omega)}\right)\right)^2}}. \tag{12.199}$$

Clearly, if $(U_i^{(\alpha,\omega)})_j$ are the components of the eigenvector associated with the eigenvalue i then $\left(U_j^{(\alpha,\omega)}\right)_i$ are the components of the eigenvector associated with the eigenvalue j.

We must also have

$$\left(U_i^{(\alpha,\omega)+}\right)_j = \left(U_i^{(\alpha,\omega)}\right)_j. \tag{12.200}$$

12.3.3 Scaling exponents

The regularized propagator is

$$\Delta_{nm,lk}(\Lambda) = \Delta_{nm,lk} \prod_{i\in\{m,n,k,l\}} K_2\left(\frac{{\mu_1}^2 i}{\Lambda^2}\right). \tag{12.201}$$

We compute

$$\begin{aligned} Q_{nm,lk}(\Lambda) &= \frac{1}{\nu_2}\Lambda\frac{\partial\Delta_{nm,lk}(\Lambda)}{\partial\Lambda} \\ &= -\sum_{j\in\{m,n,k,l\}} \frac{2{\mu_1}^2 j}{\nu_2\Lambda^2} K_2'\left(\frac{{\mu_1}^2 j}{\Lambda^2}\right) \\ &\quad\times \prod_{i\in\{m,n,k,l\}/\{j\}} K_2\left(\frac{{\mu_1}^2 i}{\Lambda^2}\right)\Delta_{nm,lk}. \end{aligned} \tag{12.202}$$

The cutoff function in two dimensions is defined by

$$K(n, m; \Lambda) = K_2\left(\frac{{\mu_1}^2 n}{\Lambda^2}\right)K_2\left(\frac{{\mu_1}^2 m}{\Lambda^2}\right). \tag{12.203}$$

We will also use the notation $K_2\left(\frac{{\mu_1}^2 m}{\Lambda^2}\right) = K_2(m)$. We choose $K_2(x)$ to be a smooth function on $\mathbf{R}^+$, i.e. $K_2 \in \mathbf{R}^+$ such that

$$K_2(x) = 1,\ \ 0 \leqslant x \leqslant 1; \quad K_2(x) = 0,\ \ x \geqslant 2. \tag{12.204}$$

In other words, for finite Λ there is only a finite number of indices m, n, k, l for which $Q_{nm,lk} \neq 0$. Clearly the support of $K_2'(x)$ is [1, 2]. In other words the indices m, n, k, l are such that their maximum is in the range given by $1 \leqslant \frac{{\mu_1}^2 i}{\Lambda^2} \leqslant 2$. Hence, $Q_{nm,lk}$ is non-zero only if

$$\frac{\Lambda^2}{{\mu_1}^2} \leqslant \max(m, n, k, l) \leqslant 2\frac{\Lambda^2}{{\mu_1}^2}. \tag{12.205}$$

The cutoff function $K_2(x)$ is also chosen such that the volume of the support of $Q_{nm,lk}$ scales as Λ^d, viz

$$\text{SCA} = \sum_m \text{sign}(\max_{n,l}|K(m, n; \Lambda)K(l + n - m, l; \Lambda)|) \leqslant C_d\left(\frac{\Lambda}{\mu_1}\right)^d. \tag{12.206}$$

Recall that the indices m, n, k, l of each term contributing to $Q_{nm,lk}$ are such that $n - m = k - l$. Note also that

$$Q_{nm,lk}(\Lambda) = \frac{1}{\nu_2}\Lambda\frac{\partial}{\partial\Lambda}\left(\ln \prod\nolimits_{j\in\{m,n,k,l\}} K_2\left(\frac{{\mu_1}^2 j}{\Lambda^2}\right)\right)\Delta_{nm,lk}(\Lambda). \tag{12.207}$$

The support of $Q_{nm,lk}$ can be estimated by

$$\sum_{m,n,k,l} \text{sign}|Q_{nm,lk}(\Lambda)| \leqslant \text{const} \times \sum_m \text{sign}\Big(\max_{n,l,k}|Q_{nm,lk}(\Lambda)|\Big). \tag{12.208}$$

We compute

$$\begin{aligned}\sum_m \text{sign}\Big(\max_{n,l,k}|Q_{nm,lk}(\Lambda)|\Big) &= \sum_m \text{sign}(\max_{n,l,k}|\Delta_{nm,lk}(\Lambda)|)\\ &= \sum_m \text{sign}(\max_{n,l}|K(m,\, n;\, \Lambda)\\ &\quad \times K(l+n-m,\, l;\, \Lambda)|)\\ &= \text{SCA}.\end{aligned} \tag{12.209}$$

It is trivial to see that

$$\begin{aligned}\text{SCA} &= \sum_m \text{sign}(\max_{n,l}|K_2(m)K_2(n)K_2(l)K_2(l+n-m)|)\\ &= \sum_m \text{sign}(|K_2(m)K_2(n_0)K_2(l_0)K_2(n_0+l_0-m)|)\\ &= \sum_{m=1}^{2\frac{\Lambda^2}{{\mu_1}^2}} 1\\ &= 2\frac{\Lambda^2}{{\mu_1}^2}.\end{aligned} \tag{12.210}$$

In other words, $d = 2$ as desired.

We define two exponents δ_0 and δ_1 by

$$\text{SCA}_0 = \max_{m,n,k,l}|Q_{nm;lk}(\Lambda)| \leqslant C_0\left(\frac{\mu_1}{\Lambda}\right)^{\delta_0}\delta_{n-m,k-l} \tag{12.211}$$

$$\text{SCA}_1 = \max_n\left(\sum_k \max_{m,l}|Q_{nm;lk}(\Lambda)|\right) \leqslant C_1\left(\frac{\mu_1}{\Lambda}\right)^{\delta_1}. \tag{12.212}$$

We compute δ_0 and δ_1. We have

$$\begin{aligned} |Q_{nm,lk}(\Lambda)| \leqslant & \sum_{j\in\{m,n,k,l\}} \left| \frac{2{\mu_1}^2 j}{\nu_2\Lambda^2} K_2'\left(\frac{{\mu_1}^2 j}{\Lambda^2}\right) \right. \\ & \left. \times \prod\nolimits_{i\in\{m,n,k,l\}/\{j\}} K_2\left(\frac{{\mu_1}^2 i}{\Lambda^2}\right)\Delta_{nm,lk} \right| \\ \leqslant & \frac{{\mu_1}^2(1+\sqrt{\omega})}{\pi}|\Delta^{\mathcal{C}}_{nm,lk}| \\ & \times \sum_{j\in\{m,n,k,l\}} \left| K_2'\left(\frac{{\mu_1}^2 j}{\Lambda^2}\right)\prod\nolimits_{i\in\{m,n,k,l\}/\{j\}} K_2\left(\frac{{\mu_1}^2 i}{\Lambda^2}\right) \right| . \end{aligned} \tag{12.213}$$

In the above we have used the facts that $j \leqslant 2\frac{\Lambda^2}{{\mu_1}^2}$ and that $\frac{\Lambda^2}{{\mu_1}^2} \leqslant \max(m,\, n,\, k,\, l) \leqslant 2\frac{\Lambda^2}{{\mu_1}^2}$ and hence the propagator $\Delta^{\mathcal{C}}_{nm,lk}$ is given by

$$\Delta^{\mathcal{C}}_{nm,lk} = \Delta_{nm,lk},\ \text{if}\ \frac{\Lambda^2}{{\mu_1}^2} \leqslant \max(m,\, n,\, k,\, l) \leqslant 2\frac{\Lambda^2}{{\mu_1}^2},\ \text{otherwise}\ \Delta^{\mathcal{C}}_{nm,lk} = 0. \tag{12.214}$$

Thus we get

$$|Q_{nm,lk}(\Lambda)| \leqslant \frac{4{\mu_1}^2(1+\sqrt{\omega})}{\pi}|\Delta^{\mathcal{C}}_{nm,lk}|\max_x|K_2'(x)|(\max_x|K_2(x)|)^3. \tag{12.215}$$

Assuming that $\max_x|K_2(x)| = 1$ we get

$$|Q_{nm,lk}(\Lambda)| \leqslant \frac{4{\mu_1}^2(1+\sqrt{\omega})}{\pi}|\Delta^{\mathcal{C}}_{nm,lk}|\max_x|K_2'(x)|. \tag{12.216}$$

Thus

$$\max_{nm,lk}|Q_{nm,lk}(\Lambda)| \leqslant \frac{4{\mu_1}^2(1+\sqrt{\omega})}{\pi}\max_{nm,lk}|\Delta^{\mathcal{C}}_{nm,lk}|\max_x|K_2'(x)| \tag{12.217}$$

$$\begin{aligned} \max_n\left(\sum_k \max_{m,l}|Q_{nm;lk}(\Lambda)|\right) \leqslant & \frac{4{\mu_1}^2(1+\sqrt{\omega})}{\pi} \\ & \times \max_n\left(\sum_k \max_{m,l}|\Delta^{\mathcal{C}}_{nm,lk}|\right)\max_x|K_2'(x)|. \end{aligned} \tag{12.218}$$

It remains to compute $\max_{nm,lk}|\Delta^{\mathcal{C}}_{nm,lk}|$ and $\max_n\left(\sum_k \max_{m,l}|\Delta^{\mathcal{C}}_{nm,lk}|\right)$. The starting point is the propagator

$$\Delta_{mm+\alpha,l+\alpha l}=\Delta^{(\alpha)}_{m,l}=\sum_{i=1}^{\mathcal{N}}\frac{1}{\mu^2+{\mu_1}^2 v_i^{(\alpha,\omega)}} \times\frac{\sqrt{\frac{m!\,l!}{\omega^{m+l}(\alpha+m)!\,(\alpha+l)!}}\,L_m^{\alpha,\omega}(v_i^{(\alpha,\omega)})L_l^{\alpha,\omega}(v_i^{(\alpha,\omega)})}{\sum_{h=0}^{\mathcal{N}-1}\frac{h!}{\omega^h(\alpha+h)!}(L_h^{\alpha,\omega}(v_i^{(\alpha,\omega)}))^2}. \tag{12.219}$$

Let us just note that for $\omega=1$ the eigenvalues $v_i^{(\alpha,\omega)}$ become the eigenvalues of the ordinary Laguerre polynomial $\mathcal{L}^{\alpha}_{\mathcal{N}}(v)$. In other words, the eigenvalues $v_i^{(\alpha,\omega)}$ become continuous variables v and the sum in the above equation becomes an integral.

We use the numerical estimations of Grosse and Wulkenhaar given by (with $\mathcal{C}=\Lambda^2/{\mu_1}^2$)

$$\begin{aligned}\max_{nm,lk}|\Delta^{\mathcal{C}}_{nm,lk}|&=\sqrt{\frac{3-2\omega}{\mu^4+4\mu^2{\mu_1}^2\mathcal{C}+4{\mu_1}^2(1-\omega)\mathcal{C}^2}}\,\delta_{m+k,n+l}\\&=\sqrt{\frac{3-2\omega}{\mu^4+4\mu^2\Lambda^2+4(1-\omega)\Lambda^4}}\,\delta_{m+k,n+l}.\end{aligned} \tag{12.220}$$

This was obtained numerically by (1) fixing $\mathcal{N}$ to a very large number (2) finding $\max_{nm,lk}|\Delta^{\mathcal{C}}_{nm,lk}|$ for indices m, n, k, l such that $\frac{\Lambda^2}{{\mu_1}^2}\leqslant\max(m,n,k,l)\leqslant 2\frac{\Lambda^2}{{\mu_1}^2}$ and for fixed parameters μ, μ_1 and ω using the exact formula (12.219), (3) varying Λ, (4) finding the best fit for the result and then (5) verify how the fit changes for other values of the parameters μ, μ_1 and ω.

We get then (with $m+k=n+l$, $0\leqslant\omega\leqslant 1$ and Λ very large)

$$\begin{aligned}\max_{nm,lk}|Q_{nm,lk}(\Lambda)|&\leqslant\frac{4{\mu_1}^2(1+\sqrt{\omega})}{\pi}\max_x|K_2'(x)|\\&\quad\times\sqrt{\frac{3-2\omega}{\mu^4+4\mu^2\Lambda^2+4(1-\omega)\Lambda^4}}\\&\leqslant\frac{{\mu_1}^2}{\Lambda^2\sqrt{1-\omega}}\frac{4\sqrt{3}}{\pi}\max_x|K_2'(x)|,\ \ \omega\neq 1\\&\leqslant\frac{{\mu_1}^2}{\mu\Lambda}\frac{4}{\pi}\max_x|K_2'(x)|,\ \ \omega=1.\end{aligned} \tag{12.221}$$

In other words, we have (with $C_0=\frac{4}{\pi}\max_x|K_2'(x)|$)

$$\begin{aligned}\max_{nm,lk}|Q_{nm,lk}(\Lambda)|&\leqslant C_0\frac{{\mu_1}^2}{\Lambda^2\sqrt{1-\omega}},\ \ \omega\neq 1\\&\leqslant C_0\frac{{\mu_1}^2}{\mu\Lambda},\ \ \omega=1.\end{aligned} \tag{12.222}$$

We also use their other numerical estimation

$$\begin{aligned}\max_n\left(\sum_k \max_{m,l}|\Delta^{\mathcal{C}}_{nm,lk}|\right) &= \frac{1}{\mu^2 + {\mu_1}^2(1-\omega)\mathcal{C}} \\ &= \frac{1}{\mu^2 + (1-\omega)\Lambda^2}.\end{aligned} \tag{12.223}$$

Thus (with $C_1 = \frac{8}{\pi}\max_x|K_2'(x)|$)

$$\begin{aligned}\max_n\left(\sum_k \max_{m,l}|Q_{nm;lk}(\Lambda)|\right) &\leqslant \frac{4{\mu_1}^2(1+\sqrt{\omega})}{\pi} \\ &\quad\times \frac{1}{\mu^2 + (1-\omega)\Lambda^2}\max_x|K_2'(x)| \\ &\leqslant C_1\frac{{\mu_1}^2}{(1-\omega)\Lambda^2},\ \ \omega \neq 1 \\ &\leqslant C_1\frac{{\mu_1}^2}{\mu^2},\ \ \omega = 1.\end{aligned} \tag{12.224}$$

12.3.4 Projection to the irrelevant part

The vertices are given by

$$L_{m_1n_1\ldots m_Nn_N}[\Lambda] = \frac{\partial^N L[M,\,\Lambda]}{\partial\tilde{M}_{m_1n_1}\ldots\partial\tilde{M}_{m_Nn_N}}|_{M=0}. \tag{12.225}$$

In the solution of the theory given by Feynman or ribbon graphs the vertices are connected to each other by internal lines $\Delta_{mn,kl}(\Lambda)$ while they are connected to sources J_{mn} by external lines $\Delta_{mn,kl}(\Lambda_0)$. It is clear that, for finite Λ, loop summations are finite. Renormalizability is the requirement that the Polchinski equation admits a regular solution which depends on a finite number of initial conditions.

We will deal with two scales. The initial scale Λ_0 and the renormalization scale Λ_R. In other words, $\Lambda_R \leqslant \Lambda \leqslant \Lambda_0$. So we can integrate from Λ_0 down to Λ or from Λ_R up to Λ.

The choice of initial conditions is at the same time determined by, and required to prove, the power-counting theorem. Thus, we have to make the correct ansatz for the initial interactions which we then verify as the correct set of relevant and marginal interactions using the power-counting theorem.

The Grosse and Wulkenhaar ansatz for the initial interactions, i.e. the set of relevant and marginal interactions, in $d = 2$ dimensions is given by

$$\begin{aligned}L[M,\,\Lambda_0,\,\Lambda_0,\,\omega,\,\rho^0] = \nu_2 &\sum_{m,n,k,l}\frac{1}{2}\rho^0_{[m]}\tilde{M}_{mn}\tilde{M}_{nm} \\ &+ \nu_2\frac{\lambda}{4!}\sum_{m,n,k,l}\tilde{M}_{mn}\tilde{M}_{nk}\tilde{M}_{kl}\tilde{M}_{lm}.\end{aligned} \tag{12.226}$$

This choice of the initial interactions is dictated by the fact that only the planar 2-point function at one-loop is divergent. Thus the interaction at $\Lambda = \Lambda_0$, i.e. the boundary condition must include a quadratic diagonal term. The coefficients $\rho^0_{[m]}$ are meant to renormalize the parameters μ^2 and ${\mu_1}^2$, i.e.

$$\rho^0_{[m]} = {\rho_1}^0 + 2m{\rho_2}^0. \tag{12.227}$$

The mass μ^2 should, therefore, be thought of as the renormalized mass and ${\rho_1}^0$ is the bare mass. Similarly, ${\rho_2}^0$ should be thought of as the bare value of the coupling ${\mu_1}^2$.

The off-diagonal quadratic term, which is proportional to ω, does not correspond to any divergences in dimension 2 so it does not need to appear in the initial interaction. It is irrelevant.

The coupling constants $\rho^0_{[m]}$ and ω must depend on Λ_0 in such a way that the limit $\Lambda_0 \longrightarrow \infty$ exists. Let us say that the difference between the two cases is that ω appears only in the propagator, whereas $\rho^0_{[m]}$ appears in the interaction.

The marginal part of the 4-point function is independent of the scale Λ_0, i.e. $\rho^0_4 = \lambda$. In other terms, the 4-point function is finite and thus it does not require a renormalization and hence λ is super-renormalizable.

The goal is to integrate the Polchinski equation (12.172) with the initial condition (12.226). The solution will depend on the current cutoff Λ, on the initial cutoff Λ_0 and on the parameters ω and $\rho^0_{[m]}$. As we have said $\omega = \omega(\Lambda_0)$ and $\rho^0_{[m]} = \rho^0_{[m]}(\Lambda_0)$. The solution will also depend on the renormalized couplings μ and λ but these are fixed numbers. The solution can always be put in the form

$$\begin{aligned} L[M, \Lambda, \Lambda_0, \omega, \rho^0] = \nu_2 \sum_{m,n,k,l} \frac{1}{2}\rho_{[m]}[\Lambda, \Lambda_0, \omega, \rho^0]\tilde{M}_{mn}\tilde{M}_{nm} \\ + \nu_2\frac{\lambda}{4!}\sum_{m,n,k,l} \tilde{M}_{mn}\tilde{M}_{nk}\tilde{M}_{kl}\tilde{M}_{lm} \\ + \text{other } M \text{ structures.} \end{aligned} \tag{12.228}$$

We have $\rho_{[m]}[\Lambda_0, \Lambda_0, \omega, \rho^0] = \rho^0_{[m]}$.

The renormalization conditions are conditions which must be satisfied by the ρ-coefficients at the renormalization cutoff Λ_R, viz $\rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0] = \text{constant}$. We will choose in particular

$$\rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0] = 0. \tag{12.229}$$

Thus, in particular $\rho_{[0]}[\Lambda_R, \Lambda_0, \omega, \rho^0] = 0$ and hence μ^2 is the renormalized mass as pointed out above.

Although the initial interaction at $\Lambda = \Lambda_0$, which is given by equation (12.226), is very simple, the interaction $L[M, \Lambda, \Lambda_0, \omega, \rho^0]$ at lower scales $\Lambda < \Lambda_0$ can become very complicated. At the renormalization scale Λ_R, which is far below Λ_0, we expect that the interaction $L[M, \Lambda_R, \Lambda_0, \omega, \rho^0]$ becomes simple again. So no matter what the initial interaction $L[M, \Lambda_0, \Lambda_0, \omega, \rho^0]$ is, the interaction $L[M, \Lambda_R, \Lambda_0, \omega, \rho^0]$

will be attracted to a p-dimensional submanifold in the infinite-dimensional space of interactions.

The p renormalizable operators in two dimensions are given by $\rho_{[m]}[\Lambda, \Lambda_0, \omega, \rho^0]$. We should also include λ and ω to the set of renormalizable operators. A good system of coordinates of the space of interactions is given by the vertices $L_{m_1 n_1 \ldots m_N n_N}[\Lambda]$. We need to determine relevant and irrelevant parts of the action.

The basic starting objects are the vectors in the space of actions given by

$$\Lambda_0 \frac{\partial L[M, \Lambda, \Lambda_0, \omega, \rho^0]}{\partial \Lambda_0} \tag{12.230}$$

$$\Lambda_0 \frac{\partial L[M, \Lambda, \Lambda_0, \omega, \rho^0]}{\partial \rho^0_{[n]}} \tag{12.231}$$

$$\Lambda_0 \frac{\partial L[M, \Lambda, \Lambda_0, \omega, \rho^0]}{\partial \omega}. \tag{12.232}$$

We will fix Λ at the renormalization scale Λ_R, i.e. $\Lambda = \Lambda_R$ and vary the initial scale Λ_0. We need first to compute the following difference

$$L[M, \Lambda_R, \Lambda_0', \omega' \rho^{0'}] - L[M, \Lambda_R, \Lambda_0'', \omega'' \rho^{0''}]. \tag{12.233}$$

We have the following

$$\begin{aligned}
&L[M, \Lambda_R, \Lambda_0', \omega' \rho^{0'}] - L[M, \Lambda_R, \Lambda_0'', \omega'' \rho^{0''}] \\
&= \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} \left(\Lambda_0 \frac{dL[M, \Lambda_R, \Lambda_0, \omega, \rho^0]}{d\Lambda_0} \right) \\
&= \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} \left(\Lambda_0 \frac{\partial L[M, \Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \Lambda_0} \right. \\
&+ \Lambda_0 \frac{d\omega}{d\Lambda_0} \frac{\partial L[M, \Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \omega} \\
&\left. + \Lambda_0 \frac{d\rho^0_{[n]}}{d\Lambda_0} \frac{\partial L[M, \Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \rho^0_{[n]}} \right).
\end{aligned} \tag{12.234}$$

From the renormalization condition $\rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0] = \text{constant}$ we have

$$d\rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0] = \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \Lambda_0} d\Lambda_0 + \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \omega} d\omega + \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \rho^0_{[n]}} d\rho^0_{[n]}. \tag{12.235}$$

If we assume that $\frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \rho^0_{[m]}}$ can be inverted then we can write

$$\frac{d\rho^0_{[n]}}{d\Lambda_0} = -\frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]} \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \Lambda_0} - \frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]} \frac{\partial \rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0]}{\partial \omega} \frac{d\omega}{d\Lambda_0}. \tag{12.236}$$

We get then

$$L[M, \Lambda_R, \Lambda_0', \omega' \rho^{0\prime}] - L[M, \Lambda_R, \Lambda_0'', \omega'' \rho^{0\prime\prime}] = \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0} R[M, \Lambda_R, \Lambda_0, \omega, \rho^0]. \tag{12.237}$$

Where R is defined by

$$\begin{aligned} R[M, \Lambda, \Lambda_0, \omega, \rho^0] = {} & \Lambda_0 \frac{\partial L[M, \Lambda, \Lambda_0, \omega, \rho^0]}{\partial \Lambda_0} \\ & + \Lambda_0 \frac{d\omega}{d\Lambda_0} \frac{\partial L[M, \Lambda, \Lambda_0, \omega, \rho^0]}{\partial \omega} \\ & - \sum_{m,n} \Lambda_0 \frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}[\Lambda, \Lambda_0, \omega, \rho^0]} \\ & \times \frac{\partial \rho_{[m]}[\Lambda, \Lambda_0, \omega, \rho^0]}{\partial \Lambda_0} \frac{\partial L[M, \Lambda, \Lambda_0, \omega, \rho^0]}{\partial \rho^0_{[n]}} \\ & - \sum_{m,n} \Lambda_0 \frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}[\Lambda, \Lambda_0, \omega, \rho^0]} \\ & \times \frac{\partial \rho_{[m]}[\Lambda, \Lambda_0, \omega, \rho^0]}{\partial \omega} \frac{d\omega}{d\Lambda_0} \frac{\partial L[M, \Lambda, \Lambda_0, \omega, \rho^0]}{\partial \rho^0_{[n]}}. \end{aligned} \tag{12.238}$$

Since at small Λ the interaction $L[M, \Lambda, \Lambda_0, \omega, \rho^0]$ becomes simple of the form (12.226) we expect that R vanishes at small Λ near Λ_R, i.e. it becomes very small in the infrared. This is equivalent to the statement that the vectors (12.230)–(12.232)

become parallel to the relevant p-dimensional submanifold and that R is the linear combination which vanishes in the p relevant directions.

Using the fact that $\frac{\partial\rho_{[m]}[\Lambda_R,\Lambda_0,\omega,\rho^0]}{\partial\rho^0_{[m]}}$ is the inverse of $\frac{\partial\rho^0_{[m]}}{\partial\rho_{[m]}[\Lambda_R,\Lambda_0,\omega,\rho^0]}$ we can show that if we replace in R the interaction L by the quadratic effective term in equation (12.228) we get zero. It is trivial that the quartic term proportional to λ leads to zero since we are assuming that λ is just a number. In other words, since R is linear in L it is trivial to see that the splitting given in equation (12.228) leads for all Λ to a vanishing R on the ρ-operators. Thus R projects to the irrelevant part of the effective action. Let us also note from the above equation that ω appears more like the cutoff Λ_0 than like the coupling constants $\rho^0_{[n]}$.

12.3.5 Flow equations

We compute

$$\begin{aligned}
\Lambda\frac{\partial R}{\partial\Lambda} = {}& \Lambda_0\frac{\partial\left(\Lambda\frac{\partial L}{\partial\Lambda}\right)}{\partial\Lambda_0} + \Lambda_0\frac{d\omega}{d\Lambda_0}\frac{\partial\left(\Lambda\frac{\partial L}{\partial\Lambda}\right)}{\partial\omega} \\
& - \sum_{m,n}\Lambda_0\frac{\partial\rho^0_{[n]}}{\partial\rho_{[m]}}\frac{\partial\rho_{[m]}}{\partial\Lambda_0}\frac{\partial\left(\Lambda\frac{\partial L}{\partial\Lambda}\right)}{\partial\rho^0_{[n]}} \\
& - \sum_{m,n}\Lambda_0\frac{\partial\rho^0_{[n]}}{\partial\rho_{[m]}}\frac{\partial\rho_{[m]}}{\partial\omega}\frac{d\omega}{d\Lambda_0}\frac{\partial\left(\Lambda\frac{\partial L}{\partial\Lambda}\right)}{\partial\rho^0_{[n]}} \\
& - \sum_{m,n}\Lambda_0\frac{\partial\rho^0_{[n]}}{\partial\rho_{[m]}}\frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\partial\Lambda}\right)}{\partial\Lambda_0}\frac{\partial L}{\partial\rho^0_{[n]}} \\
& - \sum_{m,n}\Lambda_0\frac{\partial\rho^0_{[n]}}{\partial\rho_{[m]}}\frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\partial\Lambda}\right)}{\partial\omega}\frac{d\omega}{d\Lambda_0}\frac{\partial L}{\partial\rho^0_{[n]}} \\
& + \sum_{m,n,k,l}\Lambda_0\frac{\partial\rho^0_{[n]}}{\partial\rho_{[k]}}\frac{\partial\left(\Lambda\frac{\partial\rho_{[k]}}{\partial\Lambda}\right)}{\partial\rho^0_{[l]}}\frac{\partial\rho^0_{[l]}}{\partial\rho_{[m]}}\frac{\partial\rho_{[m]}}{\partial\Lambda_0}\frac{\partial L}{\partial\rho^0_{[n]}} \\
& + \sum_{m,n,k,l}\Lambda_0\frac{\partial\rho^0_{[n]}}{\partial\rho_{[k]}}\frac{\partial\left(\Lambda\frac{\partial\rho_{[k]}}{\partial\Lambda}\right)}{\partial\rho^0_{[l]}}\frac{\partial\rho^0_{[l]}}{\partial\rho_{[m]}}\frac{\partial\rho_{[m]}}{\partial\omega}\frac{d\omega}{d\Lambda_0}\frac{\partial L}{\partial\rho^0_{[n]}}.
\end{aligned} \tag{12.239}$$

In the last two terms we have used the identity

$$-\Lambda\frac{\partial}{\partial\Lambda}\left(\frac{\partial\rho^0_{[n]}}{\partial\rho_{[m]}}\right) = \frac{\partial\rho^0_{[n]}}{\partial\rho_{[k]}}\frac{\partial\left(\Lambda\frac{\partial\rho_{[k]}}{\partial\Lambda}\right)}{\partial\rho^0_{[l]}}\frac{\partial\rho^0_{[l]}}{\partial\rho_{[m]}}. \tag{12.240}$$

We define

$$\frac{\partial L}{\partial \rho_{[m]}} = \sum_n \frac{\partial L}{\partial \rho^0_{[n]}} \frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}}. \tag{12.241}$$

Thus

$$\begin{aligned}
\Lambda \frac{\partial R}{\partial \Lambda} = {} & \Lambda_0 \frac{\partial \left(\Lambda \frac{\partial L}{\Lambda \Lambda}\right)}{\partial \Lambda_0} + \Lambda_0 \frac{d\omega}{d\Lambda_0} \frac{\partial \left(\Lambda \frac{\partial L}{\partial \Lambda}\right)}{\partial \omega} - \sum_m \Lambda_0 \frac{\partial \rho_{[m]}}{\partial \Lambda_0} \frac{\partial \left(\Lambda \frac{\partial L}{\partial \Lambda}\right)}{\partial \rho_{[m]}} \\
& - \sum_m \Lambda_0 \frac{\partial \rho_{[m]}}{\partial \omega} \frac{d\omega}{d\Lambda_0} \frac{\partial \left(\Lambda \frac{\partial L}{\partial \Lambda}\right)}{\partial \rho_{[m]}} - \sum_m \Lambda_0 \frac{\partial \left(\Lambda \frac{\partial \rho_{[m]}}{\partial \Lambda}\right)}{\partial \Lambda_0} \frac{\partial L}{\partial \rho_{[m]}} \\
& - \sum_m \Lambda_0 \frac{\partial \left(\Lambda \frac{\partial \rho_{[m]}}{\partial \Lambda}\right)}{\partial \omega} \frac{d\omega}{d\Lambda_0} \frac{\partial L}{\partial \rho_{[m]}} \\
& + \sum_{m,k} \Lambda_0 \frac{\partial \left(\Lambda \frac{\partial \rho_{[k]}}{\partial \Lambda}\right)}{\partial \rho_{[m]}} \frac{\partial \rho_{[m]}}{\partial \Lambda_0} \frac{\partial L}{\partial \rho_{[k]}} \\
& + \sum_{m,k} \Lambda_0 \frac{\partial \left(\Lambda \frac{\partial \rho_{[k]}}{\partial \Lambda}\right)}{\partial \rho_{[m]}} \frac{\partial \rho_{[m]}}{\partial \omega} \frac{d\omega}{d\Lambda_0} \frac{\partial L}{\partial \rho_{[k]}}.
\end{aligned} \tag{12.242}$$

Using the Polchinski equation we define

$$\begin{aligned}
M\left[L, \frac{\partial L}{\partial \Lambda_0}\right] &\equiv \frac{\partial}{\partial \Lambda_0}\left(\Lambda \frac{\partial L}{\partial \Lambda}\right) \\
&= \frac{1}{2\nu_2} \int_p \sum_{m,n,k,l} \Lambda \frac{\partial \Delta_{nm,lk}(\Lambda)}{\partial \Lambda} \\
&\quad \times \left[2 \frac{\partial L}{\partial \tilde{M}_{kl}} \frac{\partial \left(\frac{\partial L}{\partial \Lambda_0}\right)}{\partial \tilde{M}_{mn}} - \frac{\partial^2 \left(\frac{\partial L}{\partial \Lambda_0}\right)}{\partial \tilde{M}_{kl} \partial \tilde{M}_{mn}} \right]
\end{aligned} \tag{12.243}$$

$$\begin{aligned}
M\left[L, \frac{\partial L}{\partial \rho^0_{[q]}}\right] &\equiv \frac{\partial}{\partial \rho^0_{[q]}}\left(\Lambda \frac{\partial L}{\partial \Lambda}\right) \\
&= \frac{1}{2\nu_2} \int_p \sum_{m,n,k,l} \Lambda \frac{\partial \Delta_{nm,lk}(\Lambda)}{\partial \Lambda} \\
&\quad \times \left[2 \frac{\partial L}{\partial \tilde{M}_{kl}} \frac{\partial \left(\frac{\partial L}{\partial \rho^0_{[q]}}\right)}{\partial \tilde{M}_{mn}} - \frac{\partial^2 \left(\frac{\partial L}{\partial \rho^0_{[q]}}\right)}{\partial \tilde{M}_{kl} \partial \tilde{M}_{mn}} \right]
\end{aligned} \tag{12.244}$$

$$
\begin{aligned}
M\left[L, \frac{\partial L}{\partial \omega}\right] &\equiv \frac{\partial}{\partial \omega}\left(\Lambda \frac{\partial L}{\partial \Lambda}\right) \\
&= \frac{1}{2\nu_2}\int_p \sum_{m,n,k,l} \Lambda \frac{\partial \Delta_{nm,lk}(\Lambda)}{\partial \Lambda} \\
&\quad \times \left[2\frac{\partial L}{\partial \tilde{M}_{kl}}\frac{\partial\left(\frac{\partial L}{\partial \omega}\right)}{\partial \tilde{M}_{mn}} - \frac{\partial^2\left(\frac{\partial L}{\partial \omega}\right)}{\partial \tilde{M}_{kl}\partial \tilde{M}_{mn}}\right].
\end{aligned}
\tag{12.245}
$$

We can make the expansion

$$
M[L, \ldots] = \frac{1}{2}\sum_{m,n} M_{[m]}[L, \ldots]\tilde{M}_{mn}\tilde{M}_{nm} + \text{other } M \text{ structures}. \tag{12.246}
$$

Clearly we can make the identification

$$
M_{[m]}\left[L, \frac{\partial L}{\partial \Lambda_0}\right] = \frac{\partial}{\partial \Lambda_0}\left(\Lambda \frac{\partial \rho_{[m]}}{\partial \Lambda}\right) \tag{12.247}
$$

$$
M_{[m]}\left[L, \frac{\partial L}{\partial \rho^0_{[n]}}\right] = \frac{\partial}{\partial \rho^0_{[n]}}\left(\Lambda \frac{\partial \rho_{[m]}}{\partial \Lambda}\right) \tag{12.248}
$$

$$
M_{[m]}\left[L, \frac{\partial L}{\partial \omega}\right] = \frac{\partial}{\partial \omega}\left(\Lambda \frac{\partial \rho_{[m]}}{\partial \Lambda}\right). \tag{12.249}
$$

Recall that R is given by

$$
\begin{aligned}
R &= \Lambda_0 \frac{\partial L}{\partial \Lambda_0} + \Lambda_0 \frac{d\omega}{d\Lambda_0}\frac{\partial L}{\partial \omega} - \sum_m \Lambda_0 \frac{\partial \rho_{[m]}}{\partial \Lambda_0}\frac{\partial L}{\partial \rho_{[m]}} \\
&\quad - \sum_m \Lambda_0 \frac{\partial \rho_{[m]}}{\partial \omega}\frac{d\omega}{d\Lambda_0}\frac{\partial L}{\partial \rho_{[m]}} \\
&= \Lambda_0 \frac{\partial L}{\partial \Lambda_0} + \Lambda_0 \frac{d\omega}{d\Lambda_0}\frac{\partial L}{\partial \omega} - \sum_{m,n} \Lambda_0 \frac{\partial \rho_{[m]}}{\partial \Lambda_0}\frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}}\frac{\partial L}{\partial \rho^0_{[n]}} \\
&\quad - \sum_{m,n} \Lambda_0 \frac{\partial \rho_{[m]}}{\partial \omega}\frac{d\omega}{d\Lambda_0}\frac{\partial \rho^0_{[n]}}{\partial \rho_{[m]}}\frac{\partial L}{\partial \rho^0_{[n]}} \\
&= \hat{R}(L).
\end{aligned}
\tag{12.250}
$$

The operator $\hat{R}$ is a linear combination of the operators $\partial/\partial\Lambda_0$, $\partial/\partial\rho^0_{[n]}$ and $\partial/\partial\omega$. M is linear in the second argument. Thus

$$\begin{aligned} M[L,\,R] = \Lambda_0 \frac{\partial\left(\Lambda\frac{\partial L}{\partial\Lambda}\right)}{\partial\Lambda_0} + \Lambda_0\frac{d\omega}{d\Lambda_0}\frac{\partial\left(\Lambda\frac{\partial L}{\partial\Lambda}\right)}{\partial\omega} \\ - \sum_m \Lambda_0 \frac{\partial\rho_{[m]}}{\partial\Lambda_0}\frac{\partial\left(\Lambda\frac{\partial L}{\partial\Lambda}\right)}{\partial\rho_{[m]}} - \sum_m \Lambda_0\frac{\partial\rho_{[m]}}{\partial\omega}\frac{d\omega}{d\Lambda_0}\frac{\partial\left(\Lambda\frac{\partial L}{\partial\Lambda}\right)}{\partial\rho_{[m]}}. \end{aligned} \tag{12.251}$$

Similarly

$$\begin{aligned} M_{[m]}[L,\,R] = \Lambda_0 \frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\Lambda\Lambda}\right)}{\partial\Lambda_0} + \Lambda_0\frac{d\omega}{d\Lambda_0}\frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\partial\Lambda}\right)}{\partial\omega} \\ - \sum_k \Lambda_0 \frac{\partial\rho_{[k]}}{\partial\Lambda_0}\frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\partial\Lambda}\right)}{\partial\rho_{[k]}} - \sum_k \Lambda_0\frac{\partial\rho_{[k]}}{\partial\omega}\frac{d\omega}{d\Lambda_0}\frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\partial\Lambda}\right)}{\partial\rho_{[k]}}. \end{aligned} \tag{12.252}$$

Hence

$$\begin{aligned} \Lambda\frac{\partial R}{\partial\Lambda} &= M[L,\,R] - \sum_m \Lambda_0 \frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\partial\Lambda}\right)}{\partial\Lambda_0}\frac{\partial L}{\partial\rho_{[m]}} \\ &\quad - \sum_m \Lambda_0 \frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\partial\Lambda}\right)}{\partial\omega}\frac{d\omega}{d\Lambda_0}\frac{\partial L}{\partial\rho_{[m]}} \\ &\quad + \sum_{m,k} \Lambda_0 \frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\partial\Lambda}\right)}{\partial\rho_{[k]}}\frac{\partial\rho_{[k]}}{\partial\Lambda_0}\frac{\partial L}{\partial\rho_{[m]}} \\ &\quad + \sum_{m,k} \Lambda_0 \frac{\partial\left(\Lambda\frac{\partial\rho_{[m]}}{\partial\Lambda}\right)}{\partial\rho_{[k]}}\frac{\partial\rho_{[k]}}{\partial\omega}\frac{d\omega}{d\Lambda_0}\frac{\partial L}{\partial\rho_{[m]}} \\ &= M[L,\,R] - \sum_m M_{[m]}[L,\,R]\frac{\partial L}{\partial\rho_{[m]}}. \end{aligned} \tag{12.253}$$

We can also compute the Λ-scaling

$$\Lambda\frac{\partial}{\partial\Lambda}\left(\frac{\partial L}{\partial\rho_{[n]}}\right) = M\left[L,\,\frac{\partial L}{\partial\rho_{[n]}}\right] - \sum_m M_{[m]}\left[L,\,\frac{\partial L}{\partial\rho_{[n]}}\right]\frac{\partial L}{\partial\rho_{[m]}}. \tag{12.254}$$

12.3.6 Expansion in power series

With our normalization the effective action L is dimensionless. The dimensionless coupling is $\nu_2\lambda = \pi\theta\lambda$. The field $\tilde{M}$ is dimensionless in dimension two. We expand L in the coupling constant $\nu_2\lambda$ and in the number of fields $\tilde{M}$ as

$$L = \sum_{V=1}^{\infty}(\nu_2\lambda)^{V-1}\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{m_1n_1,\ldots,m_Nn_N} A^{(V)}_{m_1n_1,\ldots,\,m_Nn_N}\tilde{M}_{m_1n_1}\ldots\tilde{M}_{m_Nn_N}. \tag{12.255}$$

R is also dimensionless. We expand R as

$$R = \sum_{V=1}^{\infty}(\nu_2\lambda)^{V-1}\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{m_1n_1,\ldots,m_Nn_N} R^{(V)}_{m_1n_1,\ldots,\,m_Nn_N}\tilde{M}_{m_1n_1}\ldots\tilde{M}_{m_Nn_N}. \tag{12.256}$$

$\partial L/\partial\rho_{[m]}$ is of dimension $1/M^2$ since $\rho_{[m]}$ is of dimension M^2. We expand it as

$$\frac{\partial L}{\partial\rho_{[m]}} = \frac{1}{\lambda}\sum_{V=0}^{\infty}(\nu_2\lambda)^{V}\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{m_1n_1,\ldots,m_Nn_N} H^{(V)}_{m_1n_1,\ldots,\,m_Nn_N}\tilde{M}_{m_1n_1}\ldots\tilde{M}_{m_Nn_N}. \tag{12.257}$$

Define

$$Q_{nm,lk}(\Lambda) = \frac{1}{\nu_2}\Lambda\frac{\partial\Delta_{nm,lk}(\Lambda)}{\partial\Lambda}. \tag{12.258}$$

Compute

$$\begin{aligned}\frac{\partial L}{\partial\tilde{M}_{kl}} = &\sum_{V=1}^{\infty}(\nu_2\lambda)^{V-1}\sum_{N=2}^{\infty}\frac{1}{(N-1)!}\\ &\times\sum_{m_1n_1,\ldots,m_{N-1}n_{N-1}} A^{(V)}_{m_1n_1,\ldots,\,m_{N-1}n_{N-1},kl}\tilde{M}_{m_1n_1}\ldots\tilde{M}_{m_{N-1}n_{N-1}}.\end{aligned} \tag{12.259}$$

We need to compute

$$\frac{1}{2}\sum_{m,n,k,l} Q_{nm,lk}(\Lambda)\frac{\partial L}{\partial\tilde{M}_{kl}}\frac{\partial L}{\partial\tilde{M}_{mn}}, \tag{12.260}$$

and

$$\frac{1}{2}\sum_{m,n,k,l} Q_{nm,lk}(\Lambda)\frac{\partial^2 L}{\partial\tilde{M}_{kl}\partial\tilde{M}_{mn}}. \tag{12.261}$$

We compute up to the third power in the fields the following

$$
\begin{aligned}
\frac{1}{2}\sum_{m,n,k,l} Q_{nm,lk}(\Lambda)\frac{\partial L}{\partial \tilde{M}_{kl}}\frac{\partial L}{\partial \tilde{M}_{mn}} &= \sum_{V=2}^{\infty}(\nu_2\lambda)^{V-2} \\
&\times \frac{1}{2!}\sum_{m_1,n_1,m_2,n_2}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l}\frac{1}{2}Q_{nm,lk}(\Lambda) \\
&\times \left(A^{(V_1)}_{m_1n_1,mn}A^{(V-V_1)}_{m_2n_2,kl} + A^{(V_1)}_{m_2n_2,mn}A^{(V-V_1)}_{m_1n_1,kl}\right) \\
&\times \tilde{M}_{m_1n_1}\tilde{M}_{m_2n_2} + \sum_{V=2}^{\infty}(\nu_2\lambda)^{V-2} \\
&\times \frac{1}{3!}\sum_{m_1,n_1,m_2,n_2,m_3n_3}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l}\frac{1}{2}Q_{nm,lk}(\Lambda) \\
&\times \left(A^{(V_1)}_{m_1n_1,mn}A^{(V-V_1)}_{m_2n_2m_3n_3,kl} + A^{(V_1)}_{m_2n_2,mn}A^{(V-V_1)}_{m_1n_1m_3n_3,kl}\right. \\
&\left. + A^{(V_1)}_{m_3n_3,mn}A^{(V-V_1)}_{m_2n_2m_1n_1,kl}\right)\tilde{M}_{m_1n_1}\tilde{M}_{m_2n_2}\tilde{M}_{m_3n_3} + \sum_{V=2}^{\infty}(\nu_2\lambda)^{V-2} \\
&\times \frac{1}{3!}\sum_{m_1,n_1,m_2,n_2,m_3n_3}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l}\frac{1}{2}Q_{nm,lk}(\Lambda) \\
&\times \left(A^{(V_1)}_{m_1n_1m_2n_2,mn}A^{(V-V_1)}_{m_3n_3,kl} + A^{(V_1)}_{m_1n_1m_3n_3,mn}A^{(V-V_1)}_{m_2n_2,kl}\right. \\
&\left. + A^{(V_1)}_{m_3n_3m_2n_2,mn}A^{(V-V_1)}_{m_1n_1,kl}\right)\tilde{M}_{m_1n_1}\tilde{M}_{m_2n_2}\tilde{M}_{m_3n_3}.
\end{aligned}
\tag{12.262}
$$

In this above equation we have used the obvious identity

$$
\sum_{V=2}^{\infty}\sum_{V_1=1}^{V-1} I^{V_1}J^{V-V_1} = \sum_{V=1}^{\infty}\sum_{V'=1}^{\infty} I^{V}J^{V'}. \tag{12.263}
$$

This result needs to be compared with

$$
\begin{aligned}
\Lambda\frac{\partial L}{\partial \Lambda} &= \sum_{V=1}^{\infty}(\nu_2\lambda)^{V-1}\frac{1}{2!}\sum_{m_1,n_1,m_2,n_2}\Lambda\frac{\partial A^{(V)}_{m_1n_1,m_2n_2}}{\partial\Lambda}\tilde{M}_{m_1n_1}\tilde{M}_{m_2n_2} + \cdots \\
&+ \sum_{V=1}^{\infty}(\nu_2\lambda)^{V-1}\frac{1}{3!}\sum_{m_1n_1,m_2n_2,m_3n_3}\Lambda\frac{\partial A^{(V)}_{m_1n_1,m_2n_2,m_3n_3}}{\partial\Lambda} \\
&\times \tilde{M}_{m_1n_1}\tilde{M}_{m_2n_2}\tilde{M}_{m_3n_3} + \cdots \\
&= \sum_{V=2}^{\infty}(\nu_2\lambda)^{V-2}\frac{1}{2!}\sum_{m_1n_1,m_2n_2}\Lambda\frac{\partial A^{(V-1)}_{m_1n_1,m_2n_2}}{\partial\Lambda}\tilde{M}_{m_1n_1}\tilde{M}_{m_2n_2} + \cdots \\
&+ \sum_{V=2}^{\infty}(\nu_2\lambda)^{V-2}\frac{1}{3!}\sum_{m_1n_1,m_2n_2,m_3n_3}\Lambda\frac{\partial A^{(V-1)}_{m_1n_1,m_2n_2,m_3n_3}}{\partial\Lambda} \\
&\times \tilde{M}_{m_1n_1}\tilde{M}_{m_2n_2}\tilde{M}_{m_3n_3} + \cdots
\end{aligned}
\tag{12.264}
$$

We get

$$\Lambda\frac{\partial A^{(V-1)}_{m_1n_1,m_2n_2}}{\partial\Lambda} = \sum_{V_1=1}^{V-1}\sum_{m,n,k,l}\frac{1}{2}Q_{nm,lk}(\Lambda) \times\left(A^{(V_1)}_{m_1n_1,mn}A^{(V-V_1)}_{m_2n_2,kl} + A^{(V_1)}_{m_2n_2,mn}A^{(V-V_1)}_{m_1n_1,kl}\right) \tag{12.265}$$

$$\begin{aligned}\Lambda\frac{\partial A^{(V-1)}_{m_1n_1,m_2n_2,m_3n_3}}{\partial\Lambda} &= \sum_{V_1=1}^{V-1}\sum_{m,n,k,l}\frac{1}{2}Q_{nm,lk}(\Lambda) \\ &\times\left(A^{(V_1)}_{m_1n_1,mn}A^{(V-V_1)}_{m_2n_2m_3n_3,kl} + \text{permutations}\right) \\ &+ \sum_{V_1=1}^{V-1}\sum_{m,n,k,l}\frac{1}{2}Q_{nm,lk}(\Lambda) \\ &\times\left(A^{(V_1)}_{m_1n_1m_2n_2,mn}A^{(V-V_1)}_{m_3n_3,kl} + \text{permutations}\right).\end{aligned} \tag{12.266}$$

Thus, we obtain the general result

$$\begin{aligned}\Lambda\frac{\partial A^{(V-1)}_{m_1n_1,\ldots,m_Nn_N}}{\partial\Lambda} &= \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l}\frac{1}{2}Q_{nm,lk}(\Lambda) \\ &\times\left(A^{(V_1)}_{m_1n_1,\ldots,m_{N_1-1}n_{N_1-1},mn}A^{(V-V_1)}_{m_{N_1}n_{N_1},\ldots,m_Nn_N,kl}\right. \\ &\left. + \text{permutations}\right).\end{aligned} \tag{12.267}$$

For every N_1 there are $N!/(N_1!(N-N_1)!)$ permutations of the indices $m_1n_1, \ldots, m_{N_1}n_{N_1}, \ldots, m_Nn_N$. However, we should also include the effect of the second term in the Wilson–Polchinski equation. We compute

$$\begin{aligned}\sum_{m,n,k,l}Q_{nm,lk}(\Lambda)\frac{\partial^2 L}{\partial\tilde{M}_{kl}\partial\tilde{M}_{mn}} &= \sum_{V=2}^{\infty}(\nu_2\lambda)^{V-2}\left[\sum_{m,n,k,l}Q_{nm,lk}A^{(V-1)}_{mnkl}\right. \\ &+ \sum_{m,n,k,l}Q_{nm,lk}\sum_{m_1n_1}A^{(V-1)}_{m_1n_1mnkl}\tilde{M}_{m_1n_1} \\ &+ \sum_{N=2}^{\infty}\frac{1}{N!}\sum_{m_1n_1,\ldots,m_Nn_N}\sum_{m,n,k,l} \\ &\left.\times Q_{nm,lk}A^{(V-1)}_{m_1n_1\ldots m_Nn_Nmnkl}\tilde{M}_{m_1n_1}\ldots\tilde{M}_{m_Nn_N}\right].\end{aligned} \tag{12.268}$$

By neglecting the first two terms we get

$$\Lambda\frac{\partial A^{(V-1)}_{m_1n_1,\ldots,m_Nn_N}}{\partial\Lambda} = \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l}\frac{1}{2}Q_{nm,lk}(\Lambda) \times\Big(A^{(V_1)}_{m_1n_1,\ldots,m_{N_1-1}n_{N_1-1},mn}A^{(V-V_1)}_{m_{N_1}n_{N_1},\ldots,m_Nn_N,kl} + \text{permutations}\Big) - \frac{1}{2}\sum_{m,n,k,l}Q_{nm,lk}A^{(V-1)}_{m_1n_1,\ldots,m_Nn_N,mn,kl}. \tag{12.269}$$

We will set

$$H^{(V)}_{m_1n_1,\ldots,m_Nn_N} = 0. \tag{12.270}$$

This is possible because as it turns out for $\omega < 1$, the flow equations of the functions $R^{(V)}_{m_1n_1,\ldots,m_Nn_N}$ do not depend on the functions $H^{(V)}_{m_1n_1,\ldots,m_Nn_N}$. Hence, we do not need to evaluate the functions $H^{(V)}_{m_1n_1,\ldots,m_Nn_N}$. For all practical purposes we can set them equal to 0. The flow equations of $R^{(V)}_{m_1n_1,\ldots,m_Nn_N}$ are derived from the Polchinski equation for R given by equation (12.253). We find

$$\Lambda\frac{\partial R^{(V-1)}_{m_1n_1,\ldots,m_Nn_N}}{\partial\Lambda} = \sum_{N_1=2}^{N}\sum_{V_1=1}^{V-1}\sum_{m,n,k,l}Q_{nm,lk}(\Lambda) \times\Big(A^{(V_1)}_{m_1n_1,\ldots,m_{N_1-1}n_{N_1-1},mn}R^{(V-V_1)}_{m_{N_1}n_{N_1},\ldots,m_Nn_N,kl} + \text{permutations}\Big) - \frac{1}{2}\sum_{m,n,k,l}Q_{nm,lk}R^{(V-1)}_{m_1n_1,\ldots,m_Nn_N,mn,kl}. \tag{12.271}$$

12.3.7 Ribbon graphs

The coefficient $L_{m_1n_1\ldots m_Nn_N}[\Lambda]$ is represented by a circle with N outgoing lines m_i and n incoming lines n_i (figure 12.1). The arrows are only added for bookkeeping, so in fact there is no distinction between outgoing and incoming lines since we are dealing with a real scalar field. We will represent the differentiated propagator $Q_{nm,lk}(\Lambda)$ by a ribbon with two lines nk and ml (figure 12.2). We differentiate the Polchinski equation (12.172) with respect to the fields $\phi_{m_1n_1}$, … , $\phi_{m_2n_2}$[1]. The resulting equation is represented graphically on figure 12.3.

Generically, the solution $L[\phi, \Lambda]$ of the Polchinski equation (12.172) can be given by an expansion in the coupling constant λ of the form

$$L[\phi, \Lambda] = \sum_{V=1}^{\infty}\lambda^V L^{(V)}[\phi, \Lambda]. \tag{12.272}$$

[1] In this section we will denote the field $\tilde{M}_{mn}$ by ϕ_{mn}.

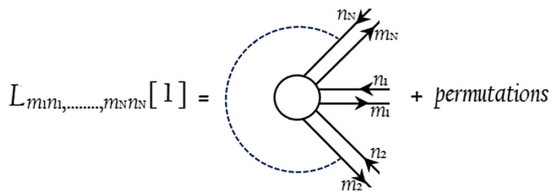

Figure 12.1. The coefficient $L_{m_1n_1\ldots m_Nn_N}[\Lambda]$.

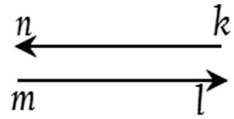

Figure 12.2. The differentiated propagator $Q_{nm,lk}(\Lambda)$.

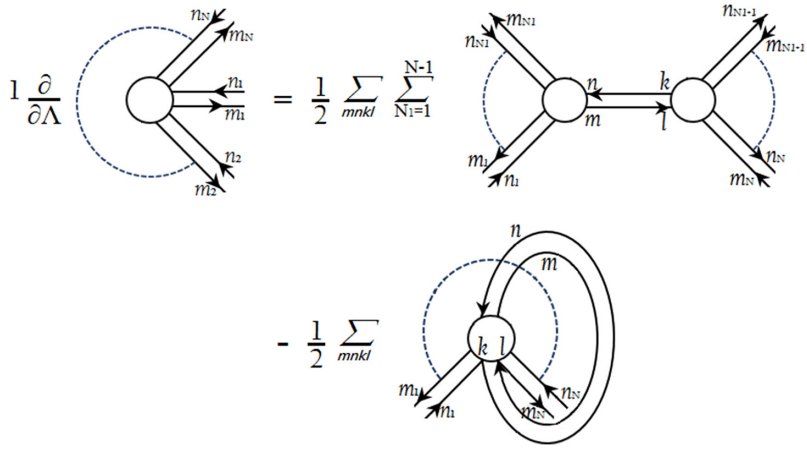

Figure 12.3. The Wilson–Polchinski equation (12.172) differentiated with respect to the fields $\phi_{m_1n_1}, \ldots, \phi_{m_2n_2}$.

For a ϕ^4-model we choose

$$L^{(1)}_{m_1n_1,m_2n_2}[\Lambda] = 0, \tag{12.273}$$

and

$$L^{(1)}_{m_1n_1,\ldots,m_Nn_N}[\Lambda] = 0, \quad N > 4. \tag{12.274}$$

Inserting these into the graphical Polchinski equation (figure 12.3) we conclude that $L^{(1)}_{m_1n_1,\ldots,m_4n_4}[\Lambda]$ is independent of Λ. We can, therefore, identify $L^{(1)}_{m_1n_1,\ldots,m_4n_4}$ with the original ϕ^4 interaction, viz

$$L^{(1)}_{m_1n_1,\ldots,m_4n_4}[\Lambda_0] = \frac{1}{4!6}(\delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}\delta_{n_4m_1} + \text{permutations}). \tag{12.275}$$

This is represented graphically on figure 12.4.

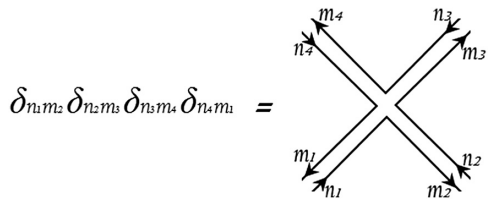

Figure 12.4. The coefficient $L^{(1)}_{m_1n_1,\ldots,m_4n_4}[\Lambda]$.

The iterative solution of the graphical Polchinski equation on figure 12.3 with the initial conditions (12.274) and (12.275) is given in terms of ribbon graphs. Clearly, ribbon graphs can become very complicated with an arbitrary number of crossing of lines so they cannot be drawn on a plane.

A general ribbon graph can be drawn on a Riemannian surface of genus g. More precisely, a given ribbon graph defines a topological Riemannian surface via Euler characteristic $\chi = 2 - 2g$. An Euler characteristic can also be computed from the formula $\chi = V - E + F$ where V is the number of polyhedron vertices, E is the number of polyhedron edges and F is the number of faces. To apply this formula we consider closed ribbon graphs where the external lines are amputated, i.e. the external lines m_i are connected with n_i. The number of faces F is the number $\bar{L}$ of single-line loops of the closed graph, viz $F = \bar{L}$, whereas the number of edges E is equal to the number I of internal double lines of the graph, viz $E = I$. Thus, we have

$$\chi = 2 - 2g = V - I + \bar{L}. \tag{12.276}$$

We can draw the ribbon graph and its Riemannian surface in a variety of equivalent ways provided V, I, $\bar{L}$ and g are kept the same. The Polchinski equation in figure 12.3 tells us which external legs of the vertices are connected. The ribbons are drawn between these legs in any convenient way. Thus, for example, there is no difference between overcrossing and undercrossing.

In a ribbon graph we have two kinds of loops. Some loops are such that they contain at least one external leg. These are the boundary components of the ribbon graph which correspond to the holes of the Riemannian surface. Their number is B. Another type of loops do not contain any external leg. They are called inner loops. Their number is $\bar{L}_0 = \bar{L} - B$. Boundary components are composed of an ensemble of trajectories from an incoming index n_i to an outgoing index m_j. An example is shown in the first graph of figure 12.5. In this case $\bar{L} = B = 2$, i.e. there are two boundary components. The inner boundary component is composed of the trajectory $n_1\vec{m}_6 = m_1n_1m_6n_6$. The outer component is composed of the two trajectories $n_5\vec{m}_2 = m_5n_5m_2n_2$ and $n_3\vec{m}_4 = m_3n_3m_4n_4$. We also have $V = 3$, $I = 3$. Other examples are given in the second and third graphs of figure 12.5.

We also need the concept of an external vertex which is a vertex with at least one external line attached to it. Let V_e be the number of external vertices. There are four possibilities for the arrangement of the external legs at an external vertex. Firstly, we can have only one external leg. Secondly, we can have three external legs. Thirdly, we can have two external legs with one starting point and one end point. Fourthly, we can have two external legs with two starting points and two end points. See the

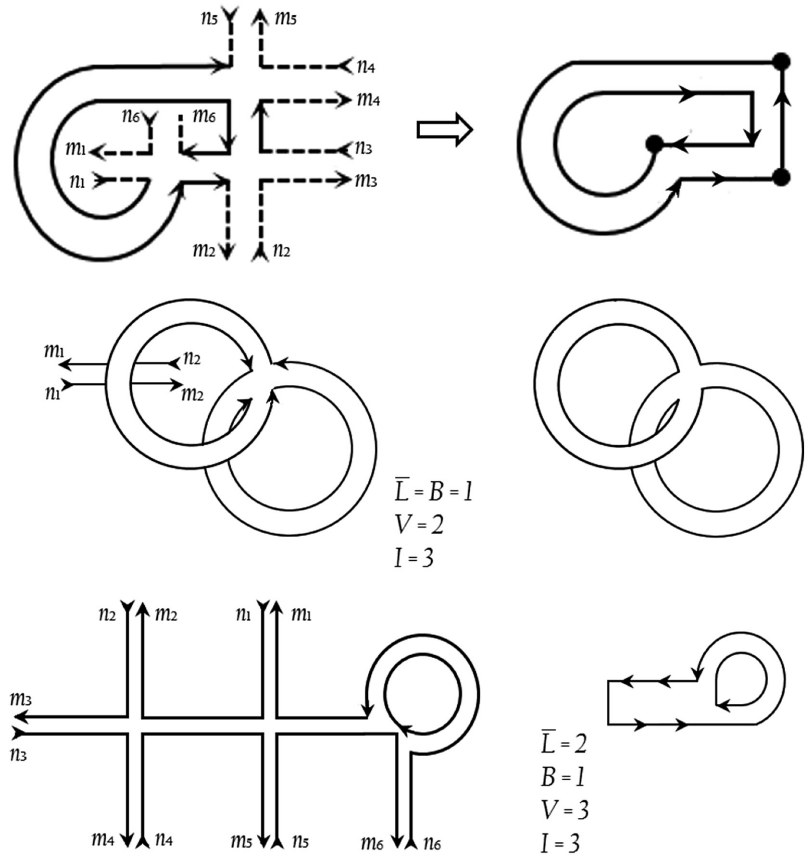

Figure 12.5. Examples of ribbon graphs.

first graph of figure 12.6. Recall that only in external legs is the index m_i connected with the index n_i. Thus, in the three first possibilities we have only one starting point and one end point for trajectories inside a ribbon graph. These are called simple vertices. In the fourth possibility we have two starting points and two end points. This is called a composed vertex. This vertex can be decomposed as in the second graph of figure 12.6.

A given ribbon graph with V_c composed vertices can be decomposed into S segments, which is generically not equal to the total number of decompositions, is $V_c + 1$. The segmentation index $\mathcal{I}$ of a given graph is the maximum number of decompositions of composed vertices which keep the graph connected. It is given by

$$\mathcal{I} = V_c - S + 1. \tag{12.277}$$

For the first graph of figure 12.5 we have $V_c = 0$, $S = 1$ and thus $\mathcal{I} = 0$. For the second graph of figure 12.5 we have $V_c = 1$, $S = 1$ and thus $\mathcal{I} = 1$. For the third graph of figure 12.5 we have $V_c = 1$, $S = 2$ and thus $\mathcal{I} = 0$.

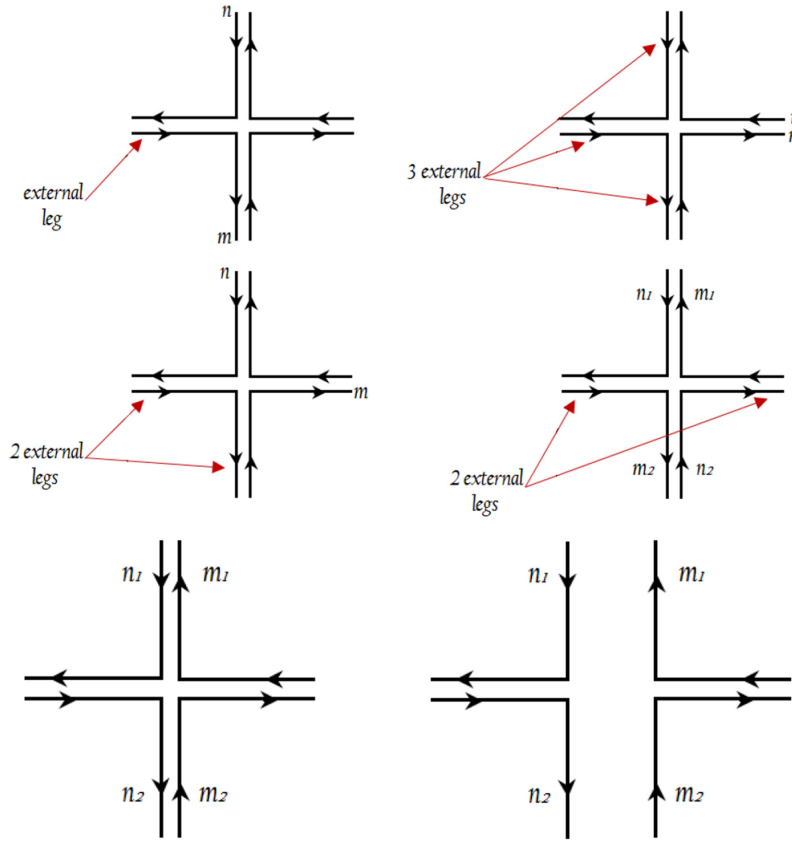

Figure 12.6. More examples of ribbon graphs.

We are going to fix all starting points of trajectories and sum over end points. Because of conservation of momentum not all summations are independent. In each segment the sum of outgoing indices is equal to the sum of incoming indices. The total number of end points in a ribbon graph is $V_e + V_c$ since the composed vertices contain two end points, whereas the simple vertices contain one end point. Hence, the number of independent index summations is

$$s \leqslant V_e + V_c - S = V_e + \mathcal{I} - 1. \tag{12.278}$$

12.3.8 Marginal, relevant and irrelevant operators

Renormalizability of a given model is the requirement that the Polchinski differential equation admits a regular solution which depends only on a finite number of initial conditions. The Polchinski equation will be integrated between the two scales Λ_R and Λ or between the two scales Λ and Λ_0. The vertices $L_{m_1 n_1 \ldots m_N n_N}[\Lambda]$ are assumed to be decomposable into parts $L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda]$ which are homogeneous. In other words, for any Λ in the interval $[\Lambda_R, \Lambda_0]$ these parts scale as

$$\left| \Lambda \frac{\partial L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda]}{\partial \Lambda} \right| \leqslant \Lambda^{r_i} P^{q_i}\left[\ln \frac{\Lambda}{\Lambda_R} \right]. \tag{12.279}$$

In the above, $P^q[x]$ is a polynomial of degree q in $x \geqslant 0$ such that $P^q[x] \geqslant 0$. Homogeneous parts of vertices $L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda]$ with $r_i >$ are called relevant. If $r_i < 0$ the parts are called irrelevant, whereas if $r_i = 0$ the parts are called marginal. Clearly we can write the vertices $L_{m_1 n_1 \ldots m_N n_N}[\Lambda]$ as

$$L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda] = L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda_0] - \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \left(\Lambda' \frac{\partial}{\partial \Lambda'} L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda'] \right) \tag{12.280}$$

$$L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda] = L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda_R] + \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \left(\Lambda' \frac{\partial}{\partial \Lambda'} L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda'] \right). \tag{12.281}$$

The largest contribution to these integrals comes from the term $\Lambda^{r-1}(\ln \Lambda/\Lambda_R)^q$. We use the integrals

$$\int dx x^{r-1} \left(\ln \frac{x}{x_R} \right)^q = \frac{(-1)^q q!}{r^{q+1}} x^r \sum_{j=0}^{q} \frac{\left(-r \ln \frac{x}{x_R} \right)^j}{j!} + \text{constant}, \quad r \neq 0 \tag{12.282}$$

$$\int dx x^{r-1} \left(\ln \frac{x}{x_R} \right)^q = \frac{1}{q+1} \left(\ln \frac{x}{x_R} \right)^{q+1} + \text{constant}, \quad r = 0. \tag{12.283}$$

First, let us consider the case $r_i < 0$. We use for the homogeneous parts $L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda]$ the definition (12.280). We get

$$\begin{aligned} |L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda]| &\leqslant |L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda_0]| \\ &\quad + \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \left| \left(\Lambda' \frac{\partial}{\partial \Lambda'} L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda'] \right) \right| . \end{aligned} \tag{12.284}$$

The contribution coming from the upper limit of the integral in the limit $\Lambda_0 \longrightarrow 0$ goes to zero. Thus, we have

$$\begin{aligned} |L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda]| &\leqslant |L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda_0]| \\ &\quad + \int_{\Lambda}^{\infty} \frac{d\Lambda'}{\Lambda'} \left| \left(\Lambda' \frac{\partial}{\partial \Lambda'} L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda'] \right) \right| \\ &\leqslant |L^{(i)}_{m_1 n_1 \ldots m_N n_N}[\Lambda_0]| + \Lambda^{r_i} Q^{q_i}\left[\ln \frac{\Lambda}{\Lambda_R} \right]. \end{aligned} \tag{12.285}$$

In the above $Q^q[x]$ is another polynomial of degree q in $x \geqslant 0$ such that $Q^q[x] \geqslant 0$. We require that for the boundary condition we have

$$|L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda_0]| \leqslant \Lambda_0^{r_i} Q^{q_i}\left[\ln \frac{\Lambda_0}{\Lambda_R}\right], \ r_i < 0. \tag{12.286}$$

We obtain

$$|L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda]| \leqslant \Lambda^{r_i} Q^{q_i}\left[\ln \frac{\Lambda}{\Lambda_R}\right]. \tag{12.287}$$

Now we consider the case $r_i > 0$. In this case we use for the homogeneous parts $L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda]$ the definition (12.281). We get

$$\begin{aligned} |L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda]| \leqslant & |L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda_R]| \\ & + \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \left| \left(\Lambda' \frac{\partial}{\partial \Lambda'} L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda']\right) \right| . \end{aligned} \tag{12.288}$$

The lower limit of the integral gives zero since $\ln \Lambda_R/\Lambda_R = 0$ for $j \neq 0$, whereas the term $j = 0$ will yield a contribution proportional to Λ_R^r which can be neglected compared to the contribution corresponding to the term $j = 0$ coming from the upper limit of the integral $\Lambda' = \Lambda$ which is proportional to Λ. We get then

$$|L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda]| \leqslant |L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda_R]| + \Lambda^{r_i} Q^{q_i}\left[\ln \frac{\Lambda}{\Lambda_R}\right]. \tag{12.289}$$

In summary, for the irrelevant interactions we perform the integration from Λ_0 to Λ starting from an initial condition $|L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda_0]| \leqslant \Lambda_0^{r_i} Q^{q_i}[\ln \frac{\Lambda_0}{\Lambda_R}]$. For relevant and marginal interactions we perform the integration from Λ_R to Λ starting from an initial condition $|L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda_R]| < \infty$. In all cases we have

$$|L^{(i)}_{m_1 n_1 \dots m_N n_N}[\Lambda]| \leqslant \Lambda^{r_i} Q^{q_i}\left[\ln \frac{\Lambda}{\Lambda_R}\right]. \tag{12.290}$$

Note that the number of conditions imposed at Λ_R must be finite for the theory to be renormalizable. Conditions imposed at Λ_0 are not normalization conditions.

12.3.9 The power-counting theorem

This provides the power-counting behaviors of non-local matrix models with ϕ^4 interactions which specify the class of all divergent functions in these theories. As an example all non-planar graphs in these models will turn out to be irrelevant.

The first ingredient is the differentiated propagator

$$Q_{nm,lk}(\Lambda) = \frac{1}{\nu_d^{\frac{d}{2}}} \Lambda \frac{\partial \Delta_{nm,lk}(\Lambda)}{\partial \Lambda}. \tag{12.291}$$

The cutoff function K is chosen so that only for a finite number of indices m, n, k and l we have $Q_{nm,lk}(\Lambda) \neq 0$. This function is normalized such that

$$\mathrm{SCA} = \sum_m \mathrm{sign}|K(m, \Lambda)| \leqslant C_d \left(\frac{\Lambda}{\mu}\right)^d, \ \mu = \frac{1}{\nu_d^{\frac{1}{d}}}. \tag{12.292}$$

The two exponents δ_0 and δ_1 (related to the summed and differentiated cutoff propagator, respectively) are defined by

$$\mathrm{SCA}_0 = \max_{m,n,k,l} |Q_{nm;lk}(\Lambda)| \leqslant C_0 \left(\frac{\mu}{\Lambda}\right)^{\delta_0} \delta_{n-m,k-l} \tag{12.293}$$

$$\mathrm{SCA}_1 = \max_n \left(\sum_k \max_{m,l} |Q_{nm;lk}(\Lambda)| \leqslant C_1 \left(\frac{\mu}{\Lambda}\right)^{\delta_1} \right). \tag{12.294}$$

We need also the product of SCA, SCA_0 given by

$$\mathrm{SCA}_2 = \mathrm{SCA} \cdot \mathrm{SCA}_0 \leqslant C_2 \left(\frac{\Lambda}{\mu}\right)^{\delta_2}. \tag{12.295}$$

Clearly $C_2 = C_d C_0$ and $\delta_2 = d - \delta_0$.

A non-local matrix model is called regular if we have $\delta_0 = \delta_1 = 2$ otherwise it is called anomalous.

We integrate the Wilson–Polchinski equation between the initial scale Λ_0 and the renormalization scale Λ_R. Recall that the Wilson–Polchinski equation is solved by ribbon graphs which are characterized by V (number of vertices), V_e (number of external vertices), B (number of boundary components), g (the genus) and $\mathcal{I}$ (segmentation index). It is necessary to sum over the $s < V_e + \mathcal{I} - 1$ indices of the external legs keeping the incoming indices fixed.

The homogeneous parts $A^{(i)}_{m_1 n_1 \ldots m_n n_N}[\Lambda]$ with $i = (V, V_e, B, g, \mathcal{I})$ of the interaction coefficients $A^{(V)}_{m_1 n_1 \ldots m_n n_N}[\Lambda]$ are defined by

$$A^{(V)}_{m_1 n_1 \ldots m_N n_N}[\Lambda] = \sum_{1 \leqslant V_e \leqslant V} \sum_{1 \leqslant B \leqslant N} \sum_{0 \leqslant g \leqslant 1 + \frac{V}{2} - \frac{N}{4} - \frac{B}{2}} \sum_{0 \leqslant \mathcal{I} \leqslant B-1} A^{(i)}_{m_1 n_1 \ldots m_n n_N}[\Lambda]. \tag{12.296}$$

The power-counting theorem for the interaction action $L[M, \Lambda]$ states that the homogeneous parts $A^{(i)}_{m_1 n_1 \ldots m_n n_N}[\Lambda]$ must scale as

$$\begin{aligned} \sum_{\mathcal{E}^s} A^{(i)}_{m_1 n_1 \ldots m_n n_N}[\Lambda] \leqslant & \left(\frac{\Lambda}{\mu}\right)^{\delta_2 (V - \frac{N}{2} + 2 - 2g - B)} \\ & \times \left(\frac{\Lambda}{\mu}\right)^{-\delta_1 (V - V_e - \mathcal{I} + 2g + B - 1 + s)} \left(\frac{\Lambda}{\mu}\right)^{-\delta_0 (V_e + \mathcal{I} - 1 - s)} \\ & \times P^{2V - \frac{N}{2}} \left[\ln \frac{\Lambda}{\Lambda_R} \right]. \end{aligned} \tag{12.297}$$

We must have $2 \leqslant N \leqslant 2V + 2$ and $\sum_{i=1}^{N}(m_i - n_i) = 0$.

The set $\mathcal{E}^s$ is the set of s independent end points of trajectories which are summed over keeping the starting points of these trajectories fixed.

For $V' < V$ and $2 \leqslant N' \leqslant 2V' + 2$ the initial conditions for relevant and marginal $A^{(i)}_{m_1 n_1 \ldots m_n n_N}[\Lambda]$ are imposed at Λ_R. For $V' = V$ and $N + 2 \leqslant N' \leqslant 2V + 2$ the initial conditions for relevant and marginal $A^{(i)}_{m_1 n_1 \ldots m_n n_N}[\Lambda]$ are imposed at Λ_0.

The choice of initial (boundary) conditions is at the same time determined by and required to prove this power-counting theorem. In other words, we have to make the correct ansatz for the initial interactions which we then verify as the correct set of relevant and marginal interactions using the power-counting theorem.

This result is actually model independent (and thus it also applies in four dimensions) and as we can see it depends only on the exponents δ_0 and δ_1 and on the topological structure of the the theory given by ribbon graphs.

12.3.10 Summary: Grosse–Wulkenhaar theorem

The theorem can be summarized as follows:

- In $d = 2$ the scaling exponents are given by equations (12.222) and (12.224) which need to be substituted into the power-counting theorem (12.297).
- We can see from equations (12.222) and (12.224) that $\delta_0 = \delta_1 = 2$ for $\omega < 1$ (non-zero harmonic oscillator term) and $\delta_0 = 1$, $\delta_1 = 0$ for $\omega = 1$ (zero harmonic oscillator term). Hence noncommutative ϕ^4 on $\mathbf{R}^2_\theta$ is expected to be regular (and thus renormalizable) for $\omega < 1$ and it is expected to be anomalous (and thus not renormalizable) for $\omega = 1$.
- We integrate the Wilson–Polchinski equation for the effective action $L[M, \Lambda]$ between the initial scale Λ_0 and the renormalization scale Λ_R. Recall that the Wilson–Polchinski equation is solved by ribbon graphs which are characterized by V (number of vertices), V_e (number of external vertices), B (number of boundary components), g (the genus) and $\mathcal{I}$ (segmentation index). We need also to sum over external indices keeping fixed incoming indices.
- The choice of initial conditions is at the same time determined by and required to prove the power-counting theorem. The Grosse and Wulkenhaar ansatz for the initial interactions, i.e. the set of relevant and marginal interactions, in $d = 2$ dimensions, is dictated by the fact that only the planar 2-point function at one-loop is divergent, is given by equation (12.226) with

$$\rho^0_{[m]} = \rho_1{}^0 + 2\, m \rho_2{}^0. \tag{12.298}$$

  The mass μ^2 should, therefore, be thought of as the renormalized mass. The coupling constant λ is super-renormalizable. The harmonic oscillator parameter ω does not correspond to any divergences in dimension 2 so it does not need to appear in the initial interaction.

  The coupling constants $\rho^0_{[m]}$ and ω must, therefore, depend on Λ_0 in such a way that the limit $\Lambda_0 \longrightarrow \infty$ exists.

- At the renormalization scale Λ_R which is far below Λ_0 we expect that the interaction $L[M, \Lambda_R, \Lambda_0, \omega, \rho^0]$ becomes simple again. Indeed, after we integrate the Polchinski equation (12.172) from Λ_0 down to Λ_R with the initial condition (12.226) we obtain a solution of the form

$$\begin{aligned} L[M, \Lambda, \Lambda_0, \omega, \rho^0] = \nu_2 \sum_{m,n,k,l} \frac{1}{2} \rho^0_{[m]} \tilde{M}_{mn} \tilde{M}_{nm} \\ + \nu_2 \frac{\lambda}{4!} \sum_{m,n,k,l} \tilde{M}_{mn} \tilde{M}_{nk} \tilde{M}_{kl} \tilde{M}_{lm} \\ + \text{other } M \text{ structures}. \end{aligned} \tag{12.299}$$

- The renormalization conditions are conditions which must be satisfied by the ρ-coefficients at the renormalization cutoff Λ_R, viz $\rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0] =$ constant. We will choose in particular

$$\rho_{[m]}[\Lambda_R, \Lambda_0, \omega, \rho^0] = 0. \tag{12.300}$$

  Thus, in particular $\rho_{[0]}[\Lambda_R, \Lambda_0, \omega, \rho^0] = 0$ and hence μ^2 is the renormalized mass as pointed out above.
- The homogeneous parts $A^{(i)}_{m_1 n_1 \ldots m_n n_N}[\Lambda]$ of the coefficients of the effective action $L[M, \Lambda]$ must scale as in equation (12.297) with $2 \leqslant N \leqslant 2V + 2$ and $\sum_{i=1}^{N}(m_i - n_i) = 0$. These homogeneous parts are, therefore, bounded and they describe a regularized noncommutative ϕ^4 on $\mathbf{R}^2_\theta$. The limit $\Lambda_0 \longrightarrow \infty$ exists.
- The homogeneous parts $R^{(i)}_{m_1 n_1 \ldots m_n n_N}[\Lambda]$ of the coefficients of the varied effective action $R[M, \Lambda]$ given by equation (12.237) must also scale as in equation (12.297) times Λ^2/Λ_0^2 with $2 \leqslant N \leqslant 2V + 2$ and $\sum_{i=1}^{N}(m_i - n_i) = 0$. These homogeneous parts are, therefore, also bounded and they approach zero in the limit $\Lambda_0 \longrightarrow \infty$.
- Hence, noncommutative ϕ^4 on $\mathbf{R}^2_\theta$ is perturbatively renormalizable order by order in λ for $\omega < 1$, i.e. with a non-zero harmonic oscillator term.
- However, in the limit $\Lambda_0 \longrightarrow \infty$ we can send $\omega[\Lambda_0] \longrightarrow 1$ in such a way that a limit of the expansion coefficients $A^{(i)}_{m_1 n_1 \ldots m_n n_N}$ also exists. A solution is given by

$$\omega[\Lambda_0] = 1 - \frac{1}{\left(1 + \ln \frac{\Lambda_0}{\Lambda_R}\right)^2}. \tag{12.301}$$

  Thus, noncommutative ϕ^4 on $\mathbf{R}^2_\theta$ is perturbatively renormalizable order by order in λ without a harmonic oscillator term.

12.4 Renormalization of scalar ϕ^4 in four dimensions with a harmonic oscillator term

12.4.1 The main result in four dimensions

Again, we will deal with two scales. The initial scale Λ_0 which is actually infinite for the noncommutative space $\mathbf{R}^4_\theta$ and the renormalization scale Λ_R at which renormalization conditions must be imposed in order to subtract divergences. In other words, $\Lambda_R \leqslant \Lambda \leqslant \Lambda_0$.

On $\mathbf{R}^4_\theta$ or equivalently $\mathbf{R}^2_\theta \times \mathbf{R}^2_\theta$ we replace mn by $m_1n_1m_2n_2$, kl by $k_1l_1k_2l_2$, nk by $n_1k_1n_2k_2$, kl by $k_1l_1k_2l_2$ and lm by $l_1m_1l_2m_2$ to obtain the Laplacian

$$\begin{aligned} G^{m_2n_2,k_2l_2}_{m_1n_1,k_1l_1} = &\Big(\mu^2 + {\mu_1}^2(m_1 + n_1 + m_2 + n_2 - 2)\Big)\delta_{n_1,k_1}\delta_{m_1,l_1}\delta_{n_2,k_2}\delta_{m_2,l_2} \\ &- {\mu_1}^2\sqrt{\omega}\Big(\sqrt{(m_1-1)(n_1-1)}\ \delta_{n_1-1,k_1}\delta_{m_1-1,l_1} \\ &+ \sqrt{m_1n_1}\ \delta_{n_1+1,k_1}\delta_{m_1+1,l_1}\Big)\delta_{n_2,k_2}\delta_{m_2,l_2} \\ &- {\mu_1}^2\sqrt{\omega}\Big(\sqrt{(m_2-1)(n_2-1)}\ \delta_{n_2-1,k_2}\delta_{m_2-1,l_2} \\ &+ \sqrt{m_2n_2}\ \delta_{n_2+1,k_2}\delta_{m_2+1,l_2}\Big)\delta_{n_1,k_1}\delta_{m_1,l_1}. \end{aligned} \tag{12.302}$$

The propagator will now involve the orthogonal Meixner polynomials.

The power-counting theory (12.297) holds also in four dimensions since it is model independent. And we integrate the Wilson–Polchinski renormalization group equation (12.172) for the effective action $L[M, \Lambda]$ between the initial scale Λ_0 and the renormalization scale Λ_R with a solution given as before by ribbon graphs.

Again, a choice of initial conditions is at the same time determined by and required to prove the power-counting theorem. The Grosse–Wulkenhaar ansatz for the initial interactions, which we then verify as the correct set of relevant and marginal interactions using the power-counting theorem, in four dimensions is given by

$$\begin{aligned} L[M, \Lambda_0, \rho^0] = &\sqrt{\det(2\pi\tilde{\theta})} \sum_{m_1,n_1,m_2,n_2,k_1,l_1,k_2,l_2} \\ &\times \frac{1}{2}\tilde{M}^{m_2n_2}_{m_1n_1}\tilde{G}^{m_2n_2,k_2l_2}_{m_1n_1,k_1l_1}\tilde{M}^{k_2l_2}_{k_1l_1} \\ &+ \sqrt{\det(2\pi\tilde{\theta})}\,\frac{\rho^0_4}{4!} \sum_{m_1,n_1,m_2,n_2,k_1,k_2,l_1,l_2} \\ &\times \tilde{M}^{m_2n_2}_{m_1n_1}\tilde{M}^{n_2k_2}_{n_1k_1}\tilde{M}^{k_2l_2}_{k_1l_1}\tilde{M}^{l_2m_2}_{l_1m_1} \end{aligned} \tag{12.303}$$

$$\begin{aligned} \tilde{G}_{m_1n_1m_2n_2,k_1l_1k_2l_2} = &\Big({\rho_1}^0 + \rho_2^0(m_1 + n_1 + m_2 + n_2)\Big)\delta_{n_1,k_1}\delta_{n_2,k_2}\delta_{m_1,l_1}\delta_{m_2,l_2} \\ &- \rho_3^0(\sqrt{m_1n_1}\ \delta_{n_1+1,k_1}\delta_{m_1+1,l_1}\delta_{n_2,k_2}\delta_{m_2,l_2} \\ &+ \sqrt{m_2n_2}\ \delta_{n_2+1,k_2}\delta_{m_2+1,l_2}\delta_{n_1,k_1}\delta_{m_1,l_1}). \end{aligned} \tag{12.304}$$

This corresponds to the fact that in four dimensions we have four divergences: the mass parameter μ^2, the coupling constant λ, the field $\tilde{M}$ (wavefunction renormalization) and also the harmonic oscillator frequency Ω.

The main result here is that noncommutative ϕ^4 on $\mathbf{R}^4_\theta$ is perturbatively renormalizable order by order in λ for $\omega < 1$, i.e. with a non-zero harmonic oscillator term. In other words, a limit $\Lambda_0 \longrightarrow \infty$ exists for $\omega < 1$.

However, in contrast with the case of $\mathbf{R}^2_\theta$, we cannot in this case scale away the frequency $\omega[\Lambda_0]$ of the harmonic oscillator with the cutoff Λ_0. The theory on $\mathbf{R}^4_\theta$ is, therefore, very different from the usual commutative ϕ^4 since the noncommutativity θ relevant at very short distances is reflected in the structure of space at large distances (the so-called UV/IR mixing).

Indeed, the harmonic oscillator frequency is seen from this renormalizability result to be a genuine physical parameter which indicates that space at very large distances is not what it seems. This is illustrated by the discrete eigenvalues of the squared momentum which are characterized by an equidistant spacing equal $4\Omega/\theta$ and thus the minimum momentum carried by the scalar field is $\sqrt{4\Omega/\theta}$. This signals that the size of the Universe in any direction is effectively finite of the order of $\sqrt{\theta/4\Omega}$.

12.4.2 The beta function

Another important result is the one-loop beta function of duality-covariant noncommutative ϕ^4 on $\mathbf{R}^4_\theta$, i.e. the theory with a non-zero harmonic oscillator term, which was calculated in [14]. The beta function of the coupling constant λ was found to be non-negative but vanishes for the self-dual theory with $\Omega = 1$. It is expected that the vanishing of the beta function at the self-dual point is exact to all orders in perturbation theory.

This result means in particular that the theory is not asymptotically free in the ultraviolet since the renormalization group flow of the coupling constant is bounded and thus the theory does not exhibit a Landau ghost, i.e. is not trivial. In contrast the commutative ϕ^4 theory, although also asymptotically free, exhibits a Landau ghost.

12.4.3 Gaussian noncommutative fixed points

In this section we give an explicit derivation of a general class of Wilson–Polchinski renormalization group equations of duality-covariant noncommutative ϕ^4 on $\mathbf{R}^4_\theta$ following [16]. Then we introduce Gaussian noncommutative fixed points (also called floating fixed points in [15]) and their renormalized trajectories.

The Wilson–Polchinski renormalization group equation

We consider the case of four dimensions. The action is given explicitly by (with $\nu_4 = \sqrt{\det(2\pi\theta)}$, $\tilde{X}_i = 2(\theta^{-1})_{ij}X_j$ and $B^2\theta^2 = 4\Omega^2$)

$$
\begin{aligned}
S &= \int d^4x \left[\Phi\left(-\frac{1}{2}\partial_i^2 + \frac{1}{2}(B_{ij}x_j)^2 + \frac{\mu^2}{2} \right)\Phi + \frac{\lambda}{4!}\Phi * \Phi * \Phi * \Phi(x) \right] \\
&= \int d^4x \left[\Phi\left(-\frac{1}{2}\partial_i^2 + \frac{1}{2}\Omega^2\tilde{x}_i^2 + \frac{\mu^2}{2} \right)\Phi + \frac{\lambda}{4!}\Phi * \Phi * \Phi * \Phi \right] \\
&= \nu_4 \,\mathrm{Tr}_{\mathcal{H}} \left[\Phi\left(-\frac{1}{2}\hat{\partial}_i^2 + \frac{1}{2}\Omega^2\tilde{X}_i^2 + \frac{\mu^2}{2} \right)\Phi + \frac{\lambda}{4!}\Phi^4 \right] \\
&= \nu_4 \,\mathrm{Tr}_{\mathcal{H}} \left[\frac{2+\omega}{\theta^2}\Phi\hat{x}_i^2\Phi + \frac{\omega}{\theta^2}\Phi\hat{x}_i\Phi\hat{x}_i + \frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 \right].
\end{aligned}
\tag{12.305}
$$

In this section ω is defined by

$$
\Omega^2 = 1 + \omega. \tag{12.306}
$$

The field Φ can be expanded as $\Phi = \sum_{m_i,n_i=1}^{\infty} \tilde{M}_{m_1n_1}^{m_2n_2} \hat{\phi}_{m_1,n_1} \otimes \hat{\phi}^{m_2,n_2}$. The action in the matrix base reads

$$
S[M] = \nu_4 \sum_{m_i,n_i,k_i,l_i} \left(\frac{1}{2}\tilde{M}_{m_1n_1}^{m_2n_2}\tilde{G}_{m_1n_1,k_1l_1}^{m_2n_2,k_2l_2}\tilde{M}_{k_1l_1}^{k_2l_2} + \frac{\lambda}{4!}\tilde{M}_{m_1n_1}^{m_2n_2}\tilde{M}_{n_1k_1}^{n_2k_2}\tilde{M}_{k_1l_1}^{k_2l_2}\tilde{M}_{l_1m_1}^{l_2m_2} \right). \tag{12.307}
$$

The Laplacian is given by

$$
\begin{aligned}
G_{m_1n_1,k_1l_1}^{m_2n_2,k_2l_2} \;=\; & \left(\mu^2 + 2\frac{2+\omega}{\theta}(m_1+n_1+m_2+n_2-2) \right)\delta_{n_1,k_1}\delta_{m_1,l_1}\delta_{n_2,k_2}\delta_{m_2,l_2} \\
& + \frac{2\omega}{\theta}\Big(\sqrt{(m_1-1)(n_1-1)}\;\delta_{n_1-1,k_1}\delta_{m_1-1,l_1} \\
& \quad + \sqrt{m_1n_1}\;\delta_{n_1+1,k_1}\delta_{m_1+1,l_1} \Big)\delta_{n_2,k_2}\delta_{m_2,l_2} \\
& + \frac{2\omega}{\theta}\Big(\sqrt{(m_2-1)(n_2-1)}\;\delta_{n_2-1,k_2}\delta_{m_2-1,l_2} \\
& \quad + \sqrt{m_2n_2}\;\delta_{n_2+1,k_2}\delta_{m_2+1,l_2} \Big)\delta_{n_1,k_1}\delta_{m_1,l_1}.
\end{aligned}
\tag{12.308}
$$

We split the action $S[M]$ into a kinetic term L_0 and an interaction L, i.e.

$$
S[M] = L_0 + L. \tag{12.309}
$$

The kinetic part is

$$
\begin{aligned}
L_0 &= \nu_4 \,\mathrm{Tr}_{\mathcal{H}} \frac{2}{\theta^2}\Phi\hat{x}_i^2\Phi \\
&= \nu_4 \sum_{m_i,n_i,k_i,l_i} \frac{1}{2}\tilde{M}_{m_1n_1}^{m_2n_2} \\
&\quad \times \left(\frac{4}{\theta}(m_1+n_1+m_2+n_2-2)\delta_{n_1,k_1}\delta_{m_1,l_1}\delta_{n_2,k_2}\delta_{m_2,l_2} \right)\tilde{M}_{k_1l_1}^{k_2l_2}.
\end{aligned}
\tag{12.310}
$$

The definition of L is obvious from the above equations. The bare propagator is given by

$$\Delta^{m_2 n_2, k_2 l_2}_{m_1 n 1, k_1 l_1} = \frac{\theta}{4(m_1 + n_1 + m_2 + n_2 - 2)} \delta_{n_1,k_1} \delta_{m_1,l_1} \delta_{n_2,k_2} \delta_{m_2,l_2}. \tag{12.311}$$

We introduce a cutoff Λ and a cutoff function $K_2(x)$ defined by $K_2(x) = 1$ for $x \leqslant 1$ and $K_2(x) \longrightarrow 0$ rapidly for $x > 1$. The regularized propagator is (with $\bar{\theta} = \theta \Lambda^2$)

$$\Delta^{m_2 n_2, k_2 l_2}_{m_1 n 1, k_1 l_1}(\Lambda) = \Delta^{m_2 n_2, k_2 l_2}_{m_1 n 1, k_1 l_1} \prod_{i \in \{m_i, n_i\}} K_2\left(\frac{4i}{\bar{\theta}}\right). \tag{12.312}$$

We consider now an action $S[M, \Lambda]$ given in terms of the inverse of the above regularized propagator and a general interaction $L[\Lambda]$ satisfying $L[\infty] = L$ by

$$S[M, \Lambda] = L_0[\Lambda] + L[\Lambda]. \tag{12.313}$$

The kinetic part is given by

$$\begin{aligned} L_0[\Lambda] = \nu_4 \sum_{m_i, n_i, k_i, l_i} & \frac{1}{2} \tilde{M}^{m_2 n_2}_{m_1 n_1} \\ & \times \left(\frac{4}{\theta} (m_1 + n_1 + m_2 + n_2 - 2) \prod_{i \in \{m_i, n_i\}} K_2^{-1}\left(\frac{4i}{\bar{\theta}}\right) \delta_{n_1,k_1} \delta_{m_1,l_1} \delta_{n_2,k_2} \delta_{m_2,l_2} \right) \\ & \times \tilde{M}^{k_2 l_2}_{k_1 l_1}. \end{aligned} \tag{12.314}$$

The general interaction $L[\Lambda]$ must solve the Polchinski equation

$$\begin{aligned} \Lambda \frac{\partial L[\Lambda]}{\partial \Lambda} = \frac{1}{2\nu_4} \sum_{m_i, n_i, k_i, l_i} & \Lambda \frac{\partial \Delta^{n_2 m_2, k_2 l_2}_{n_1 m_1, l_1 k_1}(\Lambda)}{\partial \Lambda} \\ & \times \left[\frac{\partial L[\Lambda]}{\partial \tilde{M}^{k_2 l_2}_{k_1 l_1}} \frac{\partial L[\Lambda]}{\partial \tilde{M}^{m_2 n_2}_{m_1 n_1}} - \frac{\partial^2 L[\Lambda]}{\partial \tilde{M}^{k_2 l_2}_{k_1 l_1} \partial \tilde{M}^{m_2 n_2}_{m_1 n_1}} \right]. \end{aligned} \tag{12.315}$$

For ease of notation we write this as follows

$$\begin{aligned} \Lambda \frac{\partial L[\Lambda]}{\partial \Lambda} &= \frac{1}{2\nu_4} \sum_{m,n,k,l} \Lambda \frac{\partial \Delta_{nm,lk}(\Lambda)}{\partial \Lambda} \left[\frac{\partial L[\Lambda]}{\partial \tilde{M}_{kl}} \frac{\partial L[\Lambda]}{\partial \tilde{M}_{mn}} - \frac{\partial^2 L[\Lambda]}{\partial \tilde{M}_{kl} \partial \tilde{M}_{mn}} \right] \\ &= -\frac{1}{2\nu_4} \sum_{m,n,k,l} \dot{\Delta}_{nm,lk}(\Lambda) \left[\frac{\partial L[\Lambda]}{\partial \tilde{M}_{kl}} \frac{\partial L[\Lambda]}{\partial \tilde{M}_{mn}} - \frac{\partial^2 L[\Lambda]}{\partial \tilde{M}_{kl} \partial \tilde{M}_{mn}} \right]. \end{aligned} \tag{12.316}$$

Let us define $\Sigma = S - 2L_0$. The above equation can be put into the form

$$\Lambda\frac{\partial S[M,\,\Lambda]}{\partial\Lambda}=-\frac{1}{2\nu_4}\sum_{m,n,k,l}\dot{\Delta}_{nm,lk}(\Lambda)\times\left[\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{kl}}\frac{\partial\Sigma[M,\,\Lambda]}{\partial\tilde{M}_{mn}}-\frac{\partial^2\Sigma[M,\,\Lambda]}{\partial\tilde{M}_{kl}\partial\tilde{M}_{mn}}\right]. \tag{12.317}$$

The starting point for the derivation of the Polchinski equation is

$$-\Lambda\frac{\partial}{\partial\Lambda}Z=\partial_t Z=0 \tag{12.318}$$

$$Z=\int d\tilde{M}_{mn}\,e^{-S[M,\Lambda]}. \tag{12.319}$$

The partition function does not depend on the cutoff, which means that the low-energy observables do not depend on the cutoff. The renormalization group time t is defined by $t=\ln\frac{\mu}{\Lambda}$ where μ is a fixed physical mass scale.

Let us now make a field redefinition $\tilde{M}'_{mn}=\tilde{M}_{mn}-\Theta_{mn}(\tilde{M}_{kl},\,t)$. We compute $d\tilde{M}_{mn}=\left(1+\sum_{m,n}\frac{\partial\Theta_{mn}}{\partial\tilde{M}'_{mn}}\right)d\tilde{M}'_{mn}$ and $e^{-S[M,\Lambda]}=e^{-S[M',\Lambda]}\left(1-\sum_{m,n}\Theta_{mn}\frac{\partial S}{\partial\tilde{M}'_{mn}}\right)$. Hence, from the invariance of the partition function Z under field redefinition we obtain

$$0=\int d\tilde{M}_{mn}\,\frac{\partial}{\partial\tilde{M}_{mn}}(\Theta_{mn}e^{-S[M,\Lambda]}). \tag{12.320}$$

If the field redefinition is coming from an infinitesimal change in the cutoff Λ then $\Theta=\Psi dt$. We get then

$$0=\int d\tilde{M}_{mn}\,\frac{\partial}{\partial\tilde{M}_{mn}}(\Psi_{mn}e^{-S[M,\Lambda]}). \tag{12.321}$$

Putting equations (12.318) and (12.321) together we get the general exact renormalization group equation [16]

$$\partial_t e^{-S[M,\Lambda]}=\frac{\partial}{\partial\tilde{M}_{mn}}(\Psi_{mn}e^{-S[M,\Lambda]}). \tag{12.322}$$

Thus, a change in the cutoff is actually compensated by a field redefinition. The functional Ψ_{mn} defines a Kadanoff blocking procedure. We can get the Polchinski equation (12.325) from (12.322) by choosing

$$\Psi_{mn}=\frac{1}{2\nu_4}\sum_{k,l}\dot{\Delta}_{nm,lk}(\Lambda)\frac{\partial\Sigma[M,\,\Lambda]}{\partial\tilde{M}_{kl}}. \tag{12.323}$$

Let us choose instead

$$\Psi_{mn}=\frac{Z}{2\nu_4}\sum_{k,l}\dot{\Delta}_{nm,lk}(\Lambda)\frac{\partial\Sigma[M,\,\Lambda]}{\partial\tilde{M}_{kl}}. \tag{12.324}$$

We get the Polchinski equation

$$\Lambda\frac{\partial S[M,\,\Lambda]}{\partial\Lambda}=-\frac{Z}{2\nu_4}\sum_{m,n,k,l}\dot{\Delta}_{nm,lk}(\Lambda)\times\left[\frac{\partial S[M,\,\Lambda]}{\partial\tilde{M}_{kl}}\frac{\partial\Sigma[M,\,\Lambda]}{\partial\tilde{M}_{mn}}-\frac{\partial^2\Sigma[M,\,\Lambda]}{\partial\tilde{M}_{kl}\partial\tilde{M}_{mn}}\right].\tag{12.325}$$

We rescale to dimensionless variables by setting $\tilde{M}_{mn}=\tilde{m}_{mn}\sqrt{Z}\Lambda$ where Z is the field strength renormalization. We compute $(\tilde{m}\sqrt{Z}\Lambda)_{\Lambda+d\Lambda}=(\tilde{m}\sqrt{Z}\Lambda)_{\Lambda}+((2+\eta)\tilde{m}\sqrt{Z}\,d\Lambda)/2$ where $\eta=\Lambda(\partial\ln Z/\partial\Lambda)$ is the anomalous dimension of the field. We compute

$$\begin{aligned}&\Lambda\frac{\partial S[M,\,\Lambda]}{\partial\Lambda}\\&=\Lambda\lim_{d\Lambda\longrightarrow 0}\frac{S[M,\,\Lambda+d\Lambda]-S[M,\Lambda]}{(\Lambda+d\Lambda)-\Lambda}\\&=\Lambda\lim_{d\Lambda\longrightarrow 0}\\&\quad\times\frac{S\left[(\tilde{m}\sqrt{Z}\Lambda)_{\Lambda+d\Lambda}-\frac{2+\eta}{2}\tilde{m}\sqrt{Z}\,d\Lambda,\,\Lambda+d\Lambda\right]-S[(\tilde{m}\sqrt{Z}\Lambda)_{\Lambda},\,\Lambda]}{(\Lambda+d\Lambda)-\Lambda}\\&=\Lambda\frac{\partial S[\tilde{m}\sqrt{Z}\Lambda,\,\Lambda]}{\partial\Lambda}-\frac{2+\eta}{2}\tilde{m}_{mn}\frac{\partial S[\tilde{m}\sqrt{Z}\Lambda,\,\Lambda]}{\partial\tilde{m}_{mn}}.\end{aligned}\tag{12.326}$$

The Polchinski equation is given by (with $S[\tilde{m}\sqrt{Z}\Lambda,\,\Lambda]=S[\tilde{m},\,\Lambda]$ and $\Sigma[\tilde{m}\sqrt{Z}\Lambda,\,\Lambda]=\Sigma[\tilde{m},\,\Lambda]$)

$$\left(\partial_t+\frac{2+\eta}{2}\tilde{m}_{mn}\frac{\partial}{\partial\tilde{m}_{mn}}\right)S[\tilde{m},\,\Lambda]=\frac{1}{2\pi^2\bar{\theta}}\sum_{m,n,k,l}\dot{\Delta}_{nm,lk}(\Lambda)\times\left[\frac{\partial S[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{kl}}\frac{\partial\Sigma[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{mn}}-\frac{\partial^2\Sigma[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{kl}\partial\tilde{m}_{mn}}\right].\tag{12.327}$$

Let us recall that $\dot{\Delta}_{nm,lk}(\Lambda)=\partial_t\Delta_{nm,lk}(\Lambda)$. However, the propagator $\Delta_{mn,kl}(\Lambda)=\Delta_{nm,lk}(\Lambda)$ is defined now without the θ in front, viz

$$\begin{aligned}\Delta^{m_2n_2,k_2l_2}_{m_1n1,k_1l_1}(\Lambda)&=\frac{1}{4(m_1+n_1+m_2+n_2-2)}\delta_{n_1,k_1}\delta_{m_1,l_1}\delta_{n_2,k_2}\delta_{m_2,l_2}\\&\quad\times\prod_{i\in\{m_i,n_i\}}K_2\left(\frac{4i}{\bar{\theta}}\right)\\&=\Delta^{m_2n_2,k_2l_2}_{m_1n1,k_1l_1}(\bar{\theta}).\end{aligned}\tag{12.328}$$

In the following it is understood that all other rescalings are performed using Λ and not θ in analogy with $\bar{\theta}$ and $\tilde{m}$.

Gaussian noncommutative fixed points

In commutative field theory critical fixed points are defined by the condition

$$\partial_t|_\phi S_*[\phi] = 0. \tag{12.329}$$

These fixed points correspond to conformal field theories which are non-perturbatively renormalizable. Since conformal field theory is scale independent one can send, in a trivial way, the cutoff Λ_0 (the bare scale) to infinity. Let us recall here that there is also another scale which is the renormalization scale Λ_R. The non-perturbative renormalizability of a theory is equivalent to the statement that having integrated out momenta between the bare scale Λ_0 and the effective scale Λ the limit $\Lambda_0 \longrightarrow \infty$ can be shown to exist, which means that all divergences can be absorbed in a finite number of parameters.

In noncommutative field theory critical floating points or noncommutative fixed points are defined by the condition [15]

$$\partial_t|_{\phi,\bar\theta} S_*[\phi] = 0. \tag{12.330}$$

In other words, floating points or noncommutative fixed points are points which are independent of the scale up to dependence on $\bar\theta$, i.e. we allow dependence of $S_*[\phi]$ on $\bar\theta$ which is taken to be fixed when differentiating with respect to Λ. Thus, we do not allow dependence of $S_*[\phi]$ on $\theta\Lambda_0^2 = \bar\theta\Lambda_0^2/\Lambda^2$.

Let us write the kinetic term L_0 as

$$L_0[\Lambda] = \frac{\overline{\nu_4}}{\bar\theta} Z \sum_{m_i,n_i,k_i,l_i} \frac{1}{2} \tilde m^{m_2n_2}_{m_1n_1} (\Delta^{-1})^{m_2n_2,k_2l_2}_{m_1n1,k_1l_1} \tilde m^{k_2l_2}_{k_1l_1}. \tag{12.331}$$

It is trivial to observe that (with $Z = 1$)

$$\partial_t|_{\phi,\bar\theta} L_0[\Lambda] = 0. \tag{12.332}$$

Let us next consider an interaction of the form

$$L[\Lambda] = \frac{\overline{\nu_4}}{\bar\theta} Z \sum_{m_i,n_i,k_i,l_i} \frac{1}{2} \tilde m^{m_2n_2}_{m_1n_1} \Omega^{m_2n_2,k_2l_2}_{m_1n1,k_1l_1} \tilde m^{k_2l_2}_{k_1l_1}. \tag{12.333}$$

We set $F = \Delta^{-1} + \Omega$. By using the fact that $\bar\theta = \theta\Lambda^2$, i.e. $\partial_t\bar\theta = -2\bar\theta$ we compute

$$\begin{aligned} &\left(\partial_t + \frac{2+\eta}{2} \tilde m_{mn} \frac{\partial}{\partial \tilde m_{mn}}\right) S[\tilde m, \Lambda] \\ &= \frac{\overline{\nu_4}}{\bar\theta} Z \times \sum_{m_i,n_i,k_i,l_i} \frac{1}{2} \tilde m^{m_2n_2}_{m_1n_1} (\partial_t F)^{m_2n_2,k_2l_2}_{m_1n1,k_1l_1} \tilde m^{k_2l_2}_{k_1l_1}, \end{aligned} \tag{12.334}$$

and

$$\begin{aligned} &\frac{1}{2\pi^2\bar\theta} \sum_{m,n,k,l} \dot\Delta_{nm,lk}(\Lambda) \left[\frac{\partial S[\tilde m, \Lambda]}{\partial \tilde m_{kl}} \frac{\partial \Sigma[\tilde m, \Lambda]}{\partial \tilde m_{mn}} - \frac{\partial^2 \Sigma[\tilde m, \Lambda]}{\partial \tilde m_{kl} \partial \tilde m_{mn}}\right] \\ &= \frac{\overline{\nu_4}}{\bar\theta} Z \sum_{m_i,n_i,k_i,l_i} \frac{1}{2} \tilde m^{m_2n_2}_{m_1n_1} (ZF\dot\Delta(-2\Delta^{-1} + F))^{m_2n_2,k_2l_2}_{m_1n1,k_1l_1} \tilde m^{k_2l_2}_{k_1l_1}. \end{aligned} \tag{12.335}$$

In the last equation we have neglected as usual a constant term, i.e. the vacuum energy term. We compute $\dot{\Delta}_{mn,kl}(\Lambda) = \Delta_{mn,kl}(\Lambda)\,\partial_t \ln \prod_{i\in\{m_i,n_i\}} K_2(\frac{4i}{\theta\Lambda^2})$. We write this fact as $\dot{\Delta} = \Delta K$ where K is a diagonal matrix given by $K^{m_2n_2,k_2l_2}_{m_1n_1,k_1l_1} = \partial_t \ln \prod_{i\in\{m_i,n_i\}} K_2\Big(\frac{4i}{\theta\Lambda^2}\Big)\delta_{n_1,k_1}\delta_{m_1,l_1}\delta_{n_2,k_2}\delta_{m_2,l_2}$. By comparing the above two equations we get the condition

$$\partial_t F = ZF\dot{\Delta}(-2\Delta^{-1} + F). \tag{12.336}$$

A solution is given by (with $Z = 1$, i.e. $\eta = 0$)

$$F = \frac{1}{1 - W^{-1}\Delta}\Delta^{-1}. \tag{12.337}$$

The constant W is independent of $\bar{\theta}$. We have, therefore, a line of equivalent Gaussian floating/fixed points linked by field reparametrizations.

Renormalized trajectories
We again write the action as

$$S = \nu_4 \,\mathrm{Tr}_{\mathcal{H}}\left[(2+\omega)\Phi\tilde{x}_i^2\Phi + \omega\Phi\tilde{x}_i\Phi\tilde{x}_i + \frac{\mu_0^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4\right]. \tag{12.338}$$

We have defined $\tilde{x}_i = \hat{x}_i/\theta$ which is of order $1/\sqrt{\theta}$.

Let $S_*[\phi]$ be the action at the noncommutative fixed point. It may depend on $\bar{\theta} = \theta\Lambda^2$ so we write it as $S_*[\phi][\bar{\theta}]$. The renormalization group time is $t = \ln \mu/\Lambda$ where Λ is the cutoff and μ is a physical mass scale. In the vicinity of the noncommutative fixed point the action is

$$\begin{aligned} S[\phi](t,\bar{\theta}) &= S_*[\phi](\bar{\theta}) + T_l[\phi](\bar{\theta}) \\ &= S_*[\phi](\bar{\theta}) + \bar{\nu}_4 \sum_i \alpha_i e^{\zeta_i t}\mathcal{Q}_i[\phi](\bar{\theta}). \end{aligned} \tag{12.339}$$

The Polchinski equation is

$$\begin{aligned} \left(\partial_t + \frac{2+\eta}{2}\tilde{m}_{mn}\frac{\partial}{\partial\tilde{m}_{mn}}\right)S[\tilde{m},\Lambda] = \frac{1}{2\pi^2\bar{\theta}}\sum_{m,n,k,l}\dot{\Delta}_{nm,lk}(\Lambda) \\ \times\left[\frac{\partial S[\tilde{m},\Lambda]}{\partial\tilde{m}_{kl}}\frac{\partial\Sigma[\tilde{m},\Lambda]}{\partial\tilde{m}_{mn}} - \frac{\partial^2\Sigma[\tilde{m},\Lambda]}{\partial\tilde{m}_{kl}\partial\tilde{m}_{mn}}\right]. \end{aligned} \tag{12.340}$$

We have $S = S_* + T = L_0 + T$ and $\Sigma = S - 2L_0 = -L_0 + T$. We know that

$$\partial_t S_*[\tilde{m},\Lambda] = 0, \tag{12.341}$$

and

$$\frac{2+\eta}{2}\tilde{m}_{mn}\frac{\partial}{\partial\tilde{m}_{mn}}L_0[\tilde{m},\,\Lambda] = \frac{1}{2\pi^2\bar{\theta}}\sum_{m,n,k,l}\dot{\Delta}_{nm,lk}(\Lambda) \times\left[-\frac{\partial L_0[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{kl}}\frac{\partial L_0[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{mn}}+\frac{\partial^2 L_0[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{kl}\partial\tilde{m}_{mn}}\right]. \tag{12.342}$$

Since $\dot{\Delta}_{nm,lk}=\Delta_{nm,lk}K$ and $\Delta_{nm,lk}=\Delta_{lk,nm}$ we obtain

$$\left(\partial_t+\frac{2+\eta}{2}\tilde{m}_{mn}\frac{\partial}{\partial\tilde{m}_{mn}}\right)T[\tilde{m},\,\Lambda] = \frac{1}{2\pi^2\bar{\theta}}\sum_{m,n,k,l}\dot{\Delta}_{nm,lk}(\Lambda) \times\left[\frac{\partial T[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{kl}}\frac{\partial T[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{mn}}-\frac{\partial^2 T[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{kl}\partial\tilde{m}_{mn}}\right]. \tag{12.343}$$

Also, $\eta=\Lambda\partial\ln Z/\partial\Lambda=0$ since $Z=1$. Furthermore, by neglecting quadratic terms in T we get

$$\left(\partial_t+\tilde{m}_{mn}\frac{\partial}{\partial\tilde{m}_{mn}}\right)T[\tilde{m},\,\Lambda] = -\frac{1}{2\pi^2\bar{\theta}}\sum_{m,n,k,l}\dot{\Delta}_{nm,lk}(\Lambda)\frac{\partial^2 T[\tilde{m},\,\Lambda]}{\partial\tilde{m}_{kl}\partial\tilde{m}_{mn}}. \tag{12.344}$$

We have $\partial_t=-\Lambda\partial_\Lambda$, $\partial_t\bar{\theta}=-2\bar{\theta}$ and $\partial_t=-2\bar{\theta}\partial_{\bar{\theta}}$. Thus

$$\partial_t T[\tilde{m},\,\Lambda]=\bar{\nu}_4\sum_i\alpha_i e^{\zeta_i t}(\zeta_i-4-2\bar{\theta}\partial_{\bar{\theta}})\mathcal{Q}_i[\phi](\bar{\theta}). \tag{12.345}$$

We obtain therefore

$$\begin{aligned}\left(\zeta_i-4-2\bar{\theta}\partial_{\bar{\theta}}+\tilde{m}_{mn}\frac{\partial}{\partial\tilde{m}_{mn}}\right)\mathcal{Q}_i[\phi](\bar{\theta}) &= -\frac{1}{2\pi^2\bar{\theta}}\sum_{m,n,k,l}\dot{\Delta}_{nm,lk}(\Lambda)\\ &\quad\times\frac{\partial^2\mathcal{Q}_i[\phi](\bar{\theta})}{\partial\tilde{m}_{kl}\partial\tilde{m}_{mn}}\\ &= -\dot{V}\mathcal{Q}_i.\end{aligned} \tag{12.346}$$

Define

$$\begin{aligned}\mathcal{Q}_i' &= \exp\left(\frac{1}{2\pi^2\bar{\theta}}\sum_{m,n,k,l}\Delta_{nm,lk}(\Lambda)\frac{\partial^2}{\partial\tilde{m}_{kl}\partial\tilde{m}_{mn}}\right)\mathcal{Q}_i\\ &= \exp\left(\frac{1}{2\pi^2\theta}\sum_{m,n,k,l}\Delta_{nm,lk}(\Lambda)\frac{\partial^2}{\partial\tilde{M}_{kl}\partial\tilde{M}_{mn}}\right)\mathcal{Q}_i=\exp(V)\mathcal{Q}_i.\end{aligned} \tag{12.347}$$

From the second expression it is obvious that

$$\begin{aligned}&\left(\zeta_i - 4 - 2\bar{\theta}\partial_{\bar{\theta}} + \tilde{m}_{mn}\frac{\partial}{\partial\tilde{m}_{mn}}\right)\mathcal{Q}'_i\\&= e^V\left(\zeta_i - 4 - 2\bar{\theta}\partial_{\bar{\theta}} + \tilde{m}_{mn}\frac{\partial}{\partial\tilde{m}_{mn}} + \dot{V}\right)\mathcal{Q}_i\\&= 0.\end{aligned} \tag{12.348}$$

The operators $\mathcal{Q}'_i$ are single-trace terms built out of $2J$'s ϕ and 2ξ's $\tilde{x}_i$ of the generic form

$$\mathcal{Q}'_i = \bar{\theta}^s \mathcal{Q}'^{\rho}_{\mu_1 a_1 b_1,\ldots,\mu_{2\xi} a_{2\xi} b_{2\xi}; c_1 d_1,\ldots,c_{2J} d_{2J}} \tilde{x}^{\mu_1}_{a_1 b_1} \ldots \tilde{x}^{\mu_{2\xi}}_{a_{2\xi} b_{2\xi}} \tilde{m}_{c_1 d_1} \ldots \tilde{m}_{c_{2J} d_{2J}}. \tag{12.349}$$

The index ρ denotes the independent operators which we can form from $2J$'s ϕ and 2ξ's $\tilde{x}_i$. Thus, the index i stands for $(s, \rho, 2\xi, 2J)$. We compute $-2\bar{\theta}\partial_{\bar{\theta}}\mathcal{Q}'_i = -2(s - \xi)\mathcal{Q}'_i$. Hence, we get the constraint

$$\zeta_i - 2(s - \xi) = 4 - 2J. \tag{12.350}$$

Clearly the different operators with different values of s but with the same number of ϕ and $\tilde{x}_i$ are actually the same operator. We write then

$$\mathcal{Q}'_i = \bar{\theta}^s \mathcal{Q}'_j,\ j = (\rho, 2\xi, 2J). \tag{12.351}$$

We have then

$$\begin{aligned}S[\phi](t, \bar{\theta}) &= S_*[\phi](\bar{\theta}) + \bar{\nu}_4 \sum_i \alpha_i e^{\zeta_i t}\mathcal{Q}_i[\phi](\bar{\theta})\\&= S_*[\phi](\bar{\theta}) + \bar{\nu}_4 \sum_j \int ds\alpha_j(s) e^{\zeta_j t}\bar{\theta}^s \mathcal{Q}_j[\phi](\bar{\theta}),\end{aligned} \tag{12.352}$$

where

$$\mathcal{Q}_j = \exp(-V)\mathcal{Q}'_j = \exp\left(-\frac{1}{2\pi^2\bar{\theta}}\sum_{m,n,k,l}\Delta_{nm,lk}(\Lambda)\frac{\partial^2}{\partial\tilde{m}_{kl}\partial\tilde{m}_{mn}}\right)\mathcal{Q}'_j. \tag{12.353}$$

Introducing an appropriate norm we write the above result as

$$\begin{aligned}S[\phi](t, \bar{\theta}) &= S_*[\phi](\bar{\theta}) + \bar{\nu}_4 \sum_j \int ds\alpha_j(s) e^{\zeta_j t}\bar{\theta}^s ||\mathcal{Q}_j|| \hat{\mathcal{Q}}_j[\phi](\bar{\theta})\\&= S_*[\phi](\bar{\theta}) + \bar{\nu}_4 \sum_j \hat{g}_j[\alpha](t, \bar{\theta})\hat{\mathcal{Q}}_j[\phi](\bar{\theta})\end{aligned} \tag{12.354}$$

$$\hat{g}_j[\alpha](t, \bar{\theta}) = \int ds\alpha_j(s) e^{\zeta_j t}\bar{\theta}^s ||\mathcal{Q}_j||. \tag{12.355}$$

The norm will be given by the Gaussian functional integration

$$(a,\, b) = \frac{\displaystyle\int \mathcal{D}\tilde{m}\, \exp\left(-\tfrac{1}{2}\pi^2\bar{\theta}\sum_{mnkl}\tilde{m}_{mn}W_{mn,lk}\tilde{m}_{lk}\right)ab}{\displaystyle\int \mathcal{D}\tilde{m}\, \exp\left(-\tfrac{1}{2}\pi^2\bar{\theta}\sum_{mnkl}\tilde{m}_{mn}W_{mn,lk}\tilde{m}_{lk}\right)}. \tag{12.356}$$

In the above, W is defined by

$$W^{-1} = \Delta. \tag{12.357}$$

Examples are now in order.

Example 1: The self-dual kinetic term Let us consider operators with $2J = 2$, i.e. with two fields $\tilde{m}_{mn}$ and $\tilde{m}_{kl}$. These read explicitly

$$\begin{aligned}
\mathcal{Q}_2^{(\rho,2\zeta)} &= \exp(-V)\sum_{mnkl}\mathcal{Q}'^{(\rho,2\zeta)}_{mnkl}\tilde{m}_{mn}\tilde{m}_{kl} \\
&= \left[1 - \frac{1}{2\pi^2\bar{\theta}}\sum_{m,n,k,l}\Delta_{nm,lk}(\Lambda)\frac{\partial^2}{\partial\tilde{m}_{kl}\partial\tilde{m}_{mn}}\right]\sum_{mnkl}\mathcal{Q}'^{(\rho,2\zeta)}_{mnkl}\tilde{m}_{mn}\tilde{m}_{kl} \\
&= \sum_{mnkl}\mathcal{Q}'^{(\rho,2\zeta)}_{mnkl}\left[\tilde{m}_{mn}\tilde{m}_{kl} - \frac{1}{\pi^2\bar{\theta}}\Delta_{nm,lk}\right].
\end{aligned} \tag{12.358}$$

We compute the inner product

$$\begin{aligned}
\left(\mathcal{Q}_2^{(\rho,2\zeta)},\, \mathcal{Q}_2^{(\rho',2\zeta')}\right) &= \sum_{mnkl}\sum_{m'n'k'l'}\mathcal{Q}'^{(\rho,2\zeta)}_{mnkl}\mathcal{Q}'^{(\rho',2\zeta')}_{m'n'k'l'} \\
&\quad\times\left(\tilde{m}_{mn}\tilde{m}_{kl} - \frac{1}{\pi^2\bar{\theta}}\Delta_{nm,lk},\, \tilde{m}_{m'n'}\tilde{m}_{k'l'}\right. \\
&\quad\left. - \frac{1}{\pi^2\bar{\theta}}\Delta_{n'm',l'k'}\right) \\
&= \sum_{mnkl}\sum_{m'n'k'l'}\mathcal{Q}'^{(\rho,2\zeta)}_{mnkl}\mathcal{Q}'^{(\rho',2\zeta')}_{m'n'k'l'} \\
&\quad\times\left(-\frac{1}{\pi^2\bar{\theta}}\Delta_{n'm',l'k'}\langle\tilde{m}_{mn}\tilde{m}_{kl}\rangle\right. \\
&\quad - \frac{1}{\pi^2\bar{\theta}}\Delta_{nm,lk}\langle\tilde{m}_{m'n'}\tilde{m}_{k'l'}\rangle \\
&\quad\left. + \langle\tilde{m}_{mn}\tilde{m}_{kl}\tilde{m}_{m'n'}\tilde{m}_{k'l'}\rangle + \frac{1}{(\pi^2\bar{\theta})^2}\Delta_{nm,lk}\Delta_{n'm',l'k'}\right).
\end{aligned} \tag{12.359}$$

The expectation values are defined with respect to the partition function

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\tilde{m} \exp\left(-\frac{1}{2}\pi^2\bar{\theta}\sum_{mnkl}\tilde{m}_{mn}W_{mn,lk}\tilde{m}_{lk} - \sum_{mn}J_{mn}\tilde{m}_{nm}\right) \\
&= \exp\left(\frac{1}{2\pi^2\bar{\theta}}\sum_{mn}J_{nm}W^{-1}_{nm,kl}J_{kl}\right) \\
&\quad \times \int \mathcal{D}\tilde{m} \exp\left(-\frac{1}{2}\pi^2\bar{\theta}\sum_{mnkl}\tilde{m}_{mn}W_{mn,lk}\tilde{m}_{lk}\right).
\end{aligned} \tag{12.360}
$$

Thus

$$
\langle \tilde{m}_{mn}\tilde{m}_{kl}\rangle = \frac{1}{Z[J]}\frac{\partial^2 Z[J]}{\partial J_{nm}\partial J_{lk}}|_{J=0} = \frac{1}{\pi^2\bar{\theta}}\Delta_{nm,lk} \tag{12.361}
$$

$$
\begin{aligned}
\langle \tilde{m}_{mn}\tilde{m}_{kl}\tilde{m}_{m'n'}\tilde{m}_{k'l'}\rangle &= \frac{1}{Z[J]}\frac{\partial^4 Z[J]}{\partial J_{nm}\partial J_{lk}\partial J_{n'm'}\partial J_{l'k'}}|_{J=0} \\
&= \frac{1}{(\pi^2\bar{\theta})^2}[\Delta_{nm,lk}\Delta_{n'm',l'k'} \\
&\quad + \Delta_{nm,n'm'}\Delta_{lk,l'k'} + \Delta_{nm,l'k'}\Delta_{n'm',lk}].
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\left(\mathcal{Q}_2^{(\rho,2\zeta)}, \mathcal{Q}_2^{(\rho',2\zeta')}\right) &= \frac{1}{(\pi^2\bar{\theta})^2}\sum_{mnkl}\sum_{m'n'k'l'}\mathcal{Q}'^{(\rho,2\zeta)}_{mnkl}\mathcal{Q}'^{(\rho',2\zeta')}_{m'n'k'l'} \\
&\quad \times [\Delta_{nm,n'm'}\Delta_{lk,l'k'} + \Delta_{nm,l'k'}\Delta_{n'm',lk}].
\end{aligned}
$$

The self-dual kinetic term is an example. We have

$$
\begin{aligned}
L_0 &= \nu_4 \,\mathrm{Tr}_{\mathcal{H}}\frac{2}{\theta^2}\Phi\hat{x}_i^2\Phi \\
&= \nu_4 \sum_{m_i,n_i,k_i,l_i}\frac{1}{2}\tilde{M}^{m_2n_2}_{m_1n_1} \\
&\quad \times\left(\frac{4}{\theta}(m_1+n_1+m_2+n_2-2)\delta_{n_1,k_1}\delta_{m_1,l_1}\delta_{n_2,k_2}\delta_{m_2,l_2}\right)\tilde{M}^{k_2l_2}_{k_1l_1} \\
&= \bar{\nu}_4\sum_{mnkl}\tilde{m}_{mn}\left(\frac{2}{\bar{\theta}}(m+n-2)\delta_{n,k}\delta_{m,l}\right)\tilde{m}_{kl}.
\end{aligned} \tag{12.362}
$$

In other words, we have

$$
\mathcal{Q}'^{(\rho,2\zeta)}_2 = \frac{2}{\bar{\theta}}(m+n-2)\delta_{n,k}\delta_{m,l}. \tag{12.363}
$$

We compute

$$\begin{aligned}\left(\mathcal{Q}_2^{(\rho,2\zeta)},\, \mathcal{Q}_2^{(\rho,2\zeta)}\right) &= \frac{1}{(\pi^2\bar{\theta})^2}\cdot 2\cdot\left(\frac{2}{\bar{\theta}}\right)^2 \\ &\quad\times \sum_{mnkl}(m+n-2)(k+l-2)\Delta_{nm,kl}\Delta_{mn,lk} \\ &= \frac{1}{2\pi^4\bar{\theta}^4}\sum_{mn} c^2(m,\, n,\, \bar{\theta}).\end{aligned} \tag{12.364}$$

We have used

$$\begin{aligned}\Delta_{mn,kl}(\Lambda) &= \frac{1}{4(m+n-2)}\delta_{nk}\delta_{ml}c(m,\, n,\, \bar{\theta}), \\ c(m,\, n,\, \bar{\theta}) &= \prod_{i\in\{m_i,n_i\}} K_2\left(\frac{4i}{\bar{\theta}}\right).\end{aligned} \tag{12.365}$$

In the large Λ limit the above sum can be evaluated by setting the cutoff function equal 1, replacing the sums by integrals and taking $\bar{\theta}$ as the upper limit of the integrals, i.e.

$$\begin{aligned}\left(\mathcal{Q}_2^{(\rho,\,2\zeta)},\, \mathcal{Q}_2^{(\rho,\,2\zeta)}\right) &= \frac{1}{2\pi^4\bar{\theta}^4}\int_0^{\bar{\theta}} dm_1\int_0^{\bar{\theta}} dm_2\int_0^{\bar{\theta}} dn_1\int_0^{\bar{\theta}} dn_2 \\ &= \frac{1}{2\pi^4}.\end{aligned} \tag{12.366}$$

The self-dual kinetic operator has a constant norm in the large Λ limit. The corresponding coupling is

$$\begin{aligned}\hat{g}_j[\alpha](t,\, \bar{\theta}) &= \int ds\alpha_j(s)e^{\zeta_j t}\bar{\theta}^s \\ &= \int ds\alpha_j(s)e^{-(\zeta_j-2s)\ln\Lambda}e^{s\ln\theta+\zeta_j\ln\mu}\bar{\theta}^s.\end{aligned} \tag{12.367}$$

For this operator $2\xi = 2$ since it is constructed out of two $\tilde{x}_i$. Thus $\xi = 1$. The identity $\zeta_i - 2(s-\xi) = 4 - 2J$ becomes $\zeta_i - 2s = 0$. In other words, the overall dependence on Λ vanishes which means that this is a marginal coupling.

In general, marginal and relevant couplings are given by the constraint

$$\lambda_i = \zeta_i - 2s \geqslant 0 \Leftrightarrow 2J + 2\xi \leqslant 4. \tag{12.368}$$

For marginal and relevant couplings taking the limit $\Lambda \longrightarrow 0$ brings us back into the Gaussian noncommutative fixed point since the couplings vanish in this limit. If the noncommutative renormalization group eigenvalue λ_i is negative, we obtain an irrelevant interaction.

Example 2: The mass term This is given by

$$\begin{aligned} L_0 &= \nu_4 \,\mathrm{Tr}_{\mathcal{H}} \frac{\mu_0^{\,2}}{2} \Phi\Phi \\ &= \nu_4 \sum_{m_i,n_i,k_i,l_i} \frac{\mu_0^{\,2}}{2} \tilde{M}^{m_2 n_2}_{m_1 n_1} (\delta_{n_1,k_1}\delta_{m_1,l_1}\delta_{n_2,k_2}\delta_{m_2,l_2}) \tilde{M}^{k_2 l_2}_{k_1 l_1} \\ &= \bar{\nu}_4 \sum_{mnkl} \tilde{m}_{mn} \left(\frac{\bar{\mu}_0^{\,2}}{2} \delta_{nk}\delta_{m,l} \right) \tilde{m}_{kl}. \end{aligned} \tag{12.369}$$

In this case

$$\mathcal{Q}_2'^{(\rho,2\zeta)} = \frac{\bar{\mu}_0^{\,2}}{2} \delta_{n,k}\delta_{m,l}. \tag{12.370}$$

We compute now

$$\begin{aligned} \left(\mathcal{Q}_2^{(\rho,2\zeta)},\, \mathcal{Q}_2^{(\rho,2\zeta)} \right) &= \frac{2}{(\pi^2\bar{\theta})^2} \left(\frac{\bar{\mu}_0^{\,2}}{2} \right)^2 \sum_{mnkl} \Delta_{nm,kl}\Delta_{mn,lk} \\ &= \frac{2}{(\pi^2\bar{\theta})^2} \left(\frac{\bar{\mu}_0^{\,2}}{2} \right)^2 \frac{1}{16} \sum_{mn} \frac{c^2(m,\, n,\, \bar{\theta})}{(m+n-2)^2}. \end{aligned} \tag{12.371}$$

Again we verify that in the large Λ limit we obtain

$$\begin{aligned} \left(\mathcal{Q}_2^{(\rho,2\zeta)},\, \mathcal{Q}_2^{(\rho,2\zeta)} \right) &= \frac{2}{(\pi^2\bar{\theta})^2} \left(\frac{\bar{\mu}_0^{\,2}}{2} \right)^2 \frac{1}{16} \int dm_1 \int dn_1 \int dm_2 \int dn_2 \\ &\quad \times \frac{c^2(m_1,\, n_1,\, m_2,\, n_2,\, \bar{\theta})}{(m_1+n_1+m_2+n_2-2)^2} \\ &= \frac{2}{(\pi^2\bar{\theta})^2} \left(\frac{\bar{\mu}_0^{\,2}}{2} \right)^2 \frac{1}{16} \int_0^{\bar{\theta}} dm_1 \int_0^{\bar{\theta}} dn_1 \int_0^{\bar{\theta}} dm_2 \int_0^{\bar{\theta}} dn_2 \\ &\quad \times \frac{1}{(m_1+n_1+m_2+n_2-2)^2} \\ &= \text{constant}. \end{aligned} \tag{12.372}$$

The mass term has also a constant norm in the large Λ limit. Thus, the corresponding coupling is

$$\begin{aligned}\hat{g}_j[\alpha](t,\bar{\theta}) &= \int ds\alpha_j(s)e^{\zeta_j t}\bar{\theta}^s \\ &= \int ds\alpha_j(s)e^{-(\zeta_j-2s)\ln\Lambda}e^{s\ln\theta+\zeta_j\ln\mu}\bar{\theta}^s.\end{aligned} \tag{12.373}$$

Since for the mass operator $\xi = 0$ we will have $\zeta_i - 2s = 2$ and hence

$$\hat{g}_j[\alpha](t,\bar{\theta}) = O\left(\frac{1}{\Lambda^2}\right). \tag{12.374}$$

The mass is relevant with a renormalization group eigenvalue equal to +2.

In summary, we obtain at the Gaussian floating or noncommutative fixed point the following behavior:

- The mass operator ($2J = 2$, $2\xi = 0$) is relevant.
- The self-dual kinetic term ($2J = 2$, $2\xi = 2$) is marginal.
- The harmonic oscillator term, i.e. the off-diagonal kinetic term ($2J = 2$, $2\xi = 2$) is marginal.
- The ϕ^4 coupling ($2J = 4$, $2\xi = 0$) is marginal.
- All other couplings are irrelevant.

The ϕ^4 coupling at the self-dual point is exactly marginal, as shown by the vanishing of the beta function to all orders in perturbation theory.

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Advanced topics
Badis Ydri

Chapter 13

Some exact solutions of quantum field theory

The non-perturbative physics of a quantum field theory (as we have seen) can only be probed by means of Monte Carlo methods on lattices (which can become quite intricate technically and numerically) and/or by means of the exact renormalization group equation (which is always quite intricate analytically and mathematically). But sometimes exact solutions of the quantum field theory model present themselves (lower dimensions and/or high degree of symmetries), which allow us to access the sought-after non-perturbative physics of the theory directly. In this chapter, we present as examples six models in dimension two, which all enjoy exact solutions, allowing us an unprecedented look at the true heart, i.e. the non-perturbative reality, of a quantum field theory.

13.1 The linear sigma model and Hartree–Fock approximation

A good reference here is [1].

13.1.1 The mean-field theory

We are always interested in the physics near the critical temperature of a second-order phase transition. Let us then consider the action and the partition function

$$S = \int d^d x \left[\frac{1}{2}\phi\left(-\partial_\mu^2 + r\right)\phi + u\phi^4 \right] \tag{13.1}$$

$$Z[J] = \int \mathcal{D}\phi e^{-\beta\left(S - \int d^d x J\phi\right)}, \ \beta = \frac{1}{T}. \tag{13.2}$$

The action S is the energy associated with low-energy (long-wavelength) modes with slow spatial variations. It does not describe high-energy (short-wavelength) distortions. Since the physics near the critical temperature of a second-order phase

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transition is controlled by long-wavelength fluctuations the action S will be adequate provided we suppress high-energy modes using a cutoff Λ.

The equation of motion is

$$-\partial_\mu^2\phi_c + r\phi_c + 4u\phi_c^3 = J. \tag{13.3}$$

For spatially uniform configurations we get

$$r\phi_c + 4u\phi_c^3 = J. \tag{13.4}$$

The mass parameter r is related to the temperature by

$$r = a(T - T_*). \tag{13.5}$$

Thus, for $J = 0$ we have the solutions

$$\phi_c = 0, \;\; T > T_* \tag{13.6}$$

$$\phi_c = \pm\sqrt{-\frac{r}{4u}}, \;\; T < T_*. \tag{13.7}$$

We have a second-order phase transition at $T = T_*$. Both signs have the same free energy. The sign can be fixed by allowing J to go to 0 from above or from below. The up and down phases coexist along the line $J = 0$ and $T < T_*$.

We get the first critical exponent from the behavior of the order parameter at $T = T_*$ given by

$$\langle\phi\rangle = \phi_c \sim (T_* - T)^\beta, \;\; \beta = \frac{1}{2}, \;\; T < T_*. \tag{13.8}$$

We get the second critical exponent from differentiating the equation of motion with respect to J. We obtain

$$\left(r + 12u\phi_c^2\right)\frac{\delta\phi_c}{\delta J} = 1 \tag{13.9}$$

$$\chi = \frac{\delta\phi_c}{\delta J} = \frac{1}{r + 12u\phi_c^2}. \tag{13.10}$$

We compute

$$\begin{aligned} \chi &= \frac{1}{r}, \;\; T > T_* \\ &= \frac{1}{2|r|}, \;\; T < T_*. \end{aligned} \tag{13.11}$$

In other words,

$$\chi \sim |T - T_*|^\gamma, \;\; \gamma = 1. \tag{13.12}$$

The free energy is

$$\begin{aligned} f &= 0, \ \ T > T_* \\ &= -\frac{r^2}{16u}, \ \ T < T_*. \end{aligned} \tag{13.13}$$

The specific heat

$$\begin{aligned} c_v &= -T\frac{\partial^2 f}{\partial T^2} = 0, \ \ T > T_* \\ &= \frac{a^2}{8u}T, \ \ T < T_*. \end{aligned} \tag{13.14}$$

By writing the equation of the motion at $T = T_*$ we obtain

$$\phi_c = \left(\frac{J}{4u}\right)^{\frac{1}{\delta}}, \ \ \delta = 3. \tag{13.15}$$

In order to find the correlation length we re-introduce the gradient term. In this case the susceptibility is given by

$$\chi(x, x') = \frac{\delta^2 S}{\delta\phi_c(x)\delta\phi_c(x')} = \left(-\partial_\mu^2 + r + 12u\phi_c^3\right)\delta^d(x - x'). \tag{13.16}$$

Since ϕ_c is constant we get

$$\chi(p) = \frac{1}{p^2 + r + 12u\phi_c^2}. \tag{13.17}$$

We compute

$$\chi(p) = \frac{\chi(0)}{1 + (p^2\xi^2)} \tag{13.18}$$

$$\begin{aligned} \xi &= \frac{1}{\sqrt{r + 12u\phi_c^2}} = \frac{1}{\sqrt{r}}, \ \ T > T_* \\ &= \frac{1}{\sqrt{-2r}}, \ \ T < T_*. \end{aligned} \tag{13.19}$$

Hence

$$\xi \sim |T - T_*|^{-\nu}, \ \ \nu = \frac{1}{2}. \tag{13.20}$$

13.1.2 The 1-loop effective action

The fluctuation ψ is defined by

$$\phi = \phi_c + \psi. \tag{13.21}$$

We compute

$$S[\phi] = S[\phi_c] + \int d^d x J\psi + \frac{1}{2\beta}\int d^d x d^d x' \psi(x) G_0^{-1}(x, x')\psi(x') + O(\psi^3) \tag{13.22}$$

$$\begin{aligned} G_0^{-1}(x, x') &= \beta\left(-\partial_\mu^2 + r + 12u\phi_c^2\right)\delta^d(x - x') = \frac{\delta J(x)}{\delta\phi_c(x')} \\ &= \frac{\delta^2\beta S}{\delta\phi(x)\delta\phi(x')}|_{\phi=\phi_c}. \end{aligned} \tag{13.23}$$

In the disordered phase $\phi_c = 0$. We get

$$\begin{aligned} G_0^{-1}(x, x') &= \beta\int \frac{d^d p}{(2\pi)^d}(p^2 + r)e^{ip(x-x')} \\ G_0(x, x') &= \frac{1}{\beta}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 + r}e^{ip(x-x')}. \end{aligned} \tag{13.24}$$

The classical theory (tree-level) is the mean-field theory (saddle point). In this case

$$Z_{MF}[J] = e^{-\beta\left(S[\phi_c] - \int d^d x J\phi_c\right)}. \tag{13.25}$$

At 1-loop order we have

$$\begin{aligned} Z[J] &= e^{-\beta\left(S[\phi_c] - \int d^d x J\phi_c\right)}\int \mathcal{D}\psi e^{-\frac{1}{2}\int d^d x d^d x' \psi(x) G_0^{-1}(x,x')\psi(x')} \\ &= e^{-\beta\left(S[\phi_c] - \int d^d x J\phi_c\right)}\left(\det\, G_0^{-1}\right)^{-\frac{1}{2}}. \end{aligned} \tag{13.26}$$

The vacuum energy (free energy) is defined by

$$Z[J] = e^{-W[J]}. \tag{13.27}$$

We get the free energy

$$W[J] = S[\phi_c] - \int d^d x J\phi_c + \frac{1}{2\beta}\mathrm{Tr}\,\ln G_0^{-1}(\phi_c). \tag{13.28}$$

We define the effective action as usual by the Legendre transform, viz

$$\Gamma[\phi_c] = S[\phi_c] + S_1[\phi_c],\ \ S_1[\phi_c] = \frac{1}{2\beta}\mathrm{Tr}\,\ln G_0^{-1}(\phi_c). \tag{13.29}$$

The classical field Φ_c is defined by

$$\Phi_c = -\frac{\delta W[J]}{\delta J}. \tag{13.30}$$

This is actually different from the classical solution ϕ_c.

Indeed, we compute

$$\begin{aligned}\Phi_c &= -\frac{\delta W[J]}{\delta J}\\ &= \phi_c - \frac{\delta\Gamma}{\delta J} + \int d^dx' J(x')\frac{\delta\phi_c(x')}{\delta J(x)}\\ &= \phi_c - \frac{\delta S_1}{\delta J}\\ &= \phi_c - \frac{1}{2\beta}\int d^dx_1 d^dx_2 d^dx_3 \frac{\delta\phi_c(x_3)}{\delta J(x)} G_0(x_1, x_2)\frac{\delta G_0^{-1}(x_1, x_2)}{\delta\phi_c(x_3)}\\ &= \phi_c - 12uG_0(x_2, x_2)\int d^dx_1 G_0(x, x_1)\phi_c(x_1)\\ &= \phi_c + \delta\phi_c.\end{aligned} \tag{13.31}$$

In the disordered phase $\phi_c = \Phi_c = 0$. We can immediately show that up to 2-loop effects we have (where we use the fact that ϕ_c solves the classical equation of motion)

$$W[J] = S[\Phi_c] - \int d^dx J\Phi_c + \frac{1}{2\beta}\mathrm{Tr}\,\ln G_0^{-1}(\Phi_c). \tag{13.32}$$

The effective action as we said is the Legendre transform of $W[J]$:

$$\Gamma[\Phi_c] = W[J] + \int d^dx J\Phi_c = S[\Phi_c] + \frac{1}{2\beta}\mathrm{Tr}\,\ln G_0^{-1}(\Phi_c). \tag{13.33}$$

This satisfies

$$\frac{\delta\Gamma}{\delta\Phi_c} = J. \tag{13.34}$$

The 1-point function is

$$\langle\phi(x)\rangle = \frac{1}{\beta}\frac{1}{Z}\frac{\delta Z}{\delta J(x)} = -\frac{\delta W}{\delta J} = \Phi_c. \tag{13.35}$$

The 2-point function is

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\rangle &= \frac{1}{\beta^2}\frac{1}{Z}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}\\ &= G(x_1, x_2) + \langle\phi(x_1)\rangle\langle\phi(x_2)\rangle.\end{aligned} \tag{13.36}$$

The connected 2-point function is defined by

$$G(x_1, x_2) = -\frac{1}{\beta}\frac{\delta^2 W}{\delta J(x_1)J(x_2)}. \tag{13.37}$$

The 2-point proper vertex is defined by

$$\Gamma(x_1, x_2) = \beta \frac{\delta^2 \Gamma}{\delta \Phi_c(x_1) \Phi_c(x_2)} = G^{-1}(x_1, x_2). \tag{13.38}$$

We write the effective action as

$$\Gamma[\phi] = S[\phi] + \frac{1}{2\beta} \mathrm{Tr} \ln G_0^{-1}(\phi). \tag{13.39}$$

We compute

$$\begin{aligned} \frac{\delta^2 \Gamma}{\delta \phi(x) \delta \phi(x')}|_{\phi=\Phi_c} &= \frac{1}{\beta} G^{-1}(x, x') \\ &= \frac{1}{\beta} G_0^{-1}(x, x') + \frac{1}{2\beta} \mathrm{Tr}\, G_0 \frac{\delta^2 G_0^{-1}}{\delta \phi(x) \delta \phi(x')}|_{\phi=\Phi_c} \\ &\quad + \frac{1}{2\beta} \mathrm{Tr}\, \frac{\delta^2 G_0}{\delta \phi(x')}|_{\phi=\Phi_c} \frac{\delta^2 G_0^{-1}}{\delta \phi(x)}|_{\phi=\Phi_c}. \end{aligned} \tag{13.40}$$

In the disordered phase $\Phi_c = 0$. The 2-point proper vertex is

$$\begin{aligned} G^{-1}(x, x') &= G_0^{-1}(x, x') + \frac{1}{2} \mathrm{Tr}\, G_0 \frac{\delta^2 G_0^{-1}}{\delta \phi(x) \delta \phi(x')}|_{\phi=0} \\ &= G_0^{-1}(x, x') + \frac{1}{2} \int d^d x_1 d^d x_2 G_0(x_1, x_2) \frac{\delta^2 G_0^{-1}(x_1, x_2)}{\delta \phi(x) \delta \phi(x')}|_{\phi=0} \\ &= G_0^{-1}(x, x') + 12\beta u \delta^d(x - x') G_0(x, x'). \end{aligned} \tag{13.41}$$

In momentum space

$$G^{-1}(x, x') = \int \frac{d^d p}{(2\pi)^d} G^{-1}(p) e^{ip(x-x')}. \tag{13.42}$$

We then get

$$TG^{-1}(p) = p^2 + r + 12uT \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + r}. \tag{13.43}$$

Thus, the connected 2-point function is defined by

$$\begin{aligned} G(x, x) = \langle \phi(x)^2 \rangle_{\text{connected}} &= \int \frac{d^d p}{(2\pi)^d} G(p) \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{T}{p^2 + r + 12u G_0(x, x)}. \end{aligned} \tag{13.44}$$

The free connected 2-point function is

$$G_0(x, x) = \int \frac{d^d p}{(2\pi)^d} \frac{T}{p^2 + r}. \tag{13.45}$$

13.1.3 The self-consistent approximation and critical exponents

The self-consistent Hartree–Fock approximation consists of making the replacement $G_0 \longrightarrow G$. We get the equation

$$G(x, x) = \int \frac{d^d p}{(2\pi)^d} \frac{T}{p^2 + r + 12uG(x, x)}. \tag{13.46}$$

In this case

$$TG^{-1}(p) = p^2 + r + 12u \int \frac{d^d k}{(2\pi)^d} \frac{T}{k^2 + r + 12uG(x, x)}. \tag{13.47}$$

The susceptibility is defined by

$$\begin{aligned} \chi^{-1} = TG^{-1}(0) &= r + 12u \int \frac{d^d k}{(2\pi)^d} \frac{T}{k^2 + r + 12uG(x, x)} \\ &= r + 12u \int \frac{d^d k}{(2\pi)^d} \frac{T}{k^2 + \chi^{-1}}. \end{aligned} \tag{13.48}$$

We introduce the renormalized mass $\chi^{-1} = \tau$. Thus

$$\tau = r + 12u \int \frac{d^d k}{(2\pi)^d} \frac{T}{k^2 + \tau}. \tag{13.49}$$

The integral is restricted to a sphere of radius Λ of the order of the inverse correlation length. We get (with $K_d = \Omega_d/(2\pi)^d$)

$$\tau = r + 12uTK_d \int_0^\Lambda \frac{k^{d-1} dk}{k^2 + \tau}. \tag{13.50}$$

At the critical temperature $T = T_c$ the susceptibility diverges, i.e. $\tau = 0$. We get the critical mass parameter

$$r_c = -12uT_c K_d \int_0^\Lambda \frac{k^{d-1} dk}{k^2} = -\frac{12T_c K_d \Lambda^{d-2}}{d-2} u. \tag{13.51}$$

However, $r_c = a(T_c - T_*)$ where the critical temperature T_* is the critical value obtained in mean-field theory. Hence

$$T_c = \frac{T_*}{1 + \frac{12K_d \Lambda^{d-2} u}{a(d-2)}}. \tag{13.52}$$

The critical value T_c is less than the mean-field value T_*. For $d \longrightarrow 2$, $T_c \longrightarrow 0$. In other words, there is no transition in $d < 2$. In this case the critical fluctuations are so violent that no transition is possible.

We compute now

$$\begin{aligned}\chi^{-1} &= r - r_c + 12uK_d \int k^{d-1}dk\left[\frac{T}{k^2+\tau} - \frac{T_c}{k^2}\right] \\ &= r - r_c - 12uK_dT\tau I_d(\tau) + 12uK_d(T - T_c)\int k^{d-1}dk\frac{1}{k^2} \\ &= \left(1 + \frac{12uK_d}{a}\frac{\Lambda^{d-2}}{d-2}\right)(r - r_c) - 12uK_dT_c\tau I_d(\tau) \\ &\quad - 12uK_d(T - T_c)\tau I_d(\tau) \\ &= \left(a + 12uK_d\frac{\Lambda^{d-2}}{d-2}\right)(T - T_c) - 12uK_dT_c\tau I_d(\tau) \\ &\quad - 12uK_d(T - T_c)\tau I_d(\tau).\end{aligned} \tag{13.53}$$

We consider first the case $d > 4$. Since $T \longrightarrow T_c$, $\tau \longrightarrow 0$ and therefore

$$\begin{aligned}I_d(\tau) &= \int \frac{k^{d-3}dk}{k^2+\tau} \\ &= \int \frac{k^{d-3}dk}{k^2} \\ &= \frac{\Lambda^{d-4}}{d-4}.\end{aligned} \tag{13.54}$$

Thus

$$\begin{aligned}\chi^{-1} &= \left(a + 12uK_d\frac{\Lambda^{d-2}}{d-2}\right)(T - T_c) - 12uK_dT_c\frac{1}{\chi}I_d(0) \\ &\quad - 12uK_d(T - T_c)\frac{1}{\chi}I_d(0).\end{aligned} \tag{13.55}$$

The last term can be neglected since it is of the order $(T - T_c)^2$. We get then

$$\chi^{-1} = \left(a + 12uK_d\frac{\Lambda^{d-2}}{d-2}\right)(T - T_c) - 12uK_dT_c\frac{1}{\chi}I_d(0). \tag{13.56}$$

Equivalently we get

$$\chi = \frac{1 + 12uK_dT_cI_d(0)}{\left(a + 12uK_d\dfrac{\Lambda^{d-2}}{d-2}\right)(T - T_c)}. \tag{13.57}$$

The critical exponent is $\gamma = 1$, which is the same as in the mean-field theory.

For $d < 4$ the integral $I_d(\tau)$ is divergent at $\tau = 0$. We have (with $\epsilon = 4 - d$)

$$\begin{aligned} I_d(\tau) &= \int \frac{k^{d-3}dk}{k^2+\tau} \\ &= \tau^{\frac{d-4}{2}} \int_0^{\frac{\Lambda}{\sqrt{\tau}}} \frac{x^{d-3}dx}{x^2+1} \\ &= \tau^{-\frac{\epsilon}{2}} B_d\left(\frac{\Lambda}{\sqrt{\tau}}\right) \\ &= \tau^{-\frac{\epsilon}{2}} \frac{1}{2}\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{\epsilon}{2}\right), \ \tau \longrightarrow 0. \end{aligned} \tag{13.58}$$

The inverse susceptibility

$$\chi^{-1} = \tau = \left(a + 12uK_d\frac{\Lambda^{d-2}}{d-2}\right)(T - T_c) - 12uK_dT_c\tau I_d(\tau). \tag{13.59}$$

For temperatures far from the critical temperature τ is large. Thus, $\tau^{-\epsilon/2}$ is small and as a consequence we can neglect the second term in the above equation. The susceptibility becomes

$$\chi^{-1} = \left(a + 12uK_d\frac{\Lambda^{d-2}}{d-2}\right)(T - T_c). \tag{13.60}$$

In other words, $\chi \sim 1/(T - T_c)$, i.e. the critical exponent is the same as in the mean-field theory.

For $d < 4$ and $T > T_G$ where T_G is the Ginsberg temperature we get mean-field theory prediction. The Ginsberg temperature is the temperature at which perturbation theory breaks down and fluctuations become important. It can be determined from the condition

$$(1 + 12uK_dT_cI_d(\tau))\tau = \left(a + 12uK_d\frac{\Lambda^{d-2}}{d-2}\right)(T - T_c). \tag{13.61}$$

The mean-field behavior corresponds to $12uK_dT_cI_d(\tau) \ll 1$ which is equivalent to

$$\tau \gg (12uK_dT_cB_d)^{-\frac{2}{4-d}}. \tag{13.62}$$

In other words,

$$r - r_c \gg (12uK_dT_cB_d)^{-\frac{2}{4-d}}\left(1 + \frac{12uK_d}{a}\frac{\Lambda^{d-2}}{d-2}\right)^{-1} \tag{13.63}$$

$$T \gg T_G = T_c + \frac{1}{a}(12uK_dT_cB_d)^{-\frac{2}{4-d}}\left(1 + \frac{12uK_d}{a}\frac{\Lambda^{d-2}}{d-2}\right)^{-1}. \tag{13.64}$$

For $d < 4$ and $T < T_G$ fluctuations dominate and thus $\tau \longrightarrow 0$ and we get

$$0 = \left(a + 12uK_d \frac{\Lambda^{d-2}}{d-2}\right)(T - T_c) - 12uK_d T_c \tau \tau^{-\frac{\epsilon}{2}} B_d. \tag{13.65}$$

Equivalently

$$\tau^{-1} = \chi \sim (r - r_c)^{-\frac{2}{d-2}}. \tag{13.66}$$

The critical exponent is

$$\gamma = \frac{2}{d-2}. \tag{13.67}$$

Let us write the main result one more time as

$$\chi^{-1}(p) = p^2 + r + 12uG(x, x). \tag{13.68}$$

There are no p-dependent corrections to the mean-field result $p^2 + r$. We then compute

$$\frac{\chi(p)}{\chi(0)} = \frac{1}{1 + \frac{p^2}{r + 12uG(x,x)}}. \tag{13.69}$$

The correlation length ξ is defined by

$$\frac{\chi(p)}{\chi(0)} = \frac{1}{1 + (p\xi)^2}. \tag{13.70}$$

We obtain

$$\xi^2 = \frac{1}{r + 12uG(x, x)} = \frac{1}{\tau} \sim (T - T_c)^{-2\nu}. \tag{13.71}$$

We get

$$\nu = \frac{\gamma}{2} = \frac{1}{d-2}. \tag{13.72}$$

For $d \longrightarrow 4$ we get $\nu = 1/2$ which is the mean-field result. The mean-field theory becomes exact when $d \longrightarrow \infty$. In fact mean-field theory describes well the second-order phase transitions only for $d > 4$.

13.1.4 The large N saddle point

Let us consider now

$$S = \int d^d x \left[\frac{1}{2}\phi_a\left(-\partial_\mu^2 + r\right)\phi_a + u\left(\phi_a^2\right)^2\right] \tag{13.73}$$

$$Z[J] = \int \mathcal{D}\phi_a e^{-\beta\left(S - \int d^d x J\phi_a\right)}. \tag{13.74}$$

We use the Hubbard–Stratonovich transformation given by

$$e^{-\beta u(\phi_a^2)^2} = c\int_{-i\infty}^{i\infty} \mathcal{D}\psi e^{\frac{\beta}{16u}\psi^2 - \frac{\beta}{2}\psi\phi_a^2}. \tag{13.75}$$

The contour of integration is along the imaginary rather than the real axis because the Gaussian integration over ψ has the wrong sign. We compute

$$\begin{aligned} Z[0] &= \int \mathcal{D}\psi e^{\int d^dx \frac{\beta}{16u}\psi^2}\left[\int \mathcal{D}\phi e^{-\frac{\beta}{2}\int d^dx d^dx' \phi(x)G^{-1}(x,x')\phi(x')}\right]^N \\ &= \int \mathcal{D}\psi e^{\int d^dx \frac{\beta}{16u}\psi^2 - \frac{N}{2}\mathrm{Tr}\,\ln \beta G^{-1}(x,x')} \end{aligned} \tag{13.76}$$

$$G^{-1}(x,\, x') = \left(-\partial_\mu^2 + r + \psi\right)\delta^d(x - x'). \tag{13.77}$$

We take the limit

$$N \longrightarrow \infty,\ \ u \longrightarrow 0,\ \ g = uN = \text{fixed}. \tag{13.78}$$

We get

$$Z[0] = \int \mathcal{D}\psi e^{N\left[\int d^dx \frac{\beta}{16g}\psi^2 - \frac{1}{2}\mathrm{Tr}\,\ln\beta G^{-1}(x,x')\right]}. \tag{13.79}$$

The integral is dominated by the saddle point solution. The saddle point equation is

$$\begin{aligned} \frac{\beta}{8g}\psi &= \frac{1}{2}\,\mathrm{Tr}\, G\frac{\delta}{\delta\psi}G^{-1} \\ &= \frac{1}{2}G(x,\, x). \end{aligned} \tag{13.80}$$

Since G is independent of x the configuration ψ is independent of x. Thus

$$\psi = 4gT\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2 + r + \psi}. \tag{13.81}$$

13.2 The non-linear sigma model and the $1/N$ expansion

In this section we follow the presentation of [2] and [3].

13.2.1 The $d = 2$ non-linear sigma model

We consider the following action in $d = 2$ Minkowskian spacetime

$$S = \int d^2x\mathcal{L},\ \ \mathcal{L} = \frac{1}{2g^2}(\partial_\mu\vec{n})^2, \tag{13.82}$$

where the $O(n)$-vector $\vec{n}$ satisfies also the constraint

$$\vec{n}^2 = \sum_{i=1}^{N} n_i^{\,2} = 1. \tag{13.83}$$

The coupling constant g and the field $\vec{n}$ are both dimensionless. In a perturbative quantization of this model we implement the constraint by writing

$$\vec{n} = (\pi_1, \ldots, \pi_{N-1}, \sigma), \;\; \sigma = \sqrt{1 - \vec{\pi}^2}, \tag{13.84}$$

and thus

$$(\partial_\mu \vec{n})^2 = (\partial_\mu \vec{\pi})^2 + (\partial_\mu \sigma)^2 = (\partial_\mu \vec{\pi})^2 + \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2}{1 - \vec{\pi}^2}. \tag{13.85}$$

The action up to the fourth power in the π-field becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2g^2}(\partial_\mu \vec{\pi})^2 + \frac{1}{2g^2}\frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2}{1 - \vec{\pi}^2} \\ &= \frac{1}{2g^2}(\partial_\mu \vec{\pi})^2 + \frac{1}{2g^2}(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 + \cdots \end{aligned} \tag{13.86}$$

At this stage the constraint is already being taken into consideration. To derive Feynman rules we write

$$\vec{\pi} = \int \frac{d^2k}{(2\pi)^2} \vec{\pi}(k) e^{ikx} = \vec{\pi}^*, \;\; \vec{\pi}(-k) = \vec{\pi}^*(k). \tag{13.87}$$

It is not then difficult to find that the first term in the action is

$$\begin{aligned} \frac{1}{2g^2} \int d^2x (\partial_\mu \vec{\pi})^2 &= \frac{1}{2g^2} \int \frac{d^2k}{(2\pi)^2} k^2 |\vec{\pi}(k)|^2 \\ &\equiv \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \pi_i(k) \left[(2\pi)^2 \delta^2(k + p) \left(\frac{k^2}{g^2} \delta_{ij} \right) \right] \pi_j(p). \end{aligned} \tag{13.88}$$

The propagator $D_{jl}(p, q) = \langle \pi_j(p) \pi_l(q) \rangle$ should satisfy

$$\int \frac{d^2p}{(2\pi)^2} \sum_j \left[(2\pi)^2 \delta^2(k + p) \left(\frac{k^2}{g^2} \delta_{ij} \right) \right] D_{jl}(p, q) = i \delta_{il} (2\pi)^2 \delta^2(k - q). \tag{13.89}$$

Hence

$$\langle \pi_i(k) \pi_j(p) \rangle = \frac{ig^2}{k^2} \delta_{ij} (2\pi)^2 \delta^2(k + p). \tag{13.90}$$

The interaction is, on the other hand, computed to be given by

$$\begin{aligned}\frac{1}{2g^2}\int d^2x(\vec{\pi}\partial_\mu\vec{\pi})^2 &= \frac{1}{2g^2}\int \frac{d^2k_1}{(2\pi)^2}\ldots\frac{d^2k_4}{(2\pi)^2}(2\pi)^2\delta^2\\ &\quad\times(k_1+\cdots+k_4)\pi_{i_1}(k_1)\ldots\pi_{i_4}(k_4)\\ &\quad\times(-\delta_{i_1i_2}\delta_{i_3i_4}k_2k_4)\\ &= -\frac{1}{4!g^2}\int \frac{d^2k_1}{(2\pi)^2}\ldots\frac{d^2k_4}{(2\pi)^2}(2\pi)^2\delta^2\\ &\quad\times(k_1+\cdots+k_4)\pi_{i_1}(k_1)\ldots\pi_{i_4}(k_4)\\ &\quad\times(\delta_{i_1i_2}\delta_{i_3i_4}(k_1+k_2)(k_3+k_4)\\ &\quad+\delta_{i_1i_3}\delta_{i_2i_4}(k_1+k_3)(k_2+k_4)\\ &\quad+\delta_{i_1i_4}\delta_{i_2i_3}(k_1+k_4)(k_2+k_3)),\end{aligned} \tag{13.91}$$

where we have symmetrized with respect to 1 and 2 then with respect to 3 and 4 and finally with respect to 1, 3 and 1, 4 respectively. The vertex is, therefore, given by

$$\begin{aligned}\langle\pi_{i_1}(k_1)\ldots\pi_{i_4}(k_4)\rangle = &-\frac{i}{g^2}(\delta_{i_1i_2}\delta_{i_3i_4}(k_1+k_2)(k_3+k_4)\\ &+\delta_{i_1i_3}\delta_{i_2i_4}(k_1+k_3)(k_2+k_4)\\ &+\delta_{i_1i_4}\delta_{i_2i_3}(k_1+k_4)(k_2+k_3))(2\pi)^2\delta^2(k_1+\cdots+k_4).\end{aligned} \tag{13.92}$$

The i comes from the exponent e^{iS} in the path integral while the 4! cancels because of the identical contributions from the 4! possible contractions, i.e. when we differentiate the action four times with respect to the field.

If one tries, however, to compute with these Feynman rules we will find that there is an IR divergence coming from the fact that the pion in the action (13.82) is massless. To remove this divergence one can couple the system to a constant external magnetic field h as follows [2]

$$\mathcal{L}_h = \frac{\mu^2}{g^2}\sigma = \frac{\mu^2}{g^2}\left(1-\frac{\vec{\pi}^2}{2}+\frac{(\vec{\pi}^2)^2}{8}+\cdots\right), \tag{13.93}$$

whereas before we have expanded the Lagrangian up to the fourth power in the π-field. We have also defined $\mu^2\equiv h$ to indicate that the constant magnetic field will act as a mass for the pion and thus regulate the zero momentum singularity. We compute now the action

$$\begin{aligned}S_h &= -\frac{\mu^2}{2g^2}\int d^2x\vec{\pi}^2+\frac{\mu^2}{8g^2}\int d^2x(\vec{\pi}^2)^2+\cdots\\ &= \frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\pi_i(k)\left[(2\pi)^2\delta^2(k+p)\left(\frac{-\mu^2}{g^2}\delta_{ij}\right)\right]\pi_j(p)\\ &\quad-\frac{1}{4!g^2}\int\frac{d^2k_1}{(2\pi)^2}\ldots\frac{d^2k_4}{(2\pi)^2}(2\pi)^2\delta^2(k_1+\cdots+k_4)\pi_{i_1}(k_1)\ldots\pi_{i_4}(k_4)\\ &\quad\times(\delta_{i_1i_2}\delta_{i_3i_4}+\delta_{i_1i_3}\delta_{i_2i_4}\delta_{i_1i_4}\delta_{i_2i_3})(-\mu^2).\end{aligned} \tag{13.94}$$

Note finally that the Lagrangian $\mathcal{L} + \mathcal{L}_h$ has only $O(N-1)$ symmetry and not the full $O(N)$ symmetry, which is consistent with the fact that this model describes essentially the spontaneous symmetry breaking $O(N) \longrightarrow O(N-1)$.

The action we will study is therefore

$$S = \frac{1}{2g^2}\int d^2x\Big[(\partial_\mu \vec{\pi})^2 + (\partial_\mu \sigma)^2\Big] + \frac{h}{g^2}\int d^2x\sigma. \tag{13.95}$$

It is obvious that the first term in this action (i.e. the action (13.82)) is the most general $O(N)$-symmetric Lagrangian which can be built out of the field $\vec{n}$ with constraint $\vec{n}^2 = 1$ and with dimensionless coupling g. In other words, this term is renormalizable and thus the theory can be completely made finite by renormalizing the coupling constant g and the field strength $|\vec{n}|$ in an $O(N)$-invariant fashion. The origin of the second term is in fact $\vec{h}.\,\vec{n}$ which is important both to remove the IR divergence of theory as we have just said and also to select a minimum for the model [2]. This term is, therefore, also $O(N)$-symmetric and the symmetry gets only broken to $O(N-1)$ when the $O(N)$-magnetic field $\vec{h}$ is oriented in the N direction, $\vec{h} = (0, \ldots, 0, h)$. The g, $\vec{\pi}$, σ are all dimensionless and h is of dimension mass squared. If one tries to compute Green's functions using the above action we will find that the theory is UV divergent and thus we need to regularize. We choose here dimensional regularization (DR) in which the action becomes

$$S_0 = \frac{\Lambda^{d-2}}{2g_0^2}\int d^dx\Big[(\partial_\mu \vec{\pi}_0)^2 + (\partial_\mu \sigma_0)^2\Big] + \frac{h_0\Lambda^{d-2}}{g_0^2}\int d^dx\sigma_0. \tag{13.96}$$

The g_0, $\vec{\pi}_0$, σ_0 are still dimensionless and also h_0 is still of dimension mass squared. The 0 stands for bare quantities and Λ^{d-2} is a mass parameter. Now if one computes anything with this action we will find that it depends on the bare coupling constants g_0 and h_0 as well as on dimensional regularization 'cutoff' which we can define by the pair $(\Lambda, \epsilon = 2 - d)$. Hence, the bare Green's functions as opposed to the renormalized correlations are 'cutoff'-dependent which is quite natural.

As an example we compute the 2-point function of the pion field in the 1-loop approximation which can be easily found to be given by a tadpole diagram. The value of the loop is given by the expression

$$\begin{aligned}\mathcal{X} &= \frac{1}{2}\int \frac{d^dk_3}{(2\pi)^d}\frac{d^dk_4}{(2\pi)^d}(2\pi)^d\delta^d(k_1+k_2+k_3+k_4)\left(-\frac{i}{g^2}\right)\\ &\quad\times[\delta_{i_1i_2}\delta_{i_3i_4}[(k_1+k_2)(k_3+k_4)-\mu^2] + \delta_{i_1i_3}\delta_{i_2i_4}[(k_1+k_3)(k_2+k_4)-\mu^2]\\ &\quad+\delta_{i_1i_4}\delta_{i_2i_3}[(k_1+k_4)(k_2+k_3)-\mu^2]]\frac{ig^2}{k_3^2-\mu^2}\delta_{i_3i_4}(2\pi)^d\delta^d(k_3+k_4)\\ &= \delta_{i_1i_2}(2\pi)^d\delta^d(k_1+k_2)\int\frac{d^dk_3}{(2\pi)^d}\frac{1}{k_3^2-\mu^2}\left[-k_1^2-k_3^2-\frac{1}{2}\mu^2(N+1)\right].\end{aligned} \tag{13.97}$$

The factor $\frac{1}{2}$ in front of the first line is a symmetry factor due to the fact that we have one line which starts and ends on the same vertex so the corresponding Feynman diagram is invariant under the interchange of the two ends of this line. Using now the results

$$\begin{aligned}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2-\mu^2} &= -\frac{i}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(\mu^2)^{1-\frac{d}{2}}},\\ \int \frac{d^dq}{(2\pi)^d}\frac{q^2}{q^2-\mu^2} &= \frac{i}{(4\pi)^{d/2}}\frac{d}{2}\Gamma\left(-\frac{d}{2}\right)(\mu^2)^{\frac{d}{2}},\end{aligned} \tag{13.98}$$

and $\Gamma(z+1) = z\Gamma(z)$ we obtain

$$\begin{aligned}\mathcal{X} = {}& \delta_{i_1i_2}(2\pi)^d\delta^d(k_1+k_2)\frac{ik_1^2}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(\mu^2)^{1-\frac{d}{2}}}\\ &+\delta_{i_1i_2}(2\pi)^d\delta^d(k_1+k_2)\frac{-i(N+3)}{2(4\pi)^{d/2}}\frac{d}{2}\Gamma\left(-\frac{d}{2}\right)(\mu^2)^{\frac{d}{2}}.\end{aligned} \tag{13.99}$$

Here we will assume for simplicity that the coupling constant $h \equiv \mu^2$ does not get renormalized and that its only purpose is to remove the IR divergence. This means that we take $\mu \longrightarrow 0$ wherever possible and thus the second term in the above can be simply neglected since it has no pole at $d = 0$ and for $d > 0$ this term is proportional to a positive power of μ^2. We have then

$$\mathcal{X} = \delta_{i_1i_2}(2\pi)^d\delta^d(k_1+k_2)\frac{ik_1^2}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(\mu^2)^{1-\frac{d}{2}}}. \tag{13.100}$$

The full 2-point function of the pion field is now given by (by including the amputated external propagators)

$$\begin{aligned}\langle\pi_i(k)\pi_j(p)\rangle_{1-\text{loop}} = {}& \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\left(\frac{ig^2}{k^2-\mu^2}\delta_{ii_1}(2\pi)^d\delta(k_1-k)\right)\\ &\times\left(\delta_{i_1i_2}(2\pi)^d\delta^d(k_1+k_2)\frac{ik_1^2}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(\mu^2)^{1-\frac{d}{2}}}\right)\\ &\times\left(\frac{ig^2}{p^2-\mu^2}\delta_{ji_2}(2\pi)^d\delta(k_2-p)\right)\\ ={}& \frac{ig^2}{k^2-\mu^2}\delta_{ij}(2\pi)^d\delta^d(k+p)\\ &\times\left(-\frac{g^2}{(4\pi)^{d/2}}\frac{k^2}{k^2-\mu^2}\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(\mu^2)^{1-\frac{d}{2}}}\right).\end{aligned} \tag{13.101}$$

The 1-loop 2-point function is therefore given by

$$\begin{aligned}\langle \pi_i(k)\pi_j(p)\rangle &= \frac{ig^2}{k^2-\mu^2}\delta_{ij}(2\pi)^d\delta^d(k+p)\\ &\quad\times\left(1-\frac{g^2}{(4\pi)^{d/2}}\frac{k^2}{k^2-\mu^2}\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(\mu^2)^{1-\frac{d}{2}}}\right)\\ &= \frac{ig^2}{k^2}\delta_{ij}(2\pi)^d\delta^d(k+p)\\ &\quad\times\left(1-\frac{g^2}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(\mu^2)^{1-\frac{d}{2}}}\right),\end{aligned} \tag{13.102}$$

where again we have set the IR cutoff μ equal to 0 wherever possible.

Now in dimensional regularization g^2 stands actually for $\frac{g_0^2}{\Lambda^{d-2}}$, h for h_0 (i.e. μ^2 stands for μ_0^2) and $\pi_i(x)$ for $\pi_{0i}(x)$. In momentum space, however, $\pi_i(k)$ stands for $\frac{1}{\Lambda^{d-2}}\pi_{i0}(k)$, i.e. $\pi_{i0}(k)$ is of dimension (mass)$^{-2}$ which is the same mass dimension of momentum modes in $d=2$. Hence, the bare 2-point correlation function is given by

$$\begin{aligned}\langle \pi_{0i}(k)\pi_{0j}(p)\rangle &= \frac{ig_0^2}{k^2}\delta_{ij}\Lambda^{d-2}(2\pi)^d\delta^d(k+p)\\ &\quad\times\left(1-\frac{g_0^2}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{\Lambda^2}{\mu_0^2}\right)^{1-\frac{d}{2}}\right).\end{aligned} \tag{13.103}$$

$\Lambda^{2-d}(2\pi)^d\delta^d(k+p)$ has the same mass dimension as the two-dimensional momentum space delta function $(2\pi)^2\delta^2(k+p)$ and thus we simply identify them in the limit $d\longrightarrow 2$.

Obviously this 2-point function diverges at $d=2$ (i.e, when we remove dimensional regularization) and thus we need to renormalize the coupling constant g_0 as well as scale the fields $\vec{\pi}_0$ and σ_0 in order to make this 'cutoff' dependent 2-point function finite in the limit $d\longrightarrow 2$. We introduce, therefore, the renormalized coupling constant g and the renormalized fields π_r and σ_r by

$$\begin{aligned}&g_0^2 = Z_g g_r^2,\ \ Z_g = 1+g_r^2\bar{Z}_g+\cdots\\ &\pi_{0i} = \sqrt{Z}\pi_{ri},\ \ Z = 1+g_r^2\bar{Z}+\cdots\\ &\sigma_0 = \sqrt{Z}\sigma_r,\ \ \vec{\pi}_r^2+\sigma_r^2 = z^{-1}.\end{aligned} \tag{13.104}$$

Recall that the expansion in the coupling constant g_0 is equivalent to the $\hbar$-expansion, i.e. to the loop-wise expansion [2] . The renormalization constants Z and Z_g are dimensionless and hence can only depend on the dimensionless coupling constant g_0 (or g_r), on the ratio $\frac{\Lambda}{\mu}$ and on ϵ. It is now not difficult to check that

$$\langle \pi_{ri}(k)\pi_{rj}(p)\rangle = \frac{ig_r^2}{k^2}\delta_{ij}\Lambda^{d-2}(2\pi)^d\delta^d(k+p) \times\left(1+g_r^2\left[-\frac{1}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{\Lambda^2}{\mu_0^2}\right)^{1-\frac{d}{2}}-\bar{Z}+\bar{Z}_g\right]\right)+O\left(g_r^6\right). \tag{13.105}$$

To determine $\bar{Z}$ and $\bar{Z}_g$ we need to impose at least two renormalization conditions. We impose first the condition that at the scale $\mu_0^2 = M^2$ (the renormalization scale) the 2-point function looks like that of the free theory, namely

$$\langle \pi_{ri}(k)\pi_{rj}(p)\rangle = \frac{ig_r^2}{k^2}\delta_{ij}\Lambda^{d-2}(2\pi)^d\delta^d(k+p),\ \text{at}\ \mu_0^2 = M^2. \tag{13.106}$$

This gives us the equation

$$-\frac{1}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{\Lambda^2}{M^2}\right)^{1-\frac{d}{2}}-\bar{Z}+\bar{Z}_g=0,\ \text{at}\ \mu_0^2=M^2. \tag{13.107}$$

Clearly we need one more equation to determine both $\bar{Z}$ and $\bar{Z}_g$ and thus one needs to calculate one more correlation function. We compute the expectation value $\langle\sigma\rangle$ of the field σ which is essentially like an order parameter if one recalls the spontaneous symmetry breaking which occurs in the model [3]. We have then

$$\begin{aligned}\langle\sigma(x)\rangle = \left\langle\sqrt{1-\vec{\pi}^2(x)}\right\rangle &= 1-\frac{1}{2}\langle\vec{\pi}^2(x)\rangle+\cdots\\ &= 1-\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}e^{i(k+p)x}\delta_{ij}\langle\pi_i(k)\pi_j(p)\rangle\\ &= 1-\frac{ig^2(N-1)}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\mu^2}\\ &= 1-\frac{g^2(N-1)}{2}\frac{1}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(\mu^2)^{1-\frac{d}{2}}}\\ &= 1-\frac{g_0^2(N-1)}{2}\frac{1}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{\Lambda^2}{\mu_0^2}\right)^{1-\frac{d}{2}}.\end{aligned} \tag{13.108}$$

Again, the bare correlation function is cutoff-dependent and, therefore, it needs renormalization. We get

$$\langle\sigma_r\rangle = 1-\frac{g_r^2}{2}\left[\bar{Z}+\frac{(N-1)}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{\Lambda^2}{\mu_0^2}\right)^{1-\frac{d}{2}}\right]+O\left(g_r^4\right). \tag{13.109}$$

We impose now the extra renormalization condition that at the renormalization scale $\mu_0^2 = M^2$ the order parameter $\langle \sigma_r \rangle$ is a unity, namely

$$\langle \sigma_r \rangle = 1, \text{ at } \mu_0^2 = M^2. \tag{13.110}$$

We get now the condition

$$\bar{Z} + \frac{(N-1)}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{\Lambda^2}{\mu_0^2}\right)^{1-\frac{d}{2}} = 0, \text{ at } \mu_0^2 = M^2. \tag{13.111}$$

This is solved by

$$\begin{aligned} \bar{Z} &= -\frac{(N-1)}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{\Lambda^2}{M^2}\right)^{1-\frac{d}{2}} \\ &= -\frac{(N-1)}{(4\pi)^{d/2}} \left[\frac{2}{\epsilon} - \gamma + \ln \frac{\Lambda^2}{M^2}\right], \end{aligned} \tag{13.112}$$

and hence the order parameter becomes

$$\langle \sigma_r \rangle = 1 - \frac{g_r^2 (N-1)}{8\pi} \ln \frac{M^2}{\mu_0^2} + O\left(g_r^4\right). \tag{13.113}$$

Similarly, by putting the solution (13.112) in the renormalization condition (13.107) we obtain the other renormalization constant $\bar{Z}_g$ to be given by

$$\bar{Z}_g = -\frac{N-2}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{\Lambda^2}{M^2}\right)^{1-\frac{d}{2}}. \tag{13.114}$$

The 2-point function becomes now

$$\begin{aligned} \langle \pi_{ri}(k) \pi_{rj}(p) \rangle &= \frac{i g_r^2}{k^2} \delta_{ij} (2\pi)^2 \delta^2(k+p) \\ &\quad \times \left(1 + \frac{g_r^2}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \left[\left(\frac{\Lambda^2}{M^2}\right)^{1-\frac{d}{2}} - \left(\frac{\Lambda^2}{\mu_0^2}\right)^{1-\frac{d}{2}}\right]\right) \\ &= \frac{i g_r^2}{k^2} \delta_{ij} (2\pi)^2 \delta^2(k+p) \left(1 - \frac{g_r^2}{4\pi} \ln \frac{M^2}{\mu_0^2}\right). \end{aligned} \tag{13.115}$$

13.2.2 The Callan–Symanzik equation and the beta function

The bare action we worked with is

$$S_0 = \frac{\Lambda^{d-2}}{2 g_0^2} \int d^d x \left[(\partial_\mu \vec{\pi}_0)^2 + (\partial_\mu \sigma_0)^2\right] + \frac{h_0 \Lambda^{d-2}}{g_0^2} \int d^d x \sigma_0. \tag{13.116}$$

The renormalized quantities are defined by

$$\begin{aligned} g_0^2 &= Z_g g_r^2, \;\; h_0 = Z_h \frac{Z_g}{\sqrt{Z}} h_r \\ \pi_{0i} &= \sqrt{Z}\pi_{ri}, \;\; \sigma_0 = \sqrt{Z}\sigma_r, \;\; \vec{\pi}_r^2 + \sigma_r^2 = z^{-1}. \end{aligned} \tag{13.117}$$

Using these definitions in the above action we obtain

$$S_0 = \frac{\Lambda^{d-2} Z}{2 g_r^2 Z_g} \int d^d x \left[(\partial_\mu \vec{\pi}_r)^2 + (\partial_\mu \sigma_r)^2 \right] + \frac{h_r}{g_r^2} \Lambda^{d-2} Z_h \int d^d x \sigma_r. \tag{13.118}$$

Since the renormalized action does not depend on the cutoff Λ but rather on the renormalization scale M, we can define it simply by

$$S_r = \frac{M^{d-2} Z}{2 g_r^2 Z_g} \int d^d x \left[(\partial_\mu \vec{\pi}_r)^2 + (\partial_\mu \sigma_r)^2 \right] + \frac{h_r}{g_r^2} M^{d-2} \int d^d x \sigma_r. \tag{13.119}$$

Obviously, the overall external constant magnetic field (the coefficient in front of the σ-field) should not get renormalized, i.e. it should be the same in the bare and in the renormalized theory and as a consequence we get the condition $Z_h \Lambda^{d-2} = M^{d-2}$. In fact one can write this condition in the equivalent form

$$M^{d-2} \frac{h_r}{g_r^2} = \Lambda^{d-2} Z_h \frac{h_r}{g_r^2} \equiv \Lambda^{d-2} \sqrt{Z} \frac{h_0}{g_0^2}. \tag{13.120}$$

Thus, the counter term is given by

$$\Delta S_{c\cdot t} = \frac{(\Lambda^{d-2} - M^{d-2}) Z}{2 g_r^2 Z_g} \int d^d x \left[(\partial_\mu \vec{\pi}_r)^2 + (\partial_\mu \sigma_r)^2 \right]. \tag{13.121}$$

In equation (13.120), h_0 and g_0 are the bare coupling constants and, therefore, they do not depend on the renormalization scale M, whereas the renormalized coupling constants h_r and g_r, as well as the renormalization constant Z, do depend on M. This renormalization scale M is largely arbitrary and one would like to know how much this arbitrariness affects observable physical quantities. To this end we write equation (13.120) in the form

$$M^{d-2} \frac{h_r}{g_r^2} Z^{-\frac{1}{2}} = \Lambda^{d-2} \frac{h_0}{g_0^2}. \tag{13.122}$$

Taking the logarithm of this equation and then the derivative with respect to the scale M we get

$$d - 2 + M \frac{\partial}{\partial M} \ln h_r - \frac{2}{g_r} M \frac{\partial}{\partial M} g_r + M \frac{\partial}{\partial M} \ln Z^{-\frac{1}{2}} = 0. \tag{13.123}$$

By defining

$$\begin{aligned}\beta &= M\frac{\partial g_r}{\partial M}\\ \gamma &= M\frac{\partial \ln Z^{-\frac{1}{2}}}{\partial M}\\ \rho &= M\frac{\partial \ln h_r}{\partial M},\end{aligned} \tag{13.124}$$

one can also put the above last result in the form

$$\rho = 2 - d + \frac{2}{g_r}\beta - \gamma. \tag{13.125}$$

The first equation in (13.124) is the beta function and its properties are best understood via the so-called Callan–Symanzik equation. In the renormalization conditions (13.106) and (13.110) the renormalization scale M (as we have just noticed) is completely arbitrary. However, and as it turned out, this arbitrariness is completely controlled by the beta (as well as the gamma and rho) functions in equation (13.124), which we now explain in some more detail. Indeed, any bare correlation function can only depend on spacetime points x_i (or equivalently momenta p_i if we were in momentum space), and on the bare coupling constants g_0 and h_0 and on the 'cutoff' (Λ, ϵ) (see for example equations (13.103) and (13.108)). We write then

$$G_0^{(n)}(x_1, x_2, \ldots, x_n; g_0, h_0, \Lambda, \epsilon) = \langle \phi_{01}(x_1)\phi_{02}(x_2)\ldots\phi_{0n}(x_n)\rangle, \tag{13.126}$$

where ϕ_{0i} stands for the fields π_{0i}'s and σ_0. Bare correlation functions are therefore 'cutoff' dependent, which is essentially their most important characteristic. In particular, they do not depend on the renormalization scale M. Renormalized correlations are, however, 'cutoff' independent and M-dependent and they also depend on the spacetime points x_i and on the renormalized coupling constants g_r and h_r, i.e.

$$G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M) = \langle \phi_{r1}(x_1)\phi_{r2}(x_2)\ldots\phi_{rn}(x_n)\rangle. \tag{13.127}$$

Obviously by using $\phi_0 = Z^{\frac{1}{2}}\phi_r$ we have the relation

$$G_0^{(n)}(x_1, x_2, \ldots, x_n; g_0, h_0, \Lambda, \epsilon) = Z^{\frac{n}{2}}G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M). \tag{13.128}$$

Since G_0 does not know about the renormalization scale M it must be invariant under any change $M \longrightarrow M'$, i.e. it is unaffected if one chooses to define the renormalized Green's functions at the scale M' instead of M with new renormalized coupling constants g_r', h_r' and a new renormalization constant Z'. This means that the renormalized Green's functions G_r (see for example equations (13.113) and (13.115)) change as $G_r' = G_r + \delta G_r$ when M changes as $M' = M + \delta M$ and correspondingly g_r and h_r change as $g_r' = g_r + \delta g_r$ and $h_r' = h_r + \delta h_r$ such that

$$\delta G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M) = \delta M \frac{\partial}{\partial M} G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M) + \delta g_r \frac{\partial}{\partial g_r} G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M) + \delta h_r \frac{\partial}{\partial h_r} G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M). \quad (13.129)$$

This change (13.129) in G_r will be balanced by a change in the field strength renormalization Z as follows $Z^{\frac{n}{2}} G_r^{(n)} = Z'^{\frac{n}{2}} G_r'^{(n)}$ so that the bare Green's functions G_0 remain fixed. For infinitesimal change $M \longrightarrow M' = M + \delta M$ we can write $Z' = Z(1 - 2\delta\eta)$ and thus the corresponding field strength shift is defined by

$$\phi_r' = (1 + \delta\eta)\phi_r. \quad (13.130)$$

Hence, from equation (13.127) we get

$$G_r^{(n)'}(x_1, x_2, \ldots, x_n; g_r', h_r', M') = \langle \phi_{r1}'(x_1)\phi_{r2}'(x_2) \ldots \phi_{rn}'(x_n) \rangle = (1 + n\delta\eta) G_r^{(n)} \times (x_1, x_2, \ldots, x_n; g_r, h_r, M), \quad (13.131)$$

or equivalently

$$\delta G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M) = n\delta\eta G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M). \quad (13.132)$$

Putting equations (13.129) and (13.132) together we obtain

$$\left(\delta M \frac{\partial}{\partial M} + \delta g_r \frac{\partial}{\partial g_r} + \delta h_r \frac{\partial}{\partial h_r} - n\delta\eta \right) G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M) = 0. \quad (13.133)$$

It is not difficult to check now that

$$\delta g_r = \frac{\partial g_r}{\partial M} \delta M = \beta \frac{\delta M}{M}$$
$$\delta h_r = \frac{\partial h_r}{\partial M} \delta M = \rho \frac{h_r \delta M}{M} \quad (13.134)$$
$$\delta\eta \equiv \delta ln Z^{-\frac{1}{2}} = \frac{\partial ln Z^{-\frac{1}{2}}}{\partial M} \delta M = \gamma \frac{\delta M}{M}.$$

Hence, we get the final result

$$\left(M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial g_r} + \rho h_r \frac{\partial}{\partial h_r} - n\gamma \right) G_r^{(n)}(x_1, x_2, \ldots, x_n; g_r, h_r, M) = 0. \quad (13.135)$$

The parameters β, ρ and γ are obviously x-independent (which means also that they are p-independent) as well as n-independent. They are, furthermore, cutoff independent since the renormalized Green's functions G_r do not depend on the cutoff.

The renormalization constants $\bar{Z}$ and $\bar{Z}_g$ in equations (13.112) and (13.114) do not depend on the coupling constant h_r and thus the parameters β, γ and ρ will not depend on h_r [2]. Since the theory is massless, and since β, γ and ρ are p-independent as well as cutoff independent these parameters are also (by dimensional analysis) M-independent. Thus, β, γ and ρ can only depend on the dimensionless coupling constant g_r.

The above equation, which is called the Callan–Symanzik equation, tells us that the three universal functions $\beta(g_r)$, $\rho(h_r)$ and $\gamma(g_r)$ control, respectively, the shifts of the coupling constants g_r and h_r and the shift in the field strength ϕ_{ri} which are needed to compensate the shift in the renormalization scale M.

As an example, we apply the Callan–Symanzik equation to the results (13.113) and (13.115) to obtain the beta and gamma functions of the non-linear sigma model. We have in this case

$$\left(M\frac{\partial}{\partial M}+\beta(g_r)\frac{\partial}{\partial g_r}-n\gamma(g_r)\right)G_r^{(n)}(x_1,\,x_2,\,\ldots\,,\,x_n;\,g_r,\,M)=0, \tag{13.136}$$

where we have dropped for simplicity the h_r-derivative and the h_r-dependence of the renormalized Green's functions, which also means that the bare coupling constant h_0 is now interpreted at the level of correlations as only an IR regulator μ_0^2. Therefore, we have first

$$\left(M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial g_r}-\gamma\right)\langle\sigma_r\rangle=0, \tag{13.137}$$

which leads to the condition

$$-\frac{g_r^2(N-1)}{4\pi}-\beta g_r\frac{N-1}{4\pi}\ln\frac{M^2}{\mu_0^2}-\gamma+\gamma\frac{g_r^2(N-1)}{8\pi}\ln\frac{M^2}{\mu_0^2}=0. \tag{13.138}$$

β as we will show shortly is of order g_r^3 and thus up to order g_r^2 we have for the γ function the result

$$\gamma=-\frac{g_r^2(N-1)}{4\pi}+O\left(g_r^4\right). \tag{13.139}$$

Similarly, we have

$$\left(M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial g_r}-2\gamma\right)\langle\pi_{ri}(k)\pi_{rj}(p)\rangle=0, \tag{13.140}$$

which leads to

$$-\frac{g_r^4}{2\pi}+2\beta g_r-\beta\frac{g_r^3}{\pi}\ln\frac{M^2}{\mu_0^2}+\frac{g_r^4(N-1)}{2\pi}-\frac{g_r^6(N-1)}{8\pi^2}\ln\frac{M^2}{\mu_0^2}=0. \tag{13.141}$$

Thus, the beta function of the non-linear sigma model up to order g_r^3 is given by

$$\beta = -(N-2)\frac{g_r^3}{4\pi} + O\left(g_r^5\right). \tag{13.142}$$

13.2.3 The large N limit analysis

We assume now Euclidean signature in dimension d. The partition function of the non-linear sigma model is given by

$$Z = \int \mathcal{D}\vec{n} e^{-\int d^d x \frac{1}{2g_0^2}(\partial_\mu \vec{n})^2} \delta(\vec{n}^2 - 1). \tag{13.143}$$

The g_0 is the bare coupling constant. By definition the delta function which enforces the constraint $\vec{n}^2 = 1$ is given by

$$\delta(\vec{n}^2 - 1) = \prod_x \delta(\vec{n}^2(x) - 1). \tag{13.144}$$

In one dimension we know that the delta function can be represented by

$$\delta(x) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} e^{i\alpha x} = \int_{-\infty}^{\infty} \frac{d\bar{\alpha}}{4g_0^2\pi} e^{-\frac{i}{2g_0^2}\bar{\alpha}x}. \tag{13.145}$$

Thus, we will have

$$\delta(\vec{n}(x) - 1) = \int_{-\infty}^{\infty} \frac{d\alpha(x)}{4g_0^2\pi} e^{-\frac{i}{2g_0^2}\alpha(x)(\vec{n}^2(x)-1)}, \tag{13.146}$$

and consequently

$$\delta(\vec{n}^2 - 1) = \int \mathcal{D}\alpha e^{-\frac{i}{2g_0^2}\int d^d x \alpha(x)(\vec{n}^2(x)-1)}, \ \mathcal{D}\alpha = \prod_x \frac{d\alpha(x)}{4g_0^2\pi}. \tag{13.147}$$

The partition function becomes

$$\begin{aligned} Z &= \int \mathcal{D}\vec{n}\mathcal{D}\alpha e^{-\int d^d x \frac{1}{2g_0^2}(\partial_\mu \vec{n})^2 - \frac{i}{2g_0^2}\int d^d x \alpha(\vec{n}^2-1)} \\ &= \int \mathcal{D}\alpha \left[\mathcal{D}\vec{n} e^{\int \frac{d^d x}{2g_0^2}\vec{n}(\partial^2 - i\alpha)\vec{n}}\right] e^{i\int \frac{d^d x}{2g_0^2}\alpha} \\ &= \int \mathcal{D}\alpha \prod_i^N \left[\mathcal{D}n_i e^{\int \frac{d^d x}{2g_0^2} n_i(\partial^2 - i\alpha)n_i}\right] e^{i\int \frac{d^d x}{2g_0^2}\alpha}. \end{aligned} \tag{13.148}$$

The $\vec{n}$-field measure will be defined by

$$\mathcal{D}n_i = \prod_x \frac{dn_i(x)}{\sqrt{2\pi}} \frac{1}{\sqrt{g_0^2}} = \prod_x \frac{d\tilde{n}_i(x)}{\sqrt{2\pi}}, \tag{13.149}$$

where $\tilde{n}_i = \frac{n_i}{\sqrt{g_0^2}}$. The partition function becomes then

$$Z = \int \mathcal{D}\alpha \prod_i^N \left[\prod_x \frac{d\tilde{n}_i(x)}{\sqrt{2\pi}} e^{-\frac{1}{2}\int d^dx d^dy \tilde{n}_i(x)\delta^d(x-y)(-\partial^2+i\alpha)_x \tilde{n}_i(y)} \right] e^{i\int \frac{d^dx}{2g_0^2}\alpha}. \quad (13.150)$$

By using the identity

$$\begin{aligned} Z_0[J] &= \int \prod_{i=1}^N \frac{dq_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum_{n,m} q_n M_{nm} q_m + \sum_n J_n q_n} \\ &= (\det M)^{-\frac{1}{2}} e^{\frac{1}{2}\sum_{n,m} J_n M^{-1}_{nm} J_m}, \end{aligned} \quad (13.151)$$

we obtain

$$\begin{aligned} Z &= \int \mathcal{D}\alpha [\det(\delta^d(x-y)(-\partial^2+i\alpha)_x)]^{-\frac{N}{2}} e^{i\int \frac{d^dx}{2g_0^2}\alpha} \\ &= \int \mathcal{D}\alpha [\det(-\partial^2+i\alpha(x))]^{-\frac{N}{2}} e^{i\int \frac{d^dx}{2g_0^2}\alpha} \\ &= \int \mathcal{D}\alpha e^{-\frac{N}{2}\mathrm{Tr}\ln(-\partial^2+i\alpha)+\frac{i}{2g_0^2}\int d^dx\alpha}. \end{aligned} \quad (13.152)$$

In the second line we have used the fact that $\det(AB) = \det(A)\det(B)$ and then absorbed for simplicity the infinite factor $[\det(\delta^d(x-y))]^{-\frac{N}{2}}$ into the measure $\mathcal{D}\alpha$. In the last line we have used, however, the identity $\det A = e^{Tr\ln A}$.

In the large N limit, and if we keep $g_0^2 N = h_0^2$ fixed, it is not difficult to see that both terms in the exponent in the above partition function (13.152) are of order N. One can then reasonably use the method of steepest descents to evaluate this partition function Z by effectively dominating the path integral by the configuration $\alpha(x)$ which minimizes the exponent. We write then

$$\begin{aligned} Z &= \int \mathcal{D}\alpha e^{\int d^dx F(\alpha(x))}, \ F(\alpha(x)) \\ &= -\frac{N}{2}\langle x|\ln(-\partial^2+i\alpha(x))|x\rangle + \frac{i}{2g_0^2}\alpha(x). \end{aligned} \quad (13.153)$$

The condition

$$\frac{\partial F(\alpha(x))}{\partial\alpha(x)} = 0. \quad (13.154)$$

takes also the form

$$\frac{N}{2}\langle x|\frac{1}{-\partial^2+i\alpha(x)}|x\rangle = \frac{1}{2g_0^2}. \quad (13.155)$$

If we introduce now the basis $\{|p\rangle\}$ of the operator $P_\mu = -i\partial_\mu$ which satisfies

$$
\begin{aligned}
&[x_\mu, P_\nu] = i\delta_{\mu\nu} \\
&P_\mu|p\rangle = p_\mu|p\rangle, \quad \int d^d p|p\rangle\langle p| = 1, \quad \langle p|p'\rangle = \delta^d(p-p'), \\
&\langle x|p\rangle = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{ixp}.
\end{aligned}
\tag{13.156}
$$

Then, we can write the above equation as

$$
\begin{aligned}
\frac{1}{2g_0^2} &= \frac{N}{2}\int d^d p d^d p' \langle x|p\rangle\langle p|\frac{1}{P^2 + i\alpha(x)}|p'\rangle\langle p'|x\rangle \\
&= \frac{N}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 + i\alpha(x)}.
\end{aligned}
\tag{13.157}
$$

This means that the configuration $\alpha(x)$ is such that at each point x in spacetime this configuration is a pure imaginary constant, i.e.

$$
\alpha(x) = -im^2, \tag{13.158}
$$

and thus m^2 is the solution of the equation

$$
\frac{1}{2g_0^2} = \frac{N}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 + m^2}. \tag{13.159}
$$

From the first line of equation (13.148) it is obvious that the parameter m plays exactly the role of a mass for the field $\vec{n}$. In $d = 2$ and with a cutoff Λ one can compute

$$
\begin{aligned}
\frac{1}{g_0^2} &= N\int_\Lambda \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 + m^2} \\
&= \frac{N}{2\pi}\ln\sqrt{1 + \frac{\Lambda^2}{m^2}} \\
&= \frac{N}{2\pi}\ln\frac{\Lambda}{m},
\end{aligned}
\tag{13.160}
$$

where we have assumed that the cutoff is much larger than the mass m. Motivated by the perturbative result $g_0^2 = Z_g g_r^2$, which can also be put in the form

$$
\frac{1}{g_0^2} = \frac{1}{g_r^2} - \bar{Z}_g = \frac{1}{g_r^2} + \frac{N-2}{2\pi}\ln\frac{\Lambda}{M} + \frac{N-2}{4\pi}\left(\frac{2}{\epsilon} - \gamma\right), \tag{13.161}
$$

we will renormalize in this large N expansion the bare coupling constant g_0^2 as follows

$$
\frac{1}{g_0^2} = \frac{1}{g_r^2} + \frac{N}{2\pi}\ln\frac{\Lambda}{M}. \tag{13.162}
$$

The infinite third term in equation (13.161) can be thought of as being absorbed in the infinite bare coupling constant $1/g_0^2$, i.e. we assume for simplicity that the bare coupling constants in perturbation theory and in the large N expansion are actually different. More precisely, we assume that the infinite bare coupling constant which appears in the path integral Z is in fact

$$\frac{1}{g_0^2} - \frac{N-2}{4\pi}\left(\frac{2}{\epsilon} - \gamma\right)$$

with $1/g_0^2$ in this last equation standing for the bare coupling constant we worked with in perturbation theory. Putting equations (13.160) and (13.162) together we can solve for the mass m, namely

$$m = Me^{-\frac{2\pi}{Ng_r^2}}. \tag{13.163}$$

This is an $O(N)$-symmetric mass for the (essentially) unconstrained N components of the field $\vec{n}$. This mass cannot vanish for all the non-zero values of the coupling constant $h_r^2 = Ng_r^2$ and thus in $d = 2$ there is no critical value for this coupling constant and as a consequence the symmetry is not broken, i.e. $\langle \vec{n} \rangle = 0$ always. This mass depends on the arbitrary renormalization scale M which is also a reflection of the arbitrariness of the renormalization condition (13.162). If one changes the renormalization scale M as $M \longrightarrow M'$ then from equation (13.162) the renormalized coupling constant g_r^2 must also change appropriately as $g_r^2 \longrightarrow g_r'^2$ such that the bare coupling constant g_0^2 is kept fixed. Correspondingly, and from equation (13.160), the mass m remains fixed under this change of M and hence it depends only on the cutoff and on the bare coupling constant. We write this fact in the form

$$\left[M\frac{\partial}{\partial M} + \beta(g_r)\frac{\partial}{\partial g_r}\right]m(g_r, M) = 0. \tag{13.164}$$

Using the solution (13.163) in this last equation we get the beta function

$$\beta(g_r) = -\frac{g_r^3 N}{4\pi}, \tag{13.165}$$

which is consistent with the large N limit of equation (13.142). However, this last result is exact to all orders in $h_r^2 = Ng_r^2$ [3]. This is also obvious by rewriting the above beta function in the form

$$\beta(h_r) = -\frac{h_r^3}{4\pi}. \tag{13.166}$$

13.3 The Ising model and the Onsager solution

This is perhaps the most important exact solution in all quantum field theory. The original solution is found in [4] but here we follow the presentation of [5].

13.3.1 The Ising model in *d* dimensions

We consider a d-dimensional periodic lattice with n points in every direction so that there are $N = n^d$ points in total in this lattice. In every point (lattice site) we put a spin variable s_i $(i = 1, \ldots, N)$ which can take either the value $+1$ or -1. A configuration of this system of N spins is, therefore, specified by a set of numbers $\{s_i\}$. In the Ising model the energy of this system of N spins in the configuration $\{s_i\}$ is given by

$$E_I\{s_i\} = -\sum_{\langle ij\rangle} \epsilon_{ij} s_i s_j - H \sum_{i=1}^{N} s_i. \tag{13.167}$$

The parameter H is the external magnetic field. The symbol $\langle ij\rangle$ stands for nearest neighbor spins. The sum over $\langle ij\rangle$ extends over $\frac{\gamma N}{2}$ terms where γ is the number of nearest neighbors. In $2, 3, 4$ dimensions $\gamma = 4, 6, 8$. The parameter ϵ_{ij} is the interaction energy between the spins i and j. For isotropic interactions $\epsilon_{ij} = \epsilon$. For $\epsilon > 0$ we obtain ferromagnetism while for $\epsilon < 0$ we obtain antiferromagnetism. We consider only $\epsilon > 0$. The energy is

$$E_I\{s_i\} = -\epsilon \sum_{\langle ij\rangle} s_i s_j - H \sum_{i=1}^{N} s_i. \tag{13.168}$$

The partition function is given by

$$Z = \sum_{s_1} \sum_{s_2} \cdots \sum_{s_N} e^{-\beta E_I\{s_i\}}. \tag{13.169}$$

There are 2^N terms in the sum and $\beta = \frac{1}{k_B T}$.

13.3.2 The Ising model in two dimensions

We consider $d = 2$ in the remainder. There are $N = n^2$ spins in the square lattice. The configuration $\{s_i\}$ can be viewed as an $n \times n$ matrix. We impose periodic boundary condition as follows. We consider $(n + 1) \times (n + 1)$ matrix where the $(n + 1)$th row is identified with the first row and the $(n + 1)$th column is identified with the first column. The square lattice is, therefore, a torus. We introduce the notation

$$\mu_\alpha = \{s_1, s_2, \ldots, s_n\}_{\alpha\text{th row}}, \quad \alpha = 1, \ldots, n. \tag{13.170}$$

The toroidal boundary condition means that in each row $s_{n+1} = s_n$. It also means that the $(n + 1)$th row is identical to the first row, viz $\mu_1 = \mu_{n+1}$.

The interaction energy between the spins of the αth and $(\alpha + 1)$th rows is

$$E(\mu_\alpha, \mu_{\alpha+1}) = -\epsilon \sum_{k=1}^{n} s_k^{(\alpha)} s_k^{(\alpha+1)}. \tag{13.171}$$

The interaction energy between the spins of the αth row plus the energy due to the external magnetic field is

$$E(\mu_\alpha) = -\epsilon \sum_{k=1}^{n} s_k^{(\alpha)} s_{k+1}^{(\alpha)} - H \sum_{k=1}^{n} s_k^{(\alpha)}. \tag{13.172}$$

The total energy of the lattice is

$$E_I\{s_i\} \equiv E_I\{\mu_1, \ldots, \mu_n\} = \sum_{\alpha=1}^{n} \Big[E(\mu_\alpha, \mu_{\alpha+1}) + E(\mu_\alpha) \Big]. \tag{13.173}$$

The partition function is

$$Z = \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} e^{-\beta \sum_{\alpha=1}^{n} \left[E(\mu_\alpha, \mu_{\alpha+1}) + E(\mu_\alpha) \right]}. \tag{13.174}$$

We define the $2^n \times 2^n$ matrix P by

$$\begin{aligned} \langle \mu | P | \nu \rangle &= e^{-\beta[E(\mu,\nu)+E(\mu)]} = \prod_{k=1}^{n} e^{\beta H s_k^{(\mu)}} \, e^{\beta \epsilon s_k^{(\mu)} s_{k+1}^{(\mu)}} \, e^{\beta \epsilon s_k^{(\mu)} s_k^{(\nu)}} \\ &= \langle \mu | V_3 V_2 V_1' | \nu \rangle \end{aligned} \tag{13.175}$$

$$\begin{aligned} \langle \mu | V_1' | \nu \rangle &= \prod_{k=1}^{n} e^{\beta \epsilon s_k^{(\mu)} s_k^{(\nu)}} \\ \langle \mu | V_2 | \nu \rangle &= \delta_{\mu\nu} \prod_{k=1}^{n} e^{\beta \epsilon s_k^{(\mu)} s_{k+1}^{(\mu)}} \\ \langle \mu | V_3 | \nu \rangle &= \delta_{\mu\nu} \prod_{k=1}^{n} e^{\beta H s_k^{(\mu)}}. \end{aligned} \tag{13.176}$$

Thus

$$\begin{aligned} Z &= \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} \langle \mu_1 | P | \mu_2 \rangle \langle \mu_2 | P | \mu_3 \rangle \cdots \langle \mu_{n-1} | P | \mu_n \rangle \langle \mu_n | P | \mu_1 \rangle \\ &= \mathrm{Tr}\, P^n. \end{aligned} \tag{13.177}$$

Let us introduce now the direct product of matrices defined by

$$\langle i_1 i_2 \ldots i_n | A_1 \times A_2 \cdots \times A_n | j_1 j_2 \cdots j_n \rangle = \langle i_1 | A_1 | j_1 \rangle \langle i_2 | A_2 | j_2 \rangle \cdots \langle i_n | A_n | j_n \rangle. \tag{13.178}$$

Define the 2×2 matrix a by its components

$$\langle s^{(\mu)} | a | s^{(\nu)} \rangle = e^{\beta \epsilon s^{(\mu)} s^{(\nu)}}. \tag{13.179}$$

In other words

$$\begin{aligned} a &= e^{\beta\epsilon} \mathbf{1} + e^{-\beta\epsilon} X \\ &= \frac{e^{\beta\epsilon}}{\cosh\theta} \, e^{\theta X} \\ &= \sqrt{2 \sinh 2\beta\epsilon} \; e^{\theta X}. \end{aligned} \tag{13.180}$$

In the above, X is the Pauli matrix σ_1 and θ is defined by

$$\tanh\theta = e^{-2\beta\epsilon}. \tag{13.181}$$

Then

$$\begin{aligned}\langle\mu|V_1'|\nu\rangle &= \prod_{k=1}^{n} e^{\beta\epsilon s_k^{(\mu)} s_k^{(\nu)}} \\ &= \left\langle s_1^{(\mu)}|a|s_1^{(\nu)}\right\rangle \left\langle s_2^{(\mu)}|a|s_2^{(\nu)}\right\rangle \cdots \left\langle s_n^{(\mu)}|a|s_n^{(\nu)}\right\rangle \\ &= \left\langle s_1^{(\mu)} s_2^{(\mu)} \ldots s_n^{(\mu)}|a\times a\cdots\times a|s_1^{(\nu)} s_2^{(\nu)} \ldots s_n^{(\nu)}\right\rangle.\end{aligned} \tag{13.182}$$

In other words,

$$\begin{aligned}V_1' &= a\times a\cdots\times a \\ &= (2\sinh 2\beta\epsilon)^{\frac{n}{2}} e^{\theta X}\times e^{\theta X}\cdots\times e^{\theta X}.\end{aligned} \tag{13.183}$$

We introduce the $2^n\times 2^n$ matrices

$$X_\alpha = \mathbf{1}\times\mathbf{1}\times\ \cdots\times X\times\cdots\times\mathbf{1},\ \ \alpha = 1,\ldots,n. \tag{13.184}$$

The X appears in the αth factor. Then we have

$$\begin{aligned}V_1' &= (2\sinh 2\beta\epsilon)^{\frac{n}{2}} e^{\theta X}\times e^{\theta X}\cdots\times e^{\theta X} \\ &= (2\sinh 2\beta\epsilon)^{\frac{n}{2}} e^{\theta X_1} e^{\theta X_2}\ldots e^{\theta X_n} \\ &= (2\sinh 2\beta\epsilon)^{\frac{n}{2}} V_1,\end{aligned} \tag{13.185}$$

where

$$V_1 = \prod_{\alpha=1}^{n} e^{\theta X_\alpha}. \tag{13.186}$$

Let Y and Z be the other Pauli matrices σ_2 and σ_3 respectively. Let us introduce also the $2^n\times 2^n$ matrices

$$Y_\alpha = \mathbf{1}\times\mathbf{1}\times\cdots\times Y\times\cdots\times\mathbf{1},\ \ \alpha = 1,\ldots,n \tag{13.187}$$

$$Z_\alpha = \mathbf{1}\times\mathbf{1}\times\cdots\times Z\times\cdots\times\mathbf{1},\ \ \alpha = 1,\ldots,n. \tag{13.188}$$

Now we find V_2 defined by the matrix elements

$$\langle\mu|V_2|\nu\rangle = \delta_{\mu\nu}\prod_{k=1}^{n} e^{\beta\epsilon s_k^{(\mu)} s_{k+1}^{(\mu)}}. \tag{13.189}$$

Let us remark that

$$Z_\alpha Z_{\alpha+1} = \mathbf{1}\times\cdots\times Z\times Z\times\cdots\times\mathbf{1}. \tag{13.190}$$

The Z appears in the factors α and $\alpha+1$. Since $(Z_\alpha Z_{\alpha+1})^2 = 1$ we must have

$$e^{\beta\epsilon Z_\alpha Z_{\alpha+1}} = \cosh\beta\epsilon + \sinh\beta\epsilon\ Z_\alpha Z_{\alpha+1}. \tag{13.191}$$

In other words,

$$\begin{aligned}\langle\mu|e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}|\nu\rangle &= \cosh\beta\epsilon\ \delta_{\mu\nu} \\ &\quad + \sinh\beta\epsilon\left\langle s_1^{(\mu)}\middle|s_1^{(\nu)}\right\rangle \cdots \left\langle s_\alpha^{(\mu)}|Z|s_\alpha^{(\nu)}\right\rangle\left\langle s_{\alpha+1}^{(\mu)}|Z|s_{\alpha+1}^{(\nu)}\right\rangle \cdots\left\langle s_n^{(\mu)}\middle|s_n^{(\nu)}\right\rangle \\ &= \delta_{\mu\nu}\left(\cosh\beta\epsilon + \sinh\beta\epsilon\left\langle s_\alpha^{(\mu)}s_{\alpha+1}^{(\mu)}\right\rangle\right) \\ &= \delta_{\mu\nu}\ e^{\beta\epsilon s_\alpha^{(\mu)}s_{\alpha+1}^{(\mu)}}.\end{aligned} \tag{13.192}$$

Thus we can immediately conclude that

$$V_2 = \prod_{\alpha=1}^{n}\ e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}. \tag{13.193}$$

In the above equation we have used the boundary condition $Z_{n+1} = Z_n$.

Finally, from the matrix elements of V_3 defined by

$$\langle\mu|V_3|\nu\rangle = \delta_{\mu\nu}\prod_{k=1}^{n}\ e^{\beta H s_k^{(\mu)}}, \tag{13.194}$$

we can show that

$$V_3 = \prod_{\alpha=1}^{n}\ e^{\beta H Z_\alpha}. \tag{13.195}$$

We obtain thus the result

$$P = (2\sinh 2\beta\epsilon)^{\frac{n}{2}}V_3V_2V_1. \tag{13.196}$$

For $H = 0$ we have $V_3 = 1$ and hence

$$P = (2\sinh 2\beta\epsilon)^{\frac{n}{2}}V_2V_1. \tag{13.197}$$

13.3.3 The Clifford algebra in 2*n* dimensions

A Clifford algebra in $2n$ dimensions is given in terms of $2n$ (gamma) matrices γ_μ, $\mu = 1, 2, \ldots, 2n$ which satisfy the conditions

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{13.198}$$

The irreducible representation is 2^n-dimensional. We consider the explicit representation

$$\begin{aligned}\gamma_{2\alpha-1} &= X_1X_2 \ldots X_{\alpha-1}Z_\alpha, \\ \gamma_{2\alpha} &= X_1X_2 \ldots X_{\alpha-1}Y_\alpha,\ \ \alpha = 1, 2, \ldots, n.\end{aligned} \tag{13.199}$$

Let ω be a rotation in the $2n$-dimensional space. It must satisfy $\omega^T\omega = 1$. The matrices γ_μ transform under ω to γ'_μ which are given by

$$\gamma'_\mu = \sum_{\nu=1}^{2n}\omega_{\mu\nu}\gamma_\nu. \tag{13.200}$$

It is not difficult to show that γ'_μ satisfy also the Clifford algebra (13.198), i.e. $\{\gamma'_\mu, \gamma'_\nu\} = 2\delta_{\mu\nu}$. Thus, there must exist a non-singular matrix $S(\omega)$ such that

$$\gamma'_\mu = \sum_{\nu=1}^{2n} \omega_{\mu\nu}\gamma_\nu = S(\omega)\gamma_\mu S(\omega)^{-1}. \tag{13.201}$$

The $2^n \times 2^n$ matrix $S(\omega)$ is the spin representation of the rotation ω.

Let $\omega(\mu\nu|\theta)$ be the rotation in the plane $\mu\nu$ ($\mu \neq \nu$) through an angle θ, i.e.

$$\begin{aligned} \gamma'_\lambda &= \gamma_\lambda, \ \ \lambda \neq \mu, \ \ \lambda \neq \nu \\ \gamma'_\mu &= \gamma_\mu \cos\theta - \gamma_\nu \sin\theta \\ \gamma'_\mu &= \gamma_\mu \sin\theta + \gamma_\nu \cos\theta. \end{aligned} \tag{13.202}$$

The corresponding spin representation $S_{\mu\nu}(\theta)$ is given by

$$S_{\mu\nu}(\theta) = \exp\left(-\frac{\theta}{2}\gamma_\mu\gamma_\nu\right). \tag{13.203}$$

Indeed, we check that

$$e^{-\frac{\theta}{2}\gamma_\mu\gamma_\nu}\gamma_\lambda e^{\frac{\theta}{2}\gamma_\mu\gamma_\nu} = \gamma_\lambda, \ \ \lambda \neq \mu, \ \ \lambda \neq \nu. \tag{13.204}$$

Also, since $(\gamma_\mu\gamma_\nu)^2 = -1$ we have

$$e^{-\frac{\theta}{2}\gamma_\mu\gamma_\nu} = \cos\frac{\theta}{2} - \gamma_\mu\gamma_\nu \sin\frac{\theta}{2}. \tag{13.205}$$

Thus

$$\begin{aligned} e^{-\frac{\theta}{2}\gamma_\mu\gamma_\nu}\gamma_\mu e^{\frac{\theta}{2}\gamma_\mu\gamma_\nu} &= \left(\cos\frac{\theta}{2} - \gamma_\mu\gamma_\nu \sin\frac{\theta}{2}\right)\gamma_\mu e^{\frac{\theta}{2}\gamma_\mu\gamma_\nu} \\ &= \gamma_\mu e^{\theta\gamma_\mu\gamma_\nu} \\ &= \gamma_\mu(\cos\theta - \gamma_\mu\gamma_\nu \sin\theta) \\ &= \gamma_\mu \cos\theta - \gamma_\nu \sin\theta. \end{aligned} \tag{13.206}$$

Next, we compute the eigenvalues of $S_{\mu\nu}(\theta)$. We consider the representation

$$\begin{aligned} \gamma_{2\alpha-1} &= Z_1 Z_2 \ldots Z_{\alpha-1} X_\alpha, \\ \gamma_{2\alpha} &= Z_1 Z_2 \ldots Z_{\alpha-1} Y_\alpha, \ \ \alpha = 1, 2, \ldots, n. \end{aligned} \tag{13.207}$$

We take γ_μ and γ_ν to be the matrices

$$\gamma_\mu = Z_1 X_2, \ \ \gamma_\nu = Z_1 Y_2. \tag{13.208}$$

Thus

$$\gamma_\mu\gamma_\nu = X_2 Y_2 = iZ_2 = i\mathbf{1} \times Z \times \cdots \times \mathbf{1}. \tag{13.209}$$

Hence

$$S_{\mu\nu}(\theta) = \cos\frac{\theta}{2} - \gamma_\mu\gamma_\nu \sin\frac{\theta}{2} = \mathbf{1} \times \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} \times \cdots \times \mathbf{1}. \tag{13.210}$$

Thus $S_{\mu\nu}(\theta)$ has eigenvalue $e^{-i\frac{\theta}{2}}$ with multiplicity 2^{n-1} and eigenvalue $e^{i\frac{\theta}{2}}$ with multiplicity 2^{n-1}.

We consider the product of n commuting plane rotations, viz

$$\omega = \omega(\mu_1\nu_1|\theta_1) \ldots \omega(\mu_n\nu_n|\theta_n). \tag{13.211}$$

The $2n$ eigenvalues of ω are

$$e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_n}. \tag{13.212}$$

The corresponding spin representation is given by

$$S(\omega) = \exp\left(-\frac{\theta_1}{2}\gamma_{\mu_1}\gamma_{\nu_1}\right) \ldots \exp\left(-\frac{\theta_n}{2}\gamma_{\mu_n}\gamma_{\nu_n}\right). \tag{13.213}$$

The set $\{\mu_1, \nu_1, \ldots, \mu_n, \nu_n\}$ is a permutation of the set $\{1, 2, \ldots, 2n-1, 2n\}$. The 2^n eigenvalues of $S(\omega)$ are

$$e^{\frac{i}{2}(\pm\theta_1 \pm \cdots \pm\theta_n)}. \tag{13.214}$$

Now we go back to our original problem. We need the matrices

$$V_1 = \prod_{\alpha=1}^{n} e^{\theta X_\alpha}, \ \tanh\theta = e^{-2\beta\epsilon} \tag{13.215}$$

$$V_2 = \prod_{\alpha=1}^{n} e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}. \tag{13.216}$$

We use the representation

$$\begin{aligned} \gamma_{2\alpha-1} &= X_1 X_2 \ldots X_{\alpha-1} Z_\alpha, \\ \gamma_{2\alpha} &= X_1 X_2 \ldots X_{\alpha-1} Y_\alpha, \ \alpha = 1, 2, \ldots, n. \end{aligned} \tag{13.217}$$

First, we note that $X_\alpha = -i\gamma_{2\alpha}\gamma_{2\alpha-1}$, i.e.

$$V_1 = \prod_{\alpha=1}^{n} e^{-i\theta\gamma_{2\alpha}\gamma_{2\alpha-1}}. \tag{13.218}$$

In other words, V_1 is a spin representation of a product of commuting plane rotations. Next we note that for $\alpha = 1, \ldots, n-1$ we have

$$\begin{aligned}\gamma_{2\alpha+1}\gamma_{2\alpha} &= (X_1X_2 \ldots X_\alpha Z_{\alpha+1})(X_1X_2 \ldots X_{\alpha-1}Y_\alpha)\\ &= X_\alpha Z_{\alpha+1}Y_\alpha\\ &= iZ_\alpha Z_{\alpha+1}.\end{aligned} \tag{13.219}$$

For $\alpha = n$ we have

$$\begin{aligned}\gamma_{2n+1}\gamma_{2n} = \gamma_1\gamma_{2n} &= (Z_1)(X_1X_2 \ldots X_{n-1}Y_n)\\ &= -iZ_1Z_n(X_1X_2 \ldots X_n).\end{aligned} \tag{13.220}$$

In the above we have used $\gamma_{2n+1} = \gamma_1$ and $Y_n = iX_nZ_n$. Next by using $Z_{n+1} = Z_1$ we obtain

$$\begin{aligned}V_2 &= \prod_{\alpha=1}^{n-1} e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}\, e^{\beta\epsilon Z_nZ_1}\\ &= e^{\beta\epsilon Z_nZ_1}\prod_{\alpha=1}^{n-1} e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}\\ &= e^{i\beta\epsilon U\gamma_1\gamma_{2n}}\prod_{\alpha=1}^{n-1} e^{-i\beta\epsilon\gamma_{2\alpha+1}\gamma_{2\alpha}}.\end{aligned} \tag{13.221}$$

We have also defined the chirality operator $U = X_1X_2 \ldots X_n$. For $U = 0$ the matrix V_2 is also a spin representation of a product of commuting plane rotations. The fact that $U \neq 0$ is due to the toroidal boundary condition. We can check that $U = i^n\gamma_1\gamma_2 \ldots \gamma_{2n}$, $U^2 = 1$, $[U, \gamma_1\gamma_{2n}] = 0$ and $(\gamma_1\gamma_{2n})^2 = -1$. Furthermore, we compute

$$e^{i\beta\epsilon U\gamma_1\gamma_{2n}} = \cosh\beta\epsilon + i\sinh\beta\epsilon\ U\gamma_1\gamma_{2n} = \frac{1+U}{2}e^{i\beta\epsilon\gamma_1\gamma_{2n}} + \frac{1-U}{2}e^{-i\beta\epsilon\gamma_1\gamma_{2n}}. \tag{13.222}$$

Thus

$$V_2 = \frac{1+U}{2}e^{i\beta\epsilon\gamma_1\gamma_{2n}}\prod_{\alpha=1}^{n-1} e^{-i\beta\epsilon\gamma_{2\alpha+1}\gamma_{2\alpha}} + \frac{1-U}{2}e^{-i\beta\epsilon\gamma_1\gamma_{2n}}\prod_{\alpha=1}^{n-1} e^{-i\beta\epsilon\gamma_{2\alpha+1}\gamma_{2\alpha}}. \tag{13.223}$$

In summary we obtain (with $\phi = \beta\epsilon$, $\theta = \tanh^{-1}e^{-2\phi}$)

$$Z = \mathrm{Tr}\,P^n \tag{13.224}$$

$$P = (2\sinh 2\phi)^{\frac{n}{2}}V \tag{13.225}$$

$$V = V_2V_1 = \frac{1+U}{2}V_+ + \frac{1-U}{2}V_- \tag{13.226}$$

$$V_\pm = e^{\pm i\phi\gamma_1\gamma_{2n}}\prod_{\alpha=1}^{n-1} e^{-i\phi\gamma_{2\alpha+1}\gamma_{2\alpha}}\prod_{\beta=1}^{n} e^{-i\theta\gamma_{2\beta}\gamma_{2\beta-1}}. \tag{13.227}$$

Let λ_α, $\alpha = 1, \ldots, 2^n$ be the eigenvalues of V. Then

$$Z = (2\sinh 2\phi)^{\frac{n^2}{2}} \sum_{\alpha=1}^{2^n} \lambda_\alpha^{\,n}. \tag{13.228}$$

Equivalently,

$$\frac{1}{n^2}\log Z = \frac{1}{2}\log(2\sinh 2\phi) + \frac{1}{n^2}\log\left(\sum_{\alpha=1}^{2^n}\lambda_\alpha^{\,n}\right). \tag{13.229}$$

If all the eigenvalues λ_α are positive and if $\lambda_{\max}$ is the largest eigenvalue of V then we must have

$$\lambda_{\max}^{\,n} \leqslant \sum_{\alpha=1}^{2^n}\lambda_\alpha^{\,n} \leqslant 2^n\lambda_{\max}^{\,n}. \tag{13.230}$$

As a consequence we must have

$$\begin{aligned}\frac{1}{2}\log(2\sinh 2\phi) + \frac{1}{n}\log\lambda_{\max} \leqslant \frac{1}{n^2}\log Z \leqslant \frac{1}{2}\log(2\sinh 2\phi)\\ + \frac{1}{n}\log\lambda_{\max} + \frac{1}{n}\log 2.\end{aligned} \tag{13.231}$$

In other words,

$$\frac{1}{n^2}\log Z = \frac{1}{2}\log(2\sinh 2\phi) + \frac{1}{n}\log\lambda_{\max},\ \ n \longrightarrow \infty. \tag{13.232}$$

It is expected that $\lambda_{\max}$ is of order n in the limit.

13.3.4 Diagonalization

The matrices U, V_+ and V_- are commuting matrices and thus they can be diagonalized simultaneously. Since $U^2 = 1$, the eigenvalues of U are either $+1$ or -1. It is not difficult to verify that they occur with the same multiplicity. We work in the basis in which U takes the form

$$U = \begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}. \tag{13.233}$$

The identity $\mathbf{1}$ is 2^{n-1}-dimensional. In this basis we have also

$$V_\pm = \begin{pmatrix}A_\pm & 0\\ 0 & B_\pm\end{pmatrix}. \tag{13.234}$$

The $A_\pm$ and $B_\pm$ are $2^{n-1}\times 2^{n-1}$ matrices which are not necessarily diagonal. We then compute in this basis

$$V = \begin{pmatrix}A_+ & 0\\ 0 & B_-\end{pmatrix}. \tag{13.235}$$

Clearly we need to diagonalize V_+ and V_- separately and independently. Instead of $V_\pm$ given by equation (13.227) we will consider

$$V_\pm = \Gamma\left(e^{\pm i\phi\gamma_1\gamma_{2n}}\prod_{\alpha=1}^{n-1} e^{-i\phi\gamma_{2\alpha+1}\gamma_{2\alpha}}\prod_{\beta=1}^{n} e^{-i\theta\gamma_{2\beta}\gamma_{2\beta-1}}\right)\Gamma^{-1} \tag{13.236}$$

$$\Gamma = \prod_{\beta=1}^{n} e^{-i\frac{\theta}{2}\gamma_{2\beta}\gamma_{2\beta-1}}. \tag{13.237}$$

Under the trace the extra factors Γ and Γ^{-1} will cancel. Next we write the rotations corresponding to the spin representations (13.236). These read

$$\omega_\pm = \Delta\chi_\pm\Delta \tag{13.238}$$

$$\begin{aligned}\chi_\pm &= \omega(1\ 2n|\mp 2i\phi)\prod_{\alpha=1}^{n-1}\omega(2\alpha+1,\ 2\alpha|2i\phi)\\ &= \omega(1\ 2n|\mp 2i\phi)\omega(23|-2i\phi)\\ &\quad\times\omega(45|-2i\phi)\ldots\omega(2n-2\ 2n-1|-2i\phi)\end{aligned} \tag{13.239}$$

$$\begin{aligned}\Delta &= \prod_{\beta=1}^{n}\omega(2\beta,\ 2\beta-1|i\theta)\\ &= \omega(12|-i\theta)\omega(34|-i\theta)\ldots\omega(2n-1\ 2n|-i\theta).\end{aligned} \tag{13.240}$$

We do the example $n = 3$ explicitly. We have

$$\begin{aligned}\Delta &= \omega(12|-i\theta)\omega(34|-i\theta)\omega(56|-i\theta)\\ &= \begin{pmatrix} J & 0 & 0\\ 0 & J & 0\\ 0 & 0 & J\end{pmatrix}.\end{aligned} \tag{13.241}$$

In the above, J is the $n = 1$ rotation through an angle $-i\theta$, viz

$$J = \begin{pmatrix}\cos(-i\theta) & \sin(-i\theta)\\ -\sin(-i\theta) & \cos(-i\theta)\end{pmatrix} = \begin{pmatrix}\cosh(\theta) & -i\sinh(\theta)\\ i\sinh(\theta) & \cosh(\theta)\end{pmatrix}. \tag{13.242}$$

Similarly,

$$\begin{aligned}\chi_\pm &= \omega(16|\mp 2i\phi)\omega(23|-2i\phi)\omega(45|-2i\phi)\\ &= \begin{pmatrix} a & 0 & 0 & \mp b\\ 0 & K & 0 & 0\\ 0 & 0 & K & 0\\ \pm b & 0 & 0 & a\end{pmatrix}.\end{aligned} \tag{13.243}$$

In the above, K is the $n = 1$ rotation through an angle $-2i\phi$, viz

$$K = \begin{pmatrix}\cos(-2i\phi) & \sin(-2i\phi)\\ -\sin(-2i\phi) & \cos(-2i\phi)\end{pmatrix} = \begin{pmatrix}\cosh(2\phi) & -i\sinh(2\phi)\\ i\sinh(2\phi) & \cosh(2\phi)\end{pmatrix} \equiv \begin{pmatrix} a & -b\\ b & a\end{pmatrix}. \tag{13.244}$$

The matrices $\chi_\pm$ can also be put into the form

$$\chi_\pm = \begin{pmatrix} a\mathbf{1}_2 & V & T_\pm \\ V^+ & a\mathbf{1}_2 & V \\ T_\pm^+ & V^+ & a\mathbf{1}_2 \end{pmatrix} \tag{13.245}$$

$$T_\pm = \begin{pmatrix} 0 & \mp b \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix}. \tag{13.246}$$

We can calculate

$$\omega_\pm = \begin{pmatrix} aJ^2 & JVJ & JT_\pm J \\ (JVJ)^+ & aJ^2 & JVJ \\ (JT_\pm J)^+ & (JVJ)^+ & aJ^2 \end{pmatrix} = \begin{pmatrix} A & B & \mp B^+ \\ B^+ & A & B \\ \mp B & B^+ & A \end{pmatrix} \tag{13.247}$$

$$\begin{aligned} A = aJ^2 &= \cosh 2\phi \begin{pmatrix} \cosh 2\theta & -i \sinh 2\theta \\ i \sinh 2\theta & \cosh 2\theta \end{pmatrix} \\ B = JVJ = \mp(JTJ)^+ &= \sinh 2\phi \begin{pmatrix} -\frac{1}{2}\sinh 2\theta & i \sinh^2\theta \\ -i\cosh^2\theta & -\frac{1}{2}\sinh 2\theta \end{pmatrix}. \end{aligned} \tag{13.248}$$

Generalization of this result to $n > 3$ is straightforward. We get the rotation

$$\omega_\pm = \begin{pmatrix} A & B & 0 & \cdots & \mp B^+ \\ B^+ & A & B & \cdots & 0 \\ 0 & B^+ & A & B & \cdots \\ \vdots & & & & \\ \mp B & 0 & \cdots & B^+ & A \end{pmatrix}. \tag{13.249}$$

Now we find the eigenvalues of $\omega_\pm$. We take the eigenvectors to be of the form

$$\psi = \begin{pmatrix} zu \\ z^2u \\ \vdots \\ z^nu \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{13.250}$$

We get the equations

$$\begin{aligned} &(Az + Bz^2 \mp B^+z^n)u = \lambda zu \\ &(B^+z + Az^2 + Bz^3)u = \lambda z^2u \\ &(B^+z^2 + Az^3 + Bz^4)u = \lambda z^3u \\ &\vdots \\ &(B^+z^{n-2} + Az^{n-1} + Bz^n)u = \lambda z^{n-1}u \\ &(\mp Bz + B^+z^{n-1} + Az^n)u = \lambda z^nu. \end{aligned} \tag{13.251}$$

It is not difficult to observe that only three equations are independent. These are given by the first and the last equations and any one of the others, viz

$$\begin{aligned}(A + Bz \mp B^+ z^{n-1})u &= \lambda u \\ (B^+ z^{-1} + A + Bz)u &= \lambda u \\ (\mp B z^{1-n} + B^+ z^{-1} + A)u &= \lambda u.\end{aligned} \tag{13.252}$$

A solution is given by

$$z^n = \mp 1. \tag{13.253}$$

Equivalently,

$$z = e^{\frac{i\pi k}{n}},\ k = 0, 1, \ldots, 2n-1, \tag{13.254}$$

where

$$\begin{aligned}k &= 1, 3, 5, \ldots, 2n-1, \text{ for } \omega_+ \\ k &= 0, 2, 4, \ldots, 2n-2, \text{ for } \omega_-.\end{aligned} \tag{13.255}$$

The eigenvalues λ_k are solutions of the remaining equation

$$\left(A + Bz_k + B^+ z_k^{-1}\right)u = \lambda_k u. \tag{13.256}$$

Thus, we get for each value of k two eigenvalues λ_k. We can check that

$$\det\left(A + Bz_k + B^+ z_k^{-1}\right) = 1. \tag{13.257}$$

We can conclude that

$$\lambda_k = e^{\pm r_k}. \tag{13.258}$$

Next we compute

$$\begin{aligned}\frac{1}{2}\operatorname{Tr}\left(A + Bz_k + B^+ z_k^{-1}\right) &= \cosh 2\phi \cosh 2\theta \\ &\quad - \frac{1}{2}\sinh 2\phi \sinh 2\theta \left(z_k + \frac{1}{z_k}\right) \\ &= \cosh 2\phi \cosh 2\theta \\ &\quad - \sinh 2\phi \sinh 2\theta \cos\frac{\pi k}{n}.\end{aligned} \tag{13.259}$$

On the other hand we have

$$\frac{1}{2}\operatorname{Tr}\left(A + Bz_k + B^+ z_k^{-1}\right) = \frac{1}{2}(e^{r_k} + e^{-r_k}) = \cosh r_k. \tag{13.260}$$

Thus r_k must solve the equation

$$\cosh r_k = \cosh 2\phi \cosh 2\theta - \sinh 2\phi \sinh 2\theta \cos \frac{\pi k}{n}. \tag{13.261}$$

Since $\lambda_k = e^{\pm r_k}$ we can restrict r_k to be the positive solution of the above equation, viz

$$r_k \geqslant 0. \tag{13.262}$$

It is also obvious that

$$r_{2n-k} = r_k. \tag{13.263}$$

We compute

$$\frac{\partial r_k}{\partial k} \sinh r_k = \frac{\pi}{n} \sinh 2\phi \sinh 2\theta \sin \frac{\pi k}{n}. \tag{13.264}$$

Since $r_k \geqslant 0, \theta \geqslant 0$ and $\phi \geqslant 0$ we have $\sinh r_k \geqslant 0, \sinh 2\theta \geqslant 0$ and $\sinh 2\phi \geqslant 0$. Thus $\frac{\partial r_k}{\partial k} \geqslant 0$ for all $k \leqslant n$. In other words,

$$r_0 < r_1 < \cdots < r_n. \tag{13.265}$$

In summary, we have found that the $2n$ eigenvalues of the rotation ω_+ are $e^{\pm r_k}$, $k = 1, 3, 5, \ldots, 2n-1$, whereas the $2n$ eigenvalues of the rotation ω_- are $e^{\pm r_k}$, $k = 0, 2, 4, \ldots, 2n-2$. The eigenvalues of the spin representations V_+ and V_- are, therefore, given by

$$\begin{aligned} V_+ &: e^{\frac{1}{2}(\pm r_1 \pm r_3 \pm \cdots \pm r_{2n-1})} \\ V_- &: e^{\frac{1}{2}(\pm r_0 \pm r_2 \pm \cdots \pm r_{2n-2})}. \end{aligned} \tag{13.266}$$

The next step is to find the set of eigenvalues of

$$V = \frac{1+U}{2} V_+ + \frac{1-U}{2} V_-$$

where U is the chirality operator. Since the chirality operator is an idempotent the two operators $\frac{1+U}{2}$ and $\frac{1-U}{2}$ are projection operators. Hence, half of the eigenvalues of V (i.e. 2^{n-1} of them) will come from the eigenvalues of V_+ while the other half (the other 2^{n-1}) will come from the eigenvalues of V_-. The eigenvalues of V are clearly positive since the eigenvalues of $V_\pm$ are all positive. In other words, the arguments which lead to equation (13.232) are correct and we only need the largest eigenvalue $\lambda_{\max}$ of V. The largest eigenvalues of V_+ and V_- are

$$\begin{aligned} V_+ &: \lambda_{\max +} = e^{\frac{1}{2}(+r_1 + r_3 + \cdots + r_{2n-1})} \\ V_- &: \lambda_{\max -} = e^{\frac{1}{2}(+r_0 + r_2 + \cdots + r_{2n-2})}. \end{aligned} \tag{13.267}$$

Since $r_k \geqslant 0$, $r_0 < r_1 < \cdots < r_n$ and $r_{2n-k} = r_k$ the largest eigenvalue of V is $\lambda_{\max +}$, viz

$$\lambda_{\max} \equiv \lambda_{\max +} = e^{\frac{1}{2}(+r_1 + r_3 + \cdots + r_{2n-1})}. \tag{13.268}$$

Hence

$$\frac{1}{n}\log\lambda_{\max} = \frac{1}{2n}(+r_1 + r_3 + \cdots + r_{2n-1}) = \frac{1}{2n}\sum_{k=1}^{n} r_{2k-1}. \tag{13.269}$$

13.3.5 The largest eigenvalue $\lambda_{\max}$

We are interested in the continuum limit $n \longrightarrow \infty$. We can then use the following continuous variables

$$\begin{aligned} \nu &= \frac{\pi}{n}(2k-1) \\ r(\nu) &= r_{2k-1}. \end{aligned} \tag{13.270}$$

We make the replacement

$$\frac{1}{n}\sum_{k=1}^{n} r_{2k-1} = \frac{1}{2\pi}\int_0^{2\pi} d\nu\; r(\nu). \tag{13.271}$$

We get

$$\mathcal{L} = \lim_{n\longrightarrow\infty}\frac{1}{n}\log\lambda_{\max} = \frac{1}{4\pi}\int_0^{2\pi} d\nu\; r(\nu). \tag{13.272}$$

The condition $r_k = r_{2n-k}$ becomes $r(\nu) = r(2\pi - \nu)$. Thus

$$\mathcal{L} = \lim_{n\longrightarrow\infty}\frac{1}{n}\log\lambda_{\max} = \frac{1}{2\pi}\int_0^{\pi} d\nu\; r(\nu). \tag{13.273}$$

The $r(\nu)$ must solve the equation

$$\cosh r(\nu) = \cosh 2\phi\cosh 2\theta - \sinh 2\phi\sinh 2\theta\cos\nu. \tag{13.274}$$

Recall that $\phi = \beta\epsilon$ and $\tanh\theta = e^{-2\phi}$ from which we compute $\sinh 2\theta = 1/\sinh 2\phi$ and $\cosh 2\theta = \coth 2\phi$. Thus

$$\cosh r(\nu) = \cosh 2\phi\coth 2\phi - \cos\nu. \tag{13.275}$$

Before we proceed any further we introduce the integral

$$I(z) = \frac{1}{\pi}\int_0^{\pi} dt\,\log(2\cosh z - 2\cos t). \tag{13.276}$$

We compute

$$\frac{dI(z)}{dz} = \frac{1}{\pi}\int_0^{\pi} dt\frac{\sinh z}{\cosh z - \cos t}. \tag{13.277}$$

We use the result

$$\frac{1}{\pi}\int_0^\pi dt \frac{1}{1+a\cos t} = \frac{1}{\sqrt{1-a^2}}. \tag{13.278}$$

Thus

$$\frac{dI(z)}{dz} = \frac{\sinh z}{|\sinh z|} = \frac{z}{|z|}. \tag{13.279}$$

In other words,

$$I(z) = |z| \tag{13.280}$$

An immediate application of this result is the following

$$\begin{aligned} r(\nu) &= \frac{1}{\pi}\int_0^\pi d\nu' \log(2\cosh r(\nu) - 2\cos\nu') \\ &= \frac{1}{\pi}\int_0^\pi d\nu' \log(2\cosh 2\phi \coth 2\phi - 2\cos\nu - 2\cos\nu') \end{aligned} \tag{13.281}$$

$$\mathcal{L} = \frac{1}{2\pi^2}\int_0^\pi d\nu \int_0^\pi d\nu' \log(2\cosh 2\phi \coth 2\phi - 2\cos\nu - 2\cos\nu'). \tag{13.282}$$

We introduce the variables $\delta_1 = \frac{\nu+\nu'}{2}$ and $\delta_2 = \nu - \nu'$. We have $4\cos\delta_1\cos\frac{\delta_2}{2} = 2\cos\nu + 2\cos\nu'$ and $d\nu d\nu' = d\delta_1 d\delta_2$. We obtain (with $D = \cosh 2\phi \coth 2\phi$)

$$\begin{aligned} \mathcal{L} &= \frac{1}{2\pi^2}\int_0^\pi d\delta_1 \int_0^\pi d\delta_2 \log\left(2D - 4\cos\delta_1\cos\frac{\delta_2}{2}\right) \\ &= \frac{1}{\pi^2}\int_0^\pi d\delta_1 \int_0^{\frac{\pi}{2}} d\delta_2 \log(2D - 4\cos\delta_1\cos\delta_2) \\ &= \frac{1}{\pi^2}\int_0^\pi d\delta_1 \int_0^{\frac{\pi}{2}} d\delta_2 \log(2\cos\delta_2) \\ &\quad + \frac{1}{\pi^2}\int_0^\pi d\delta_1 \int_0^{\frac{\pi}{2}} d\delta_2 \log\left(\frac{D}{\cos\delta_2} - 2\cos\delta_1\right) \\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\delta_2 \log(2\cos\delta_2) + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\delta_2 \cosh^{-1}\frac{D}{2\cos\delta_2}. \end{aligned} \tag{13.283}$$

Using $\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$ we obtain (with $\kappa = 2/D$)

$$\mathcal{L} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\delta_2 \log\left[D\left(1 + \sqrt{1-\kappa^2\cos^2\delta}\right)\right]. \tag{13.284}$$

We make the change of variable $\delta_2 = \frac{\pi}{2} - \rho$. We get

$$\begin{aligned}\mathcal{L} &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\rho \log\left[D\left(1+\sqrt{1-\kappa^2\sin^2\rho}\right)\right] \\ &= \frac{1}{2}\log\frac{2\cosh^2 2\beta\epsilon}{\sinh 2\beta\epsilon} + \frac{1}{2\pi}\int_0^{\pi} d\rho \log\left[\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2\rho}\right)\right].\end{aligned} \tag{13.285}$$

13.3.6 Thermodynamical quantities

We have already found that

$$\begin{aligned}\frac{1}{n^2}\log Z &= \frac{1}{2}\log(2\sinh 2\beta\epsilon) + \frac{1}{n}\log\lambda_{\max} \\ &= \frac{1}{2}\log(2\sinh 2\beta\epsilon) + \mathcal{L} \\ &= \frac{1}{2}\log 4\cosh^2 2\beta\epsilon \\ &\quad + \frac{1}{2\pi}\int_0^{\pi} d\rho \log\left[\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2\rho}\right)\right].\end{aligned} \tag{13.286}$$

Thus, the Helmholtz free energy is given by

$$\begin{aligned}-\frac{\beta F}{n^2} = \frac{1}{n^2}\log Z &= \log 2\cosh 2\beta\epsilon \\ &\quad + \frac{1}{2\pi}\int_0^{\pi} d\rho \log\left[\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2\rho}\right)\right].\end{aligned} \tag{13.287}$$

The internal energy is given by (with $\Delta = \sqrt{1-\kappa^2\sin^2\rho}$)

$$\begin{aligned}-\frac{U}{n^2} = -\frac{\partial}{\partial\beta}\left(\frac{\beta F}{n^2}\right) &= 2\epsilon\tanh 2\beta\epsilon - \frac{\kappa}{2\pi}\frac{\partial\kappa}{\partial\beta}\int_0^{\pi} d\rho\frac{\sin^2\rho}{\Delta(1+\Delta)} \\ &= 2\epsilon\tanh 2\beta\epsilon + \frac{1}{2\kappa}\frac{\partial\kappa}{\partial\beta} - \frac{1}{2\pi\kappa}\frac{\partial\kappa}{\partial\beta}\int_0^{\pi}\frac{d\rho}{\sqrt{1-\kappa^2\sin^2\rho}} \\ &= 2\epsilon\tanh 2\beta\epsilon + \frac{1}{2\kappa}\frac{\partial\kappa}{\partial\beta} - \frac{1}{\pi\kappa}\frac{\partial\kappa}{\partial\beta}\int_0^{\frac{\pi}{2}}\frac{d\rho}{\sqrt{1-\kappa^2\sin^2\rho}}.\end{aligned} \tag{13.288}$$

We introduce the complete elliptic integral of the first kind

$$K_1(\kappa) = \int_0^{\frac{\pi}{2}}\frac{d\rho}{\sqrt{1-\kappa^2\sin^2\rho}} \tag{13.289}$$

We note that

$$\kappa = \frac{2\sinh 2\beta\epsilon}{\cosh^2 2\beta\epsilon} \tag{13.290}$$

$$\frac{1}{\kappa}\frac{\partial \kappa}{\partial \beta} = -2\epsilon\kappa' \coth 2\beta\epsilon, \;\; \kappa' = 2\tanh^2 2\beta\epsilon - 1 \tag{13.291}$$

$$\kappa^2 + \kappa'^2 = 1. \tag{13.292}$$

Hence

$$-\frac{U}{n^2} = -\frac{\partial}{\partial\beta}\left(\frac{\beta F}{n^2}\right) = \epsilon \coth 2\beta\epsilon\left(1 + \frac{2\kappa'}{\pi}K_1(\kappa)\right). \tag{13.293}$$

The specific heat is given by

$$\frac{C_v}{n^2} = \frac{\partial}{\partial T}\left(\frac{U}{n^2}\right) = -k_B\beta^2\frac{\partial}{\partial\beta}\left(\frac{U}{n^2}\right). \tag{13.294}$$

A straightforward calculation yields

$$\begin{aligned}\frac{C_v}{n^2} &= k_B\frac{2}{\pi}(\beta\epsilon \coth 2\beta\epsilon)^2 \\ &\times\left(2K_1(\kappa) - 2E_1(\kappa) - (1-\kappa')\left(\frac{\pi}{2} + \kappa' K_1(\kappa)\right)\right).\end{aligned} \tag{13.295}$$

The $E_1(\kappa)$ is the complete elliptic integral of the second kind given by

$$E_1(\kappa) = \int_0^{\frac{\pi}{2}} d\rho\sqrt{1 - \kappa^2\sin^2\rho}\,. \tag{13.296}$$

The complete elliptic integral of the first kind is divergent at $\kappa = 1$. Indeed we compute

$$K_1(1) = \int_0^{\alpha}\frac{d\rho}{\sqrt{1-\sin^2\rho}} = \log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right). \tag{13.297}$$

This divergence is logarithmic. This can be cheked as follows. We compute

$$K_1(\kappa) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2x^2)}} = \frac{1}{k}\int_0^k \frac{dy}{\sqrt{(1-y^2)\left(1 - \frac{y^2}{\kappa^2}\right)}}. \tag{13.298}$$

Near $\kappa = 1$ we can approximate the integral with

$$K_1(\kappa) = \int_0^k \frac{dy}{1-y^2} = -\frac{1}{2}\log(1-k) + \frac{1}{2}\log 2. \tag{13.299}$$

Near $\kappa = 1$ the complete elliptic integral of the second kind is finite given by

$$E_1(1) = 1. \tag{13.300}$$

The value $\kappa = 1$ or equivalently $\kappa' = 0$ corresponds to a temperature T_c which is given by the solution of any of the following equivalent equations

$$\frac{2\sinh 2\beta_c\epsilon}{\cosh^2 2\beta_c\epsilon} = 1,\ 2\tanh^2 2\beta_c\epsilon = 1 \tag{13.301}$$

$$\sinh 2\beta_c\epsilon = 1,\ \cosh 2\beta_c\epsilon = \sqrt{2} \tag{13.302}$$

$$e^{2\beta_c\epsilon} = 1 + \sqrt{2}\,. \tag{13.303}$$

The behaviour near $\kappa = 1$ is given by

$$e^{2\beta\epsilon} = (1 + \sqrt{2})\left(1 + 2\epsilon\beta\frac{T_c - T}{T_c} + \cdots\right) \tag{13.304}$$

$$\sinh 2\beta\epsilon = 1 + \sqrt{2}\,(2\epsilon\beta)\frac{T_c - T}{T_c} + \ldots \tag{13.305}$$

Thus

$$\begin{aligned} K_1(\kappa) &= -\frac{1}{2}\log(1 - k) + \frac{1}{2}\log 2 \\ &= -\log\frac{|\sinh 2\beta\epsilon - 1|}{\cosh 2\beta\epsilon} + \frac{1}{2}\log 2 \\ &= -\log\left|1 - \frac{T}{T_c}\right| + \log\frac{k_B T}{2\epsilon} + \frac{1}{2}\log 2. \end{aligned} \tag{13.306}$$

The specific heat near $\kappa = 1$ behaves as

$$\begin{aligned} \frac{C_v}{n^2} &= k_B\frac{2}{\pi}\left(\frac{2\epsilon}{k_B T_c}\right)^2 \\ &\times\left(-\log\left|1 - \frac{T}{T_c}\right| + \log\frac{k_B T_c}{2\epsilon} + \frac{1}{2}\log 2 - 1 - \frac{\pi}{4}\right). \end{aligned} \tag{13.307}$$

This diverges logarithmically at the critical point $T = T_c$. It is not difficult to observe that the internal energy is continuous at $T = T_c$. Thus, there is no latent heat and the transition is second-order.

13.3.7 Spontaneous magnetization

This is defined by

$$\langle M\rangle = \left\langle \sum_i s_i \right\rangle = \frac{1}{\beta}\frac{\partial \log Z}{\partial H} = -\frac{\partial F}{\partial H} \tag{13.308}$$

$$Z = \text{Tr } P^n \tag{13.309}$$

$$P = (2 \sinh 2\beta\epsilon)^{\frac{n}{2}} V_3 V_2 V_1 \tag{13.310}$$

$$V_1 = \prod_{\alpha=1}^{n} e^{\theta X_\alpha}, \quad V_2 = \prod_{\alpha=1}^{n} e^{\beta\epsilon Z_\alpha Z_{\alpha+1}}, \quad V_3 = \prod_{\alpha=1}^{n} e^{\beta H Z_\alpha}. \tag{13.311}$$

A very long calculation yields at $H = 0$ the spontaneous magnetization given by [6]

$$\frac{\langle M \rangle}{n^2} = 0, \quad T > T_c \tag{13.312}$$

$$\frac{\langle M \rangle}{n^2} = \left(1 - \frac{1}{\sinh^4 2\beta\epsilon}\right)^{\frac{1}{8}}, \quad T < T_c. \tag{13.313}$$

13.4 QED$_2$ and the Thirring and sine-Gordon models

The original solution of quantum electrodynamics in two dimensions is found in [7, 8] but here we will follow the presentation of [9, 10].

13.4.1 QED in two dimensions

Quantum electrodynamics in two dimensions, also knows as the Schwinger model, is given by the path integral

$$\begin{aligned} Z[J] &= \int [dA][d\bar{\psi}][d\psi] e^{i\int d^2x \mathcal{L} + i\int d^2x J_\mu(x) A_\mu(x)} \\ \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i\gamma^\mu D_\mu \psi \\ D_\mu &= \partial_\mu + ieA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{13.314}$$

The metric is $g = (1, -1)$ and the coordinates are x^0 and x^1 ($x = (x^0, x^1)$). The Gamma matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Explicitly we have

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{13.315}$$

The chirality operator is $\gamma^5 = \gamma^0\gamma^1 = \text{diag}(1, -1)$. We will also need the epsilon symbol defined by $\epsilon^{01} = -\epsilon^{10} = 1$.

The classical equations of motion are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\nu \frac{\partial \mathcal{L}}{\partial_\nu \bar{\psi}} = 0 &\Leftrightarrow i\gamma^\mu D_\mu \psi = 0 \\ \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial_\nu A_\mu} = 0 &\Leftrightarrow \partial_\nu F^{\nu\mu} = j^\mu - J^\mu, \quad j^\mu = e\bar{\psi}\gamma^\mu\psi. \end{aligned} \tag{13.316}$$

The quantum equations of motion (the Dyson–Schwinger equations) corresponding to the second equation of (13.316) reads

$$\left[(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)\frac{1}{i}\frac{\delta}{\delta J_\nu} + \left[J_\mu - j_\mu\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]\right] Z[J] = 0. \tag{13.317}$$

In the above we have made the substitution $A_\mu \longrightarrow \frac{1}{i}\frac{\delta}{\delta J^\mu}$ and also taken into account the fact that the current j_μ becomes, in the quantum theory, a function of the gauge field A_μ. In the Lorentz gauge $\partial_\mu A^\mu = 0$ the quantum equations of motion become

$$\left[(\partial^2)\frac{1}{i}\frac{\delta}{\delta J^\mu} + \left[J_\mu - j_\mu\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]\right] Z[J] = 0. \tag{13.318}$$

Let $|\Omega\rangle$ be the vacuum of the theory. The fermion 2-point function is defined by

$$\begin{aligned} G(x, y)_{\mu\alpha} &= \langle\Omega|T\psi_\mu(x)\bar{\psi}_\alpha(y)|\Omega\rangle \\ &= \frac{\int [dA][d\bar{\psi}][d\psi]\psi_\mu(x)\bar{\psi}_\alpha(y)e^{i\int d^2x\mathcal{L}+i\int d^2xJ_\mu(x)A_\mu(x)}}{\int [dA][d\bar{\psi}][d\psi]e^{i\int d^2x\mathcal{L}+i\int d^2xJ_\mu(x)A_\mu(x)}}. \end{aligned} \tag{13.319}$$

Physically this is the amplitude of probability for propagation between x and y. T is the usual time-ordering operator defined by $T\psi_\mu(x)\bar{\psi}_\alpha(y) = \psi_\mu(x)\bar{\psi}_\alpha(y)$ if $x^0 > y^0$ and $T\psi_\mu(x)\bar{\psi}_\alpha(y) = -\bar{\psi}_\alpha(y)\psi_\mu(x)$ if $y^0 > x^0$.

The fermion 2-point function in a background gauge field is given by

$$\begin{aligned} G(x, y)^A_{\mu\alpha} &= \langle\Omega|T\psi_\mu(x)\bar{\psi}_\alpha(y)|\Omega\rangle^A \\ &= \frac{\int [d\bar{\psi}][d\psi]\psi_\mu(x)\bar{\psi}_\alpha(y)e^{i\int d^2x\bar{\psi}i\gamma^\mu D_\mu\psi}}{\int [d\bar{\psi}][d\psi]e^{i\int d^2x\bar{\psi}i\gamma^\mu D_\mu\psi}}. \end{aligned} \tag{13.320}$$

This satisfies the equations of motion

$$i\gamma^\rho_{\lambda\mu}D_\rho(x)G(x, y)^A_{\mu\alpha} = i\delta^2(x - y)\delta_{\lambda\alpha}. \tag{13.321}$$

These are the quantum equations corresponding to the first equation of (13.316). They can be derived from the invariance under $\bar{\psi} \longrightarrow \bar{\psi}' = \bar{\psi} + \bar{\epsilon}$ (keeping ψ fixed) of the 1-point function

$$\langle\Omega|\bar{\psi}_\alpha(y)|\Omega\rangle^A = \frac{\int [d\bar{\psi}][d\psi]\bar{\psi}_\alpha(y)e^{i\int d^2x\bar{\psi}i\gamma^\mu D_\mu\psi}}{\int [d\bar{\psi}][d\psi]e^{i\int d^2x\bar{\psi}i\gamma^\mu D_\mu\psi}}. \tag{13.322}$$

The free propagator is the solution of equation (13.321) with $A_\mu = 0$. This reads

$$\gamma^\mu\partial_\mu G_0(x, y) = \delta^2(x - y). \tag{13.323}$$

We get the solution

$$\begin{aligned} G_0(x, y) &= \int \frac{d^2p}{(2\pi)^2} \frac{\gamma^\mu p_\mu}{p^2} e^{-ip(x-y)} \\ &= i\gamma^\mu \partial_\mu \left(\int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2} e^{-ip(x-y)} \right). \end{aligned} \tag{13.324}$$

In Euclidean signature ($x^0 = -ix_E^0$) when $y \longrightarrow x$ we get the result

$$\begin{aligned} G_0(x, y)^E &= -\gamma^\mu \partial_\mu \left(\int \frac{d^2\vec{p}}{4\pi^2} \frac{1}{\vec{p}^2} e^{i\vec{p}\,(\vec{x}-\vec{y})} \right) \\ &= -\gamma^\mu \partial_\mu \left(-\frac{1}{4\pi} \ln(\vec{x} - \vec{y})^2 + \cdots \right). \end{aligned} \tag{13.325}$$

The terms $\cdots$ will not contribute after we perform the differentiation (see below for detailed calculation). Rotating back to Minkowskian signature we get

$$\begin{aligned} G_0(x, y) &= -\gamma^\mu \partial_\mu \left(\frac{i}{4\pi} \ln(x - y)^2 + \cdots \right) \\ &= -\frac{i}{2\pi} \frac{\gamma^\mu (x_\mu - y_\mu)}{(x - y)^2}. \end{aligned} \tag{13.326}$$

13.4.2 Schwinger's solution

This consists of finding the solutions of equations (13.318) and (13.321) for any configuration A_μ. A general gauge field in two dimensions can be written in the form (this does not hold in four dimensions or noncommutative two dimensions)

$$A^\mu = \epsilon^{\mu\nu} \partial_\mu \phi + \partial^\mu \Lambda. \tag{13.327}$$

The scalar field ϕ is the physical degree of freedom, whereas Λ is a pure gauge. Since we have already fixed the gauge we can set $\Lambda = 0$. The general solution of equation (13.321) is given by the 2-point function (for any A_μ)

$$G(x, y)^A = G_0(x, y) e^{ie\gamma^5(\phi(x)-\phi(y))}. \tag{13.328}$$

Indeed, by using the identities $\{G_0, \gamma^5\} = 0$ and $\gamma^\mu \gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$ we calculate

$$\gamma^\mu \partial_\mu G(x, y)^A = \delta^2(x - y) - ie\gamma^\mu A_\mu(x) G(x, y)^A. \tag{13.329}$$

Next, we need to evaluate the current $j^\mu = e\bar{\psi}\gamma^\mu\psi$ in the background gauge field A_μ. This quantity is needed in equation (13.318). The gauge-invariant definition of this current employees a Wilson line, viz

$$\begin{aligned}\langle\Omega|j^\mu(x)|\Omega\rangle^A &= e\langle\Omega|\bar\psi(y)\gamma^\mu e^{-ie\int_x^y dz_\mu A^\mu}\psi(x)|\Omega\rangle^A,\ \ y\longrightarrow x\\ &= -\,e\ \mathrm{tr}_2\left(\gamma^\mu e^{-ie\int_x^y dz_\mu A^\mu}G(x,\,y)^A\right)\end{aligned}\tag{13.330}$$

The limit $y\longrightarrow x$ should be taken in a symmetrical way, i.e. $\epsilon^\mu/\epsilon^2\longrightarrow 0$ and $\epsilon^\mu\epsilon^\nu/\epsilon^2\longrightarrow g^{\mu\nu}/d$ when $\epsilon=y-x\longrightarrow 0$. Recall that gauge transformations are given by $A_\mu\longrightarrow A_\mu+\partial_\mu\Lambda$, $\psi\longrightarrow\exp(-ie\Lambda)\psi$ and $\bar\psi\longrightarrow\bar\psi\exp(ie\Lambda)$. This dictates the form of the above expression. Now we compute

$$\begin{aligned}\langle\Omega|j^\mu(x)|\Omega\rangle^A &= -\,e\ \mathrm{tr}_2\left(\gamma^\mu\left(1-ie\epsilon^\mu A_\mu\right)\left(\frac{i}{2\pi}\frac{\gamma^\nu\epsilon_\nu}{\epsilon^2}\right)\left(1-ie\gamma^5\epsilon^\mu\partial_\mu\phi\right)\right)\\ &= -\frac{ie}{2\pi}\frac{\epsilon_\nu}{\epsilon^2}\left[\mathrm{tr}_2\gamma^\mu\gamma^\nu-ie\epsilon^\rho\partial_\rho\phi\,\mathrm{tr}_2\gamma^\mu\gamma^\nu\gamma^5-ie\epsilon^\rho A_\rho\,\mathrm{tr}_2\gamma^\mu\gamma^\nu+O(\epsilon^2)\right]\\ &= -\frac{e^2}{2\pi}\left(A^\mu+\epsilon^{\mu\nu}\partial_\nu\phi\right).\end{aligned}\tag{13.331}$$

We have used $\mathrm{tr}_2\gamma^\mu\gamma^\nu=2g^{\mu\nu}$, $\mathrm{tr}_2\gamma^5\gamma^\mu\gamma^\nu=2\epsilon^{\mu\nu}$. By using equation (13.327) we obtain

$$\langle\Omega|j^\mu(x)|\Omega\rangle^A=-\frac{e^2}{\pi}A^\mu.\tag{13.332}$$

By using this result we can rewrite equation (13.318) as

$$\left(\partial^2+\frac{e^2}{\pi}\right)\frac{1}{i}\frac{\delta Z[J]}{\delta J^\mu(x)}+J_\mu(x)Z[J]=0.\tag{13.333}$$

Introduce the 2-point function

$$D(x,\,y)=\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2-\dfrac{e^2}{\pi}}e^{-ip(x-y)}.\tag{13.334}$$

This describes the propagation of a scalar particle with mass e^2/π. It satisfies the equation of motion

$$\left(-\partial^2-\frac{e^2}{\pi}\right)D(x,\,y)=\delta^2(x-y).\tag{13.335}$$

The solution of the Schwinger model is given exactly by

$$Z[J]=e^{\frac{i}{2}\int d^2xd^2yJ_\mu(x)D(x,y)J^\mu(y)}.\tag{13.336}$$

13.4.3 The chiral anomaly

The axial current is defined by

$$j^{\mu 5}=e\bar\psi\gamma^\mu\gamma^5\psi.\tag{13.337}$$

Since $\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$ we must have

$$j^{\mu 5} = -\epsilon^{\mu\nu} j_\nu . \tag{13.338}$$

Hence

$$\langle\Omega|j^{\mu 5}(x)|\Omega\rangle^A = \frac{e^2}{\pi}\epsilon^{\mu\nu}A_\nu . \tag{13.339}$$

Immediately we get

$$\langle\Omega|\partial_\mu j^{\mu 5}(x)|\Omega\rangle^A = \frac{e^2}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{13.340}$$

13.4.4 The sine-Gordon model

This is given by the Lagrangian density

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\alpha_0}{\beta^2}\cos\beta\phi + \gamma_0 \\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \left(\frac{\alpha_0}{\beta^2} - \frac{1}{2}\alpha_0\phi^2 + \frac{\alpha_0\beta^2}{4!}\phi^4 + \cdots\right) + \gamma_0.\end{aligned} \tag{13.341}$$

The parameters α_0 and β are positive real numbers. The parameter α_0 is a 'squared mass' associated with small oscillations about the minimum $\phi = 0$ and β measures the strength of the interaction. We will give the scalar field a small mass μ in order to remove the IR divergences which plague any perturbation calculations about a massless theory (especially about a massless scalar field theory in two dimensions). The Largrangian density is then given by

$$\begin{aligned}\mathcal{L} = \mathcal{L}_0 + \frac{\alpha_0}{\beta^2}\cos\beta\phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\ - \frac{1}{2}\mu^2\phi^2 + \frac{\alpha_0}{\beta^2}\cos\beta\phi + \gamma_0.\end{aligned} \tag{13.342}$$

The corresponding Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\alpha_0}{\beta^2}\cos\beta\phi - \gamma_0. \tag{13.343}$$

Clearly we want the limit $\mu \longrightarrow 0$ where (obviously) we will get the usual IR divergences. We handle this problem by multiplying the interaction by a spacetime function $f(x)$ of compact support. To find the desired Green's functions we must (1) compute all orders in perturbation theory, (2) take the limit $\mu \longrightarrow 0$, (3) sum up the series then (4) set $f = 1$.

Next, we need to renormalize the above theory. For any scalar field theory in two dimensions with non-derivative interactions the only UV divergences which occur in any order of perturbation theory come from tadpole graphs (those which contain a

closed loop consisting of a single internal line). Hence, all UV divergences can be removed by normal ordering the Hamiltonian. We will denote by N_m the normal ordering operator defined by a mass m which is generally different from the mass μ appearing in the original model.

We will use Wick's theorem in the form

$$Te^{i\int d^2xJ(x)\phi(x)} = N_m e^{i\int d^2xJ(x)\phi(x)} e^{-\frac{1}{2}\int J(x)\Delta(x-y;m)J(y)d^2xd^2y}. \tag{13.344}$$

The proof will be given by computing some examples. Towards this end, define $J\phi = i\int d^2xJ(x)\phi(x)$ and $\Delta = -\Big(\int d^2xd^2yJ(x)\Delta(x-y;m)J(y)\Big)/2$. We can then compute

$$\begin{aligned}
T(J\phi) &= N_m(J\phi)\\
T\frac{(J\phi)^2}{2!} &= N_m\frac{(J\phi)^2}{2!} + \Delta\\
T\frac{(J\phi)^3}{3!} &= N_m\frac{(J\phi)^3}{3!} + N_m\frac{(J\phi)^1}{1!}\Delta\\
T\frac{(J\phi)^4}{4!} &= N_m\frac{(J\phi)^4}{4!} + N_m\frac{(J\phi)^2}{2!}\Delta + \frac{\Delta^2}{2!}\\
T\frac{(J\phi)^5}{5!} &= N_m\frac{(J\phi)^5}{5!} + N_m\frac{(J\phi)^3}{3!}\Delta + N_m\frac{(J\phi)^1}{1!}\frac{\Delta^2}{2!}.
\end{aligned} \tag{13.345}$$

This is sufficient to show the desired structure. In the above, $\Delta(x-y;m)$ is the contraction of two fields $\phi(x)$ and $\phi(y)$, i.e. it is the Feynman propagator (if we remove the time-ordering operator T then we will get the Wightman function instead). It is given by the equation

$$\Delta(x-y;m) = \int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2 - m^2}e^{-ip(x-y)}. \tag{13.346}$$

To stress one more time we are using in the normal ordering a mass m different from the mass μ appearing in the original Lagrangian density. The Euclidean version of this integral is (with $y \neq x$)

$$\begin{aligned}
\Delta(x-y;m)^E &= i\int \frac{d^2\vec{p}}{(2\pi)^2}\frac{1}{\vec{p}^2 + m^2}e^{i\vec{p}\,(\vec{x}-\vec{y})}\\
&= i\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dp^2d\alpha}{4\pi^2}\frac{\cos(p|\vec{x}-\vec{y}|\cos\alpha)}{p^2 + m^2}\\
&= i\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dp^2d\alpha}{4\pi^2}\frac{\cos(p\cos\alpha)}{p^2 + m^2|\vec{x}-\vec{y}|^2}.
\end{aligned} \tag{13.347}$$

In the limit $y \longrightarrow x$ we can set the cosine equal to 1 and we find the following dominant contribution

$$\Delta(x-y;m)^E = i\int \frac{d^2\vec{p}}{(2\pi)^2}\frac{1}{\vec{p}^2+m^2|\vec{x}-\vec{y}|^2}. \tag{13.348}$$

In other words, we have (with $d = 2-\epsilon$) [3]

$$\begin{aligned}\Delta(x-y;m)^E &= i\int \frac{d^d\vec{p}}{(2\pi)^d}\frac{1}{\vec{p}^2+m^2|\vec{x}-\vec{y}|^2}\\ &= \frac{i}{4\pi}\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{4\pi}{m^2|\vec{x}-\vec{y}|^2}\right)^{\frac{\epsilon}{2}}\\ &= \frac{i}{4\pi}\left(\frac{2}{\epsilon}-\gamma\right)\left(1+\frac{\epsilon}{2}\log\left(\frac{4\pi}{m^2|\vec{x}-\vec{y}|^2}\right)\right)\\ &= \frac{i}{4\pi}\left(\frac{2}{\epsilon}-\log cm^2|\vec{x}-\vec{y}|^2+O(\epsilon^2)\right).\end{aligned} \tag{13.349}$$

Hence

$$\Delta(x-y;m) = \frac{1}{4\pi}\left(\frac{2}{\epsilon}-\log(-cm^2(x-y)^2)+O(\epsilon^2)\right). \tag{13.350}$$

By dropping the singular term we get when $y \longrightarrow x$ the propagator

$$\begin{aligned}\Delta(x-y;m) = &-\frac{1}{4\pi}\log\left(-cm^2(x-y)^2\right)\\ &+O((x-y)^2),\ \ c=\frac{e^\gamma}{4\pi}.\end{aligned} \tag{13.351}$$

Let us do this calculation for $x = y$. From equation (13.347) we have

$$\Delta(0;m)^E = i\int \frac{d^2\vec{p}}{(2\pi)^2}\frac{1}{\vec{p}^2+m^2}. \tag{13.352}$$

This gives, using an ordinary cutoff Λ, the result

$$\Delta(0;m) = -\frac{1}{4\pi}\log\frac{m^2}{\Lambda^2},\ \ y=x. \tag{13.353}$$

This agrees with Coleman's result, although he got it using a different method. Now we use this result in Wick's theorem (13.344) with $J(x) = \beta\delta(x-x_1)$. We get immediately

$$e^{i\beta\phi(x_1)} = \left(\frac{m^2}{\Lambda^2}\right)^{\frac{\beta^2}{8\pi}} N_m e^{i\beta\phi(x_1)}. \tag{13.354}$$

Thus

$$\alpha_0 \cos\beta\phi = \alpha N_m \cos\beta\phi, \;\; \alpha = \alpha_0 \left(\frac{m^2}{\Lambda^2}\right)^{\frac{\beta^2}{8\pi}}. \tag{13.355}$$

We remark the identity (where N_μ is the normal ordering operator defined by the mass μ)

$$N_m \cos\beta\phi = \left(\frac{\mu^2}{m^2}\right)^{\frac{\beta^2}{8\pi}} N_\mu \cos\beta\phi. \tag{13.356}$$

Let us now introduce

$$\begin{aligned}
\phi(x^1) &= \phi_+(x^1) + \phi_-(x^1) \\
\phi_+(x^1) &= \int \frac{dk^1}{2\pi} \frac{1}{\sqrt{2\omega(k^1, m)}} a(k^1, m) e^{-ik^1 x_1} \\
\phi_-(x^1) &= \int \frac{dk^1}{2\pi} \frac{1}{\sqrt{2\omega(k^1, m)}} a^+(k^1, m) e^{ik^1 x_1}.
\end{aligned} \tag{13.357}$$

As usual $\omega(k^1, m)^2 = (k^0)^2 = (k^1)^2 + m^2$ and $[a(k^1, m), a^+(p^1, m)] = 2\pi\delta(k^1 - p^1)$. We can compute

$$\begin{aligned}
T(\partial_1\phi)^2 &= N_m(\partial_1\phi)^2 + [\partial_1\phi_+, \partial_1\phi_-] = N_m(\partial_1\phi)^2 + \int \frac{dk^1}{2\pi} \frac{(k^1)^2}{2\omega(k^1, m)} \\
T(\phi)^2 &= N_m(\phi)^2 + [\phi_+, \phi_-] = N_m(\phi)^2 + \int \frac{dk^1}{2\pi} \frac{1}{2\omega(k^1, m)}.
\end{aligned} \tag{13.358}$$

Similarly,

$$\begin{aligned}
\pi(x^1) &= \pi_+(x^1) + \pi_-(x^1) \\
\pi_+(x^1) &= -i \int \frac{dk^1}{2\pi} \sqrt{\frac{\omega(k^1, m)}{2}} a(k^1, m) e^{-ik^1 x_1}, \\
\pi_-(x^1) &= i \int \frac{dk^1}{2\pi} \sqrt{\frac{\omega(k^1, m)}{2}} a^+(k^1, m) e^{ik^1 x_1}.
\end{aligned} \tag{13.359}$$

Then

$$T(\pi)^2 = N_m(\pi)^2 + [\pi_+, \pi_-] = N_m(\pi)^2 + \int \frac{dk^1}{2\pi} \frac{(k^1)^2 + m^2}{2\omega(k^1, m)}. \tag{13.360}$$

Hence we get

$$T\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \gamma_0\right) = N_m\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \gamma\right). \tag{13.361}$$

The renormalized coupling γ is given by

$$\gamma = \gamma_0 - \int \frac{dk^1}{8\pi} \frac{2(k^1)^2 + m^2 + \mu^2}{\omega(k^1, m)}. \tag{13.362}$$

The renormalized Hamiltonian is

$$\mathcal{H} = N_m \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\alpha}{\beta^2}\cos\beta\phi - \gamma \right). \tag{13.363}$$

The mass μ does not get renormalized (anyway we want to take the limit $\mu \longrightarrow 0$ at the end of calculation). Similarly, the parameter β does not get renormalized. The parameters α_0 and γ_0 were renormalized (multiplicatively and additively respectively) to obtain the finite values α and γ. All that was required is normal ordering. In all this the coupling α is the only crucial parameter.

We now compute the following quantity (where ϕ is a field of mass μ)

$$\begin{aligned} \langle 0, \mu | T \prod_i N_m e^{i\beta_i \phi(x_i)} | 0, \mu \rangle &= \left(\frac{\mu^2}{m^2} \right)^{\sum_i \frac{\beta_i^2}{8\pi}} \langle 0, \mu | T \prod_i N_\mu e^{i\beta_i \phi(x_i)} | 0, \mu \rangle \\ &= \left(\frac{\Lambda^2}{m^2} \right)^{\sum_i \frac{\beta_i^2}{8\pi}} \langle 0, \mu | T \prod_i e^{i\beta_i \phi(x_i)} | 0, \mu \rangle. \end{aligned} \tag{13.364}$$

We have used the identities (13.356) and (13.354). Thus, using Wick's theorem (13.344) we have the result

$$\begin{aligned} \langle 0, \mu | T \prod_i N_m e^{i\beta_i \phi(x_i)} | 0, \mu \rangle &= \left(\frac{\Lambda^2}{m^2} \right)^{\sum_i \frac{\beta_i^2}{8\pi}} \langle 0, \mu | T e^{i \sum_i \beta_i \phi(x_i)} | 0, \mu \rangle \\ &= \left(\frac{\Lambda^2}{m^2} \right)^{\sum_i \frac{\beta_i^2}{8\pi}} e^{-\sum_{i,j} \frac{1}{2}\beta_i \beta_j \Delta(x_i - x_j; \mu)}. \end{aligned} \tag{13.365}$$

Assuming that the spacetime points x_i are restricted to a finite region (determined by the regulator $f(x)$) we can use the short-distance formula (13.351). We get by neglecting contractions of fields at the same point the expression

$$\langle 0, \mu | T \prod_i N_m e^{i\beta_i \phi(x_i)} | 0, \mu \rangle = \left(\frac{\Lambda^2}{m^2} \right)^{\sum_i \frac{\beta_i^2}{8\pi}} \prod_{i>j} (-c\mu^2 (x_i - x_j)^2)^{\frac{\beta_i \beta_j}{4\pi}}. \tag{13.366}$$

This expression is proportional to $\mu^{(\sum_i \beta_i)^2/4\pi}$ (including the contribution from the neglected contractions). Hence, if $\sum_i \beta_i \neq 0$ we get the result that this expression must vanish when $\mu \longrightarrow 0$. In other words, when $\mu \longrightarrow 0$ we must have $\sum_i \beta_i = 0$ otherwise the expectation value is 0.

We are interested in the perturbation series of the theory in powers of α_0 given by (with $A_\pm = \exp(\pm i\beta\phi)$)

$$\begin{aligned}&\int [d\phi] e^{i\int d^2x \mathcal{L}_0} e^{i\frac{\alpha_0}{2\beta^2}\int d^2x (A_+ + A_-)} \\ &= \sum_n \frac{\left(i\frac{\alpha_0}{2\beta^2}\right)^{2n}}{(n!)^2} \prod_{i=1}^n \int d^2x_i d^2y_i \langle 0, \mu | T \prod_{i=1}^n A_+(x_i) A_-(y_i) | 0, \mu \rangle,\end{aligned} \tag{13.367}$$

where

$$\langle 0, \mu | T \prod_{i=1}^n A_+(x_i) A_-(y_i) | 0, \mu \rangle = \int [d\phi] e^{i\int d^2x \mathcal{L}_0} A_+(x_i) A_-(y_i). \tag{13.368}$$

We have used the fact that the only non-zero terms in the expansion are those with equal numbers of A_+ and A_-. We have, by using Wick's theorem (13.344) and by dropping contractions at the same point, the result

$$\begin{aligned}&\langle 0, \mu | T \prod_i A_+(x_i) A_-(y_i) | 0, \mu \rangle = \langle 0, \mu | T e^{i\beta \sum_i (\phi(x_i) - \phi(y_i))} | 0, \mu \rangle \\ &= e^{-\frac{\beta^2}{2} \sum_{i,j} \left[\Delta(x_i - x_j; m) - \Delta(x_i - y_j; m) - \Delta(y_i - x_j; m) + \Delta(y_i - y_j; m)\right]} \\ &= \frac{\prod_{i>j} \left[c^2 m^4 (x_i - x_j)^2 (y_i - y_j)^2\right]^{\frac{\beta^2}{4\pi}}}{\prod_{i,j} \left[c m^2 (x_i - y_j)^2\right]^{\frac{\beta^2}{4\pi}}}.\end{aligned} \tag{13.369}$$

The last calculation we want to do is to compute the commutators

$$\begin{aligned}[\partial_\nu \phi(x), A_\pm(y)] = e^{\pm i\beta\phi_+(y)} [\partial_\nu \phi_+(x), e^{\pm i\beta\phi_-(y)}] \\ + [\partial_\nu \phi_-(x), e^{\pm i\beta\phi_+(y)}] e^{\pm i\beta\phi_-(y)}.\end{aligned} \tag{13.370}$$

In the above we have used the fact that $\phi = \phi_+ + \phi_-$, $\partial_\nu \phi = \partial_\nu \phi_+ + \partial_\nu \phi_-$. Using the expressions (with $k^0 = \omega(k^1, m)$)

$$\begin{aligned}\phi_+(x) &= \int \frac{dk^1}{2\pi} \frac{1}{\sqrt{2\omega(k^1, m)}} a(k^1, m) e^{-ikx}, \\ \phi_-(x) &= \int \frac{dk^1}{2\pi} \frac{1}{\sqrt{2\omega(k^1, m)}} a^+(k^1, m) e^{ikx}.\end{aligned} \tag{13.371}$$

The expressions (13.371) are the fields in the Heisenberg picture, whereas (13.357) are the fields in the Schrödinger picture. We can compute (with $k^0 = \omega(k^1, m)$)

$$[\phi_+(x), \phi_-(y)] = \int \frac{dk^1}{2\pi} \frac{1}{2\omega(k^1, m)} e^{-ik(x-y)} = i\Delta_+(x - y). \tag{13.372}$$

Thus

$$[\phi_+(x),\, \phi_-(y)^n] = in\phi_-^{n-1}(y)\Delta_+(x-y). \tag{13.373}$$

In other words, we have

$$[\phi_+(x),\, e^{\pm i\beta\phi_-(y)}] = \mp\beta\Delta_+(x-y)e^{\pm i\beta\phi_-(y)}. \tag{13.374}$$

Similarly, we compute

$$[\phi_-(x),\, e^{\pm i\beta\phi_+(y)}] = \mp\beta\Delta_-(x-y)e^{\pm i\beta\phi_+(y)}. \tag{13.375}$$

Now (with $k^0 = -\omega(k^1, m)$)

$$-[\phi_-(x),\, \phi_+(y)] = \int \frac{dk^1}{2\pi}\frac{1}{2\omega(k^1, m)}e^{-ik(x-y)} = -i\Delta_-(x-y). \tag{13.376}$$

Hence

$$[\phi(x),\, A_\pm(y)] = \mp\beta\Delta(x-y)A_\pm(y). \tag{13.377}$$

We have used the identity

$$\begin{aligned} i\Delta_+(x-y) + i\Delta_-(x-y) &= \int \frac{dk^1}{2\pi}\frac{1}{2\omega(k^1,m)}e^{-ik(x-y)}|_{k^0=\omega(k^1,m)} \\ &\quad + \int \frac{dk^1}{2\pi}\frac{1}{-2\omega(k^1, m)}e^{-ik(x-y)}|_{k^0=-\omega(k^1, m)} \\ &= i\Delta(x-y). \end{aligned} \tag{13.378}$$

The propagator $\Delta(x-y)$ is given by equation (13.346). The sign of equation (13.377) agrees with Coleman's sign in his paper on the Schwinger model but not with his sign in his paper on the sine-Gordon model.

From equation (13.377) we deduce

$$[\epsilon^{\mu\nu}\partial_\nu\phi(x),\, A_\pm(y)] = \mp\beta\epsilon^{\mu\nu}\partial_\nu\Delta(x-y)A_\pm(y). \tag{13.379}$$

The final point we mention about the sine-Gordon model is that the quantum theory does not make sense for $\beta > 8\pi$ since the energy becomes unbounded from below and the theory has no ground state.

13.4.5 The Thirring model

The massless Thirring model is given by the action

$$\mathcal{L} = \bar\psi i\gamma_\mu\partial^\mu\psi - \frac{1}{2}g\hat j_\mu\hat j^\mu,\ \hat j^\mu = \bar\psi\gamma^\mu\psi. \tag{13.380}$$

By requiring that the current $\hat j^\mu$ must obey the appropriate Ward identities we find that the Hamiltonian of the above massless Thirring model requires no further renormalization. The model is exactly solvable and makes sense for all $g > -\pi$. We

are interested in the scalar density $\bar{\psi}\psi$. The renormalized scalar density involves a cutoff-dependent constant Z, viz

$$\sigma = Z\bar{\psi}\psi, \;\; \sigma_+ = Z\bar{\psi}_0\psi_0, \;\; \sigma_- = Z\bar{\psi}_1\psi_1. \tag{13.381}$$

The massive Thirring model is defined by

$$\mathcal{L}_m = \mathcal{L} - m'\sigma. \tag{13.382}$$

The perturbation series of the theory in powers of m' is well defined (aside from trivial infrared divergences associated with the massless Thirring model). We are thus interested in perturbation theory in m', viz

$$\begin{aligned} &\int [d\psi][d\bar{\psi}] e^{i\int d^2x \mathcal{L}} e^{-im'\int d^2x(\sigma_+ + \sigma_-)} \\ &= \sum_n \frac{(-im')^{2n}}{(n!)^2} \prod_{i=1}^n \int d^2x_i d^2y_i \langle 0|T \prod_{i=1}^n \sigma_+(x_i)\sigma_-(y_i)|0\rangle, \end{aligned} \tag{13.383}$$

where

$$\langle 0|T \prod_{i=1}^n \sigma_+(x_i)\sigma_-(y_i)|0\rangle = \int [d\psi][d\bar{\psi}] e^{i\int d^2x \mathcal{L}} \sigma_+(x_i)\sigma_-(y_i). \tag{13.384}$$

In the above, all terms which contain unequal numbers of σ_+ and σ_- vanish by chiral invariance of the massless Thirring model. This is obvious at least for $g = 0$ where we can use the free propagators given by equation (13.387). Coleman computes these expectation values using Klaiber's results and he finds

$$\langle 0|T \prod_{i=1}^n \sigma_+(x_i)\sigma_-(y_i)|0\rangle = \left(\frac{1}{2}\right)^{2n} \frac{\prod_{i>j} \left[M^4(x_i - x_j)^2(y_i - y_j)^2\right]^{1+\frac{b}{\pi}}}{\prod_{i,j} \left[M^2(x_i - y_j)^2\right]^{1+\frac{b}{\pi}}}. \tag{13.385}$$

The parameter M is an arbitrary mass which is the reflection of the arbitrary finite renormalization in the definition of σ. In other words, σ in this formula should be given by equation (13.381). The Klaiber's parameter is given by

$$b = -\frac{g}{1 + \frac{g}{\pi}}. \tag{13.386}$$

Let us say that this result should be easily checked for $g = 0$, $b = 0$. For $g = 0$ we have computed the following 2-point functions (from equation (13.326))

$$\begin{aligned} \langle 0|T\psi_0(x)\bar{\psi}_1(y)|0\rangle &= -\frac{1}{2\pi} \frac{(x_0 - y_0) - (x_1 - y_1)}{(x - y)^2} \\ \langle 0|T\psi_1(x)\bar{\psi}_0(y)|0\rangle &= \frac{1}{2\pi} \frac{(x_0 - y_0) + (x_1 - y_1)}{(x - y)^2} \\ \langle 0|T\psi_0(x)\bar{\psi}_0(y)|0\rangle &= 0 \\ \langle 0|T\psi_1(x)\bar{\psi}_1(y)|0\rangle &= 0. \end{aligned} \tag{13.387}$$

By comparing equations (13.368), (13.369) with equations (13.384), (13.385) we see that the perturbative series of the Thirring model in m' is identical to the perturbative series of the sine-Gordon model in α_0 provided we match the couplings and fields in the two theories as follows

$$1 + \frac{b}{\pi} = \frac{1}{1 + \frac{g}{\pi}} = \frac{\beta^2}{4\pi} \tag{13.388}$$

$$-m'\sigma = \frac{\alpha_0}{\beta^2}\cos\beta\phi \Leftrightarrow -m' Z\bar{\psi}\psi = \frac{\alpha}{\beta^2}N_m \cos\beta\phi. \tag{13.389}$$

These two matching conditions do not depend on renormalization conventions. In other words, they do not depend on the equation

$$M^2 = cm^2. \tag{13.390}$$

We can choose

$$\begin{aligned} m' &= \frac{\alpha_0}{\beta^2} \\ -\sigma_\pm &= \frac{1}{2}A_\pm \Leftrightarrow -\sigma = -Z\bar{\psi}\psi = \cos\beta\phi, \end{aligned} \tag{13.391}$$

or

$$\begin{aligned} m' &= \frac{\alpha}{\beta^2} \\ -\sigma_\pm &= \frac{1}{2}N_m A_\pm \Leftrightarrow -\sigma = -Z\bar{\psi}\psi = N_m\cos\beta\phi. \end{aligned} \tag{13.392}$$

Coleman computes also the commutators

$$\left[\hat{j}^\mu(x), \sigma_\pm(y)\right] = \mp\frac{2}{1 + \frac{g}{\pi}}\epsilon^{\mu\nu}\partial_\nu\Delta(x - y)\sigma_\pm(y). \tag{13.393}$$

Comparing with equation (13.379) we get the extra identity

$$\hat{j}^\mu = \bar{\psi}\gamma^\mu\psi = \frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi. \tag{13.394}$$

The sign does not agree with Coleman's sign in the paper on the sine-Gordon model. This is related to the sign of (13.379). However, this sign agrees with his sign in the paper on the Schwinger model. Compare this equation with equation (13.332).

13.4.6 Coleman's solution of QED in 2D

We take the model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - e\gamma^{\mu}A_{\mu} - m\right)\psi. \tag{13.395}$$

We use the (axial or radiation) gauge

$$A_1 = 0. \tag{13.396}$$

The equation of motion of the remaining field A_0 becomes a constraint equation given by

$$\partial_1^2 A_0 = -j_0 = -e\psi^{+}\psi. \tag{13.397}$$

In one spatial dimension there is no photon since there are no transverse dimensions. We solve the above equation by introducing the following Green's function

$$-\partial_1^2 D(x^1, y^1) = \delta(x^1 - y^1). \tag{13.398}$$

It has the solution

$$D(x^1, y^1) = \int \frac{dp_1}{2\pi}\frac{1}{p_1^2}e^{-ip_1(x^1-y^1)}. \tag{13.399}$$

Then we have the solutions for A_0 and F_{01} given by

$$\begin{aligned} A_0 &= \int D(x^1, y^1)j_0(y^1)dy^1 - Fx^1 - G, \\ F_{01} &= -\int \partial_1 D(x^1, y^1)j_0(y^1)dy^1 + F. \end{aligned} \tag{13.400}$$

The constant G is irrelevant but F (a background electric field) is physically significant. It is essentially the famous theta angle. Indeed, as we will see we can make the identification

$$F = e\frac{\theta}{2\pi}. \tag{13.401}$$

The parameter θ which is an angle in $[-\pi, \pi]$ is an extra parameter in the model which is completely independent of the charge e and the mass m. In four dimensions a background electric field will be cancelled by pair production from the vacuum.

We need the quantity

$$-\partial_1 D(x^1, y^1) = \int \frac{dp_1}{2\pi}\frac{i}{p_1}e^{-ip_1(x^1-y^1)} = \frac{1}{2}\theta(x^1 - y^1). \tag{13.402}$$

This can be checked by using the identity

$$\int dy^1 \theta(x^1 - y^1) = 2x^1. \tag{13.403}$$

Hence, we must have

$$D(x^1, y^1) = -\frac{1}{2}|x^1 - y^1|. \tag{13.404}$$

The canonical momentum associated with the field ψ is $\bar{\psi} i\gamma^0$. All other canonical momenta are zero. Thus we have the Hamiltonian density (by using also the equation of motion and integration by parts)

$$\mathcal{H} = \bar{\psi}(i\gamma_1\partial_1 + m)\psi + \frac{1}{2}(\partial_1 A_0)^2. \tag{13.405}$$

Restriction on the states of charge zero (which seems to be necessary for translational invariance) means that we must have

$$\int j_0(x^1)dx^1 = 0. \tag{13.406}$$

The Hamiltonian takes, in these states, the form

$$\begin{aligned} H = &\int \bar{\psi}(i\gamma_1\partial_1 + m)\psi - \frac{1}{4}\int dx^1 dy^1 j_0(x^1) j_0(y^1)|x^1 - y^1| \\ &- \frac{F}{2}\int dx^1 j_0(x^1)x^1. \end{aligned} \tag{13.407}$$

Let us specialize our results of the sine-Gordon and Thirring models to the values $g = 0$ (the Thirring model becomes a theory of a free massive Dirac field) and $\beta^2 = 4\pi$ (in the sine-Gordon model). The identity (13.388) is trivially satisfied. The identity (13.389) reads with $m'Z = m$ (which is the mass used in normal ordering) as follows

$$-m\bar{\psi}\psi = \frac{\alpha}{4\pi} N_m \cos 2\sqrt{\pi}\phi. \tag{13.408}$$

If we choose

$$\frac{\alpha}{4\pi} \equiv cm^2 = M^2. \tag{13.409}$$

Then we will obtain instead the relation

$$\bar{\psi}\psi = -cmN_m \cos 2\sqrt{\pi}\phi. \tag{13.410}$$

We get, therefore, the two equivalent theories (we put also $\mu = 0$ and $\gamma = 0$ in our expressions)

$$\mathcal{L} = \bar{\psi}\left(i\gamma_\mu\partial^\mu - m\right)\psi \Leftrightarrow \mathcal{H} = \bar{\psi}(i\gamma_1\partial_1 + m)\psi \tag{13.411}$$

$$\begin{aligned} \mathcal{H} &= N_m\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 - \frac{\alpha}{4\pi}\cos 2\sqrt{\pi}\phi\right) \\ &= N_m\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 - cm^2 \cos 2\sqrt{\pi}\phi\right). \end{aligned} \tag{13.412}$$

We have also the equivalence between the local operators in the two theories

$$\bar{\psi}\gamma^{\mu}\psi = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_{\nu}\phi. \tag{13.413}$$

States of charge 0 (analogue of states of color 0 in QCD) satisfy

$$\int dx^1 j^0(x^1) = e\int dx^1\bar{\psi}\gamma^0\psi = \frac{e}{\sqrt{\pi}}\int dx^1\partial_1\phi = 0 \tag{13.414}$$

On these states we can compute

$$\begin{aligned} F_{01} &= -e\int \partial_1 D(x^1, y^1)\bar{\psi}(y^1)\gamma^0\psi(y^1)dy^1 + \frac{e\theta}{2\pi} \\ &= -\frac{e}{\sqrt{\pi}}\int \partial_1 D(x^1, y^1)\partial_1\phi(y^1)dy^1 + \frac{e\theta}{2\pi} \\ &= -\frac{e}{\sqrt{\pi}}\left(\phi - \frac{\theta}{2\sqrt{\pi}}\right). \end{aligned} \tag{13.415}$$

The Hamiltonian (13.405) becomes

$$\mathcal{H} = N_m\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 - cm^2\cos 2\sqrt{\pi}\phi + \frac{e^2}{2\pi}\left(\phi - \frac{\theta}{2\sqrt{\pi}}\right)^2\right). \tag{13.416}$$

Finally, we make the shift $\phi \longrightarrow -\phi + \frac{\theta}{2\sqrt{\pi}}$ and set

$$\mathcal{M}^2 = \frac{e^2}{\pi}. \tag{13.417}$$

We get

$$\begin{aligned} \mathcal{H} &= N_m\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 - cm^2\cos(2\sqrt{\pi}\phi - \theta) + \frac{\mathcal{M}^2}{2}\phi^2\right) \\ &= N_{\mathcal{M}}\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 - cm\mathcal{M}\cos(2\sqrt{\pi}\phi - \theta) + \frac{\mathcal{M}^2}{2}\phi^2\right). \end{aligned} \tag{13.418}$$

We get exactly the correct Schwinger mass $\mathcal{M} = e/\sqrt{\pi}$. From this we can see immediately that θ is indeed an angle. The physics is periodic in θ. We can restrict this angle to the interval $[-\pi, \pi]$. This angle is independent from the mass m and the electromagnetic coupling e^2 (or $\mathcal{M}^2$) and thus it is another fundamental parameter of QED_2.

13.4.7 Flavor and condensate

The two-flavor model is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{2}\bar{\psi}_i\left(i\gamma^{\mu}\partial_{\mu} - e\gamma^{\mu}A_{\mu} - m\right)\psi_i. \tag{13.419}$$

We use the (axial or radiation) gauge $A_1 = 0$. The equation of motion of the remaining field A_0 becomes a constraint equation given by

$$\partial_1^2 A_0 = -j_0 = -e \sum_{i=1}^{2} \psi_i^+ \psi_i. \tag{13.420}$$

Only the definition of j_0 has changed. Thus, as before, we have the solutions for A_0 and F_{01} given by

$$\begin{aligned} A_0 &= \int D(x^1, y^1) j_0(y^1) dy^1 - Fx^1 - G, \\ F_{01} &= -\int \partial_1 D(x^1, y^1) j_0(y^1) dy^1 + F, \;\; F = e\frac{\theta}{2\pi}. \end{aligned} \tag{13.421}$$

The canonical momentum associated with the fields ψ_i is $\bar{\psi}_i i\gamma^0$. All other canonical momenta are zero. Thus we have the Hamiltonian density

$$\mathcal{H} = \sum_{i=1}^{2} \bar{\psi}_i (i\gamma_1 \partial_1 + m)\psi_i + \frac{1}{2}(\partial_1 A_0)^2. \tag{13.422}$$

We introduce two scalar fields ϕ_i by the equations

$$-m\bar{\psi}_1\psi_1 = \frac{\alpha}{4\pi} N_m \cos 2\sqrt{\pi}\phi_1, \; -m\bar{\psi}_2\psi_2 = \frac{\alpha}{4\pi} N_m \cos 2\sqrt{\pi}\phi_2. \tag{13.423}$$

As before we choose

$$\frac{\alpha}{4\pi} = cm^2 = M^2. \tag{13.424}$$

We get, therefore, the two equivalent theories

$$\mathcal{L} = \sum_{i=1}^{2} \bar{\psi}_i \Big(i\gamma_\mu \partial^\mu - m \Big) \psi_i \Leftrightarrow \mathcal{H} = \sum_{i=1}^{2} \bar{\psi}_i (i\gamma_1 \partial_1 + m)\psi_i \tag{13.425}$$

$$\begin{aligned} \mathcal{H} &= N_m \Big(\frac{1}{2}\pi_1^2 + \frac{1}{2}\pi_2^2 + \frac{1}{2}(\partial_1\phi_1)^2 + \frac{1}{2}(\partial_1\phi_2)^2 \\ &\quad - \frac{\alpha}{4\pi}\cos 2\sqrt{\pi}\phi_1 - \frac{\alpha}{4\pi}\cos 2\sqrt{\pi}\phi_1 \Big) \\ &= N_m \Big(\frac{1}{2}\pi_1^2 + \frac{1}{2}\pi_2^2 + \frac{1}{2}(\partial_1\phi_1)^2 + \frac{1}{2}(\partial_1\phi_2)^2 \\ &\quad - cm^2 \cos 2\sqrt{\pi}\phi_1 - cm^2 \cos 2\sqrt{\pi}\phi_1 \Big). \end{aligned} \tag{13.426}$$

We have also the equivalence between the local operators in the two theories

$$\bar{\psi}_1 \gamma^\mu \psi_1 = \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi_1, \;\; \bar{\psi}_2 \gamma^\mu \psi_2 = \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi_2. \tag{13.427}$$

States of charge 0 satisfy

$$\int dx^1 j^0(x^1) = e\sum_{i=1}^{2}\int dx^1 \bar{\psi}_i\gamma^0\psi_i = \frac{e}{\sqrt{\pi}}\sum_{i=1}^{2}\int dx^1\partial_1\phi_i = 0 \tag{13.428}$$

On these states we can compute

$$\begin{aligned} F_{01} &= -e\sum_{i=1}^{2}\int \partial_1 D(x^1, y^1)\bar{\psi}_i(y^1)\gamma^0\psi_i(y^1)dy^1 + \frac{e\theta}{2\pi} \\ &= -\frac{e}{\sqrt{\pi}}\sum_{i=1}^{2}\int \partial_1 D(x^1, y^1)\partial_1\phi_i(y^1)dy^1 + \frac{e\theta}{2\pi} \\ &= -\frac{e}{\sqrt{\pi}}\left(\phi_1 + \phi_2 - \frac{\theta}{2\sqrt{\pi}}\right). \end{aligned} \tag{13.429}$$

The Hamiltonian (13.422) becomes

$$\begin{aligned} \mathcal{H} = N_m\Bigg(&\frac{1}{2}\pi_1^2 + \frac{1}{2}\pi_2^2 + \frac{1}{2}(\partial_1\phi_1)^2 + \frac{1}{2}(\partial_1\phi_2)^2 \\ &- cm^2\cos 2\sqrt{\pi}\phi_1 - cm^2\cos 2\sqrt{\pi}\phi_1 \\ &+ \frac{e^2}{2\pi}\left(\phi_1 + \phi_2 - \frac{\theta}{2\sqrt{\pi}}\right)^2\Bigg). \end{aligned} \tag{13.430}$$

Let us define

$$\phi_+ = \frac{1}{\sqrt{2}}\left(\phi_1 + \phi_2 - \frac{\theta}{2\sqrt{\pi}}\right),\ \ \phi_- = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2), \tag{13.431}$$

and

$$\mathcal{M}^2 = \frac{2e^2}{\pi}. \tag{13.432}$$

We get then the Hamiltonian

$$\begin{aligned} \mathcal{H} = N_m\Bigg(&\frac{1}{2}\pi_+^2 + \frac{1}{2}\pi_-^2 + \frac{1}{2}(\partial_1\phi_+)^2 + \frac{1}{2}(\partial_1\phi_-)^2 \\ &- 2cm^2\cos\left(\sqrt{2\pi}\phi_+ + \frac{\theta}{2}\right)\cos(\sqrt{2\pi}\phi_-) + \frac{\mathcal{M}^2}{2}\phi_+^2\Bigg). \end{aligned} \tag{13.433}$$

In the weak coupling region $m \gg e$ there is no difference between the one-flavor and the two-flavor models (see Coleman). Indeed, when e/m goes to zero (i.e. $m \gg e$) the one-flavor and two-flavor models become the exactly soluble free Dirac theories with one Dirac fermion and two Dirac fermions respectively.

For $m \ll e$ (or equivalently e/m goes to infinity) the one-flavor model becomes the soluble Schwinger model. For the two-flavor model the physics of the strong coupling region $m \ll e$ is significantly different. Indeed, for $m \ll e$ we see from the above Hamiltonian (13.433) that we have two scalar fields (the ϕ_- is light, whereas the ϕ_+ is heavy) with a weak interaction. Perturbation theory will not work here (in contrast to the case of one-flavor) since the scale of the ϕ_- mass and the magnitude of the ϕ_- coupling are given by the same small parameter m. Although we are sure of the existence of a ϕ_+ meson with mass close to $\mathcal{M}$ it is very difficult to compute corrections to this mass (naive calculations will show that the 1-loop correction is of the same order as the tree-level). However, if we look for particles of mass much less then $\mathcal{M}$ and restrict our attention to ϕ_- Green's functions then graphs with internal ϕ_+ are down by powers of $m/\mathcal{M}$. Thus, they can be neglected and perturbation theory will be reliable. The only possible source of problems are tadpole graphs involving ϕ_+ which again can be removed by normal ordering, viz (with $\beta = \sqrt{2\pi}$)

$$\begin{aligned} N_m \cos\left(\beta\phi_+ + \frac{\theta}{2}\right) &= \left(\frac{\mathcal{M}^2}{m^2}\right)^{\frac{\beta^2}{8\pi}} N_{\mathcal{M}} \cos\left(\beta\phi_+ + \frac{\theta}{2}\right) \\ &= \left(\frac{\mathcal{M}}{m}\right)^{\frac{1}{2}} N_{\mathcal{M}} \cos\left(\beta\phi_+ + \frac{\theta}{2}\right). \end{aligned} \tag{13.434}$$

The Hamiltonian becomes (the remaining normal ordering N_m acts on the field ϕ_- only)

$$\begin{aligned} \mathcal{H} = N_m\Big(&\frac{1}{2}N_{\mathcal{M}}\pi_+^2 + \frac{1}{2}\pi_-^2 + \frac{1}{2}N_{\mathcal{M}}(\partial_1\phi_+)^2 \\ &+ \frac{1}{2}(\partial_1\phi_-)^2 - 2cm^{\frac{3}{2}}\mathcal{M}^{\frac{1}{2}}N_{\mathcal{M}} \\ &\times \cos\left(\sqrt{2\pi}\phi_+ + \frac{\theta}{2}\right)\cdot\cos(\sqrt{2\pi}\phi_-) + \frac{\mathcal{M}^2}{2}\phi_+^2\Big). \end{aligned} \tag{13.435}$$

Clearly we can set $\phi_+ = 0$ in this regime. We get the Hamiltonian

$$\begin{aligned} \mathcal{H} &= N_m\left(\frac{1}{2}\pi_-^2 + \frac{1}{2}(\partial_1\phi_-)^2 - 2cm^{\frac{3}{2}}\mathcal{M}^{\frac{1}{2}}\cos\left(\frac{\theta}{2}\right)\cos(\sqrt{2\pi}\phi_-)\right) \\ &= N_{\mathcal{M}'}\left(\frac{1}{2}\pi_-^2 + \frac{1}{2}(\partial_1\phi_-)^2 - \mathcal{M}'^2\cos\frac{\theta}{2}\cos(\sqrt{2\pi}\phi_-)\right), \end{aligned} \tag{13.436}$$

where

$$\mathcal{M}' = \left(2cm\mathcal{M}^{\frac{1}{2}}\cos\frac{\theta}{2}\right)^{\frac{2}{3}}. \tag{13.437}$$

The only effect of θ is through an overall mass scale (in contrast to the one-flavor model). This model is a special case of the sine-Gordon theory (again in contrast to the one-flavor model).

Another (more important) difference between the one-flavor model and the two-flavor model is with regard to the chiral condensate and spontaneous symmetry breaking of chiral symmetry. In the two-flavor model there is an extra $SU(2)_A$ (axial, flavor, isospin) chiral symmetry. Indeed, although it is not obvious from the above Hamiltonians, these Hamiltonians are in fact all isospin invariant. Since we are in two dimensions this $SU(2)_A$ symmetry cannot suffer from spontaneous symmetry breaking due to Coleman's theorem (this applies only in two dimensions). Thus, the chiral condensate $\langle\bar{\psi}\psi\rangle$ is protected from acquiring a non-zero vacuum value. In the one-flavor model this condensate acquires a non-zero value because the $U(1)_A$ axial symmetry is anomalous (so it is not really a symmetry) and since there is no extra symmetry which protects the condensate to form. Thus, we get in the one-flavor model $\langle\bar{\psi}\psi\rangle \neq 0$. In the two-flavor model there is also an axial $U(1)_A$ symmetry which is also anomalous but since $\langle\bar{\psi}\psi\rangle \neq 0$ would also mean that the $SU(2)_A$ is spontaneously broken (which it cannot be in two dimensions) we conclude that we will have a zero condensate despite the presence of this anomaly. Seeing it this way, the condensate is really tied to the spontaneous symmetry breakdown of chiral $SU(2)_A$ symmetry more than it is associated with the chiral $U(1)_A$ anomaly. Explicitly we have the following mass terms

$$\begin{aligned} m\bar{\psi}\psi &= -cm^2 N_m \cos(2\sqrt{\pi}\phi - \theta), \quad N_f = 1 \\ m\sum_{i=1}^{2}\bar{\psi}_i\psi_i &= -2cm^{\frac{3}{2}}\mathcal{M}^{\frac{1}{2}}\cos\frac{\theta}{2}N_m\cos(\sqrt{2\pi}\phi_-), \quad N_f = 2. \end{aligned} \tag{13.438}$$

By differentiating with respect to m we get the condensates

$$\begin{aligned} \bar{\psi}\psi &= -2c\mathcal{M}N_{\mathcal{M}}\cos(2\sqrt{\pi}\phi - \theta), \quad N_f = 1 \\ \sum_{i=1}^{2}\bar{\psi}_i\psi_i &= -\frac{3}{2}m^{\frac{1}{3}}\left(2c\mathcal{M}^{\frac{1}{2}}\cos\frac{\theta}{2}\right)^{\frac{4}{3}}N_{\mathcal{M}'}\cos(\sqrt{2\pi}\phi_-), \quad N_f = 2. \end{aligned} \tag{13.439}$$

This is indeed the correct behaviour. For the one-flavor model the condensate approaches a constant value $-2c\mathcal{M}$ as m is sent to zero (due to the $U(1)_A$ anomaly), whereas in the two-flavor model the condensate vanishes as $m^{\frac{1}{3}}$ when $m \longrightarrow 0$ (due to the exact $SU(2)_A$ chiral symmetry).

The axial (chiral) symmetry $U(2)_A = U(1)_A \times SU(2)_A$ in four dimensions is fundamental to QCD with two flavors u and d (the gauge group is now non-abelian $SU(3)$). There is also a vector $U(2)_V = U(1)_V \times SU(2)_V$ which is always exact. The chiral $SU(2)_A$ symmetry is not afflicted with anomalies and it is spontaneously broken in the quantum theory (the vacuum condensate acquires a non-zero value and the massless quarks become effectively massive). The corresponding Goldstone's bosons are the three pions (they are massless in the limit of vanishing mass of the quarks). The symmetry $U(1)_A$ (related to the famous '$U(1)$ problem') is

actually anomalous in QCD (due to instanton effects) and thus it is not really a symmetry of the action and there is neither spontaneous symmetry breaking nor Goldstone's bosons. If there was no instanton (i.e. if this symmetry was not anomalous) then it would spontaneously break as the $SU(2)_A$ and the corresponding boson would be the η. The difference between the spontaneously broken $SU(2)_A$ and the anomalous $U(1)_A$ is the reason why the η particle is much heavier than the pions. The quarks of QCD couple also to electromagnetic interactions and as a consequence the $SU(2)_A$ is electromagnetically anomalous. This anomaly provides the dominant contribution to the electromagnetic decay $\pi^0 \longrightarrow 2\gamma$.

13.5 Other exactly solvable models

Other very interesting exactly solvable models include for example:

(1) The Schwinger model on the sphere [11] which can be thought of as a regularization/compactification of the Schwinger model studied in this chapter.
(2) 't Hooft model [12] which deals with quantum chromodynamics in two dimensions in the large N limit.
(3) Yang–Mills theory in two dimensions [13–15] (see also [16] for a comparison between the exact solution and perturbation theory) which is intimately related to string theory [17–19]. This is a very important case which is as important, and perhaps even more important, than the Ising model.

All these solutions are in fact possible because they all live in two dimensions (one time and one space direction) and perhaps because of the large N expansion in many instances. By going to four dimensions (one time and three space directions) the possibility of an exact solution becomes so rare that we can single out only (aside from large N solutions) supersymmetric gauge theory with $\mathcal{N} = 2$ which is due originally to Seiberg and Witten.

The fundamental question is always: can we solve exactly quantum chromodynamics which is the gauge theory of the strong nuclear force?

The answer is no. This is a highly non-perturbative theory with highly non-trivial properties such as confinement, the mass gap, chiral symmetry breaking and asymptotic freedom (in the perturbative regime). At this time only the powerful tools of Monte Carlo methods on lattices and the renormalization group equation are available to us to tackle this fundamental theory.

But another powerful complete trick was discovered which allows us to solve the theory. Namely, by adding a lot of supersymmtery (more precisely we add $\mathcal{N} = 2$ supersymmetry) and using holomorphy the theory becomes exactly soluble, as shown explicitly by Seiberg and Witten in 1994 [20, 21].

Nekrasov in 2002 rederived the Seiberg–Witten prepotential, which encodes the full low-energy physics of the system, from microscopic considerations and first principles [22]. He managed to compute the path integral/partition integral of the system exactly and explicitly. The prepotential was found to be equal to a

perturbative part, computed from renormalization group equation considerations, and a non-perturbative part computed by summing up all instanton sectors.

It was the first time a sum over all instanton contributions of some gauge theory could be preformed so explicitly. Some of the technical machineries used by Nekrasov to achieve this feat are (1) supergravity, the so-called omega background, to regulate the infrared behavior of the theory and (2) noncommutative geometry to smooth out the moduli space of instantons.

Thus, by means of a lot of supersymmetry, holomorphy, instantons (and monopoles) we can also solve models in four dimensions exactly.

13.6 Exercise

Exercise 1:

Set up the explicit and detailed solution of the following models:

(1) The Schwinger model on the sphere [11].
(2) 't Hooft model [12].
(3) Yang–Mills theory in two dimensions [13–15].

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Chapter 14

The monopoles and instantons

Monopoles are non-trivial topological gauge field configurations which appear in spontaneously broken gauge theory via the Higgs mechanism. These are particle-like solitonic configurations characterized by stability and finite energy among other properties. Their stability is of a topological origin characterized by the so-called winding numbers or magnetic charges. For this reason monopoles are one of the best examples in which physics and topology become intertwined. The existence of the monopole requires the embedding of electromagnetism, i.e. the group $U(1)$, as a subgroup in a larger non-abelian group G with compact cover which then becomes broken spontaneously via the usual Higgs mechanism.

The original literature on the subject consists of 't Hooft [1] and Polyakov [2]. Some of the pedagogical (from my perspective) lectures I can mention here: Lenz [3], 't Hooft [4], Coleman [5] and Tong [6]. A comprehensive book is Shnir [7] and a comprehensive review is given by Weinberg and Yi [8].

Instantons are also fundamental topological gauge configurational, perhaps more fundamental than monopoles, which are given by events localized in spacetime and hence the other name given for them: pseudo-particles (in contrast with particles such as monopoles which are events localized in space). Instantons are also the gauge field configurations which dominate the path integral in the semi-classical limit with the trivial instanton identified precisely with the perturbative vacuum $A = 0$.

We will discuss here in great detail the theta term, the role of vacuum degeneracy, the quantization of the topological charge and the role of topology in instanton physics. More precisely, the instanton is defined as a solution of the self-duality equation with zero/finite energy which happens to saturate the Bogomol'nyi bound. The BPST instanton solution is then derived explicitly. The original literature on the BPST instanton is the paper by Belavin, Polyakov, Schwartz and Tyupkin [9]. We then discuss in some detail the moduli space, the collective coordinates, the zero modes, the ADHM construction, the one-loop quantization in the background of

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instantons as well as the connection of instantons to quantum tunneling. We have benefited here greatly from the pedagogical presentations found in [6, 10, 11].

14.1 Monopoles

14.1.1 The Georgi–Glashow model

Since the Higgs mechanism plays an essential role here, we will consider a non-abelian gauge field A_μ coupled to a scalar field ϕ in the adjoint representation, with a Lagrangian density given by the Georgi–Glashow model [12]. We will assume an $SU(N)$ gauge group with gauge coupling constant g. The covariant derivative and the field strength are given, respectively, by $D_\mu = \partial_\mu + igA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. Explicitly, the Georgi–Glashow model is given by

$$\mathcal{L}_{GG} = -\frac{1}{2}\text{tr}F_{\mu\nu}F^{\mu\nu} + \text{tr}(D_\mu\phi)(D^\mu\phi) - V(\phi). \tag{14.1}$$

The first term is the Yang–Mills Lagrangian density for the pure gauge field A_μ, whereas the second term is the covariant kinetic term for the scalar field ϕ, and since the scalar field is in the adjoint representation, the covariant derivative is defined via commutators, viz $D_\mu\phi = \partial_\mu\phi + ig[A_\mu, \phi]$. The Yang–Mills term can also be rewritten in terms of the electric and magnetic fields as

$$-\frac{1}{2}\text{tr}F_{\mu\nu}F^{\mu\nu} = \text{tr}(\vec{E}^2 - \vec{B}^2), \quad E^i = -F^{0i}, \quad B^i = -\frac{1}{2}\epsilon^{ijk}F^{jk}. \tag{14.2}$$

The third and final term in the above Lagrangian density is the potential term which specifies the self-interaction of the scalar Higgs field ϕ and is taken to be a ϕ^4-interaction, viz

$$V(\phi) = \frac{1}{4}\lambda(\phi^a\phi^a - F^2)^2, \quad \lambda > 0. \tag{14.3}$$

Since this is a non-abelian gauge theory, the gauge field A_μ, the field strength $F_{\mu\nu}$, the scalar Higgs field ϕ and the covariant derivative $D_\mu\phi$ are actually matrix-valued functions of spacetime. If we let λ_a be the generators of the gauge group $SU(N)$, viz $\text{tr}\lambda_a\lambda_b = 2\delta_{ab}$, and $[\lambda^a, \lambda^b] = 2if_{abc}\lambda^c$, where f_{abc} are the structure constants of the group, then we can expand all the fields as

$$A_\mu = A_\mu^a\frac{\lambda^a}{2}, \quad \phi = \phi^a\frac{\lambda^a}{2}, \quad F_{\mu\nu} = F_{\mu\nu}^a\frac{\lambda^a}{2}, \quad D_\mu\phi = (D_\mu\phi)^a\frac{\lambda^a}{2}. \tag{14.4}$$

The index a is called the color index, which runs from 1 to $N^2 - 1$, transforms as a vector under the gauge group. The components A_μ^a, ϕ^a, $(D_\mu\phi)^a$ and $F_{\mu\nu}^a$ are called the color components of the fields A_μ, ϕ, $(D_\mu\phi)$ and $F_{\mu\nu}$ respectively. For $SU(2)$ gauge group, we have three generators which are precisely the Pauli matrices, and $f_{abc} = \epsilon_{abc}$.

The equations of motion are given by the inhomogeneous field equations

$$\left[D_\mu,\, F^{\mu\nu}\right] = j^\nu\,, \quad j^\nu = g f^{abc}\phi^b (D^\nu \phi)^c, \tag{14.5}$$

and the homogeneous field equations (Jacobi identities)

$$\left[D_\mu,\, \tilde{F}^{\mu\nu}\right] = 0, \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{14.6}$$

The field A_μ transforms under the gauge transformation $U \in SU(N)$ in such a way as to make the transformation law for the matter field $D_\mu\phi$ covariant, i.e. to make $D_\mu\phi$ transforms in the same way as ϕ. This transformation law is given explicitly by

$$A_\mu \longrightarrow U\left(A_\mu + \frac{1}{ig}\partial_\mu\right)U^\dagger. \tag{14.7}$$

The fact that the fields ϕ, $D_\mu\phi$ and $F_{\mu\nu}$ transform in the adjoint representation under the gauge transformation $U \in SU(N)$ means that

$$\phi \longrightarrow U\phi U^\dagger, \quad D_\mu\phi \longrightarrow U D_\mu\phi U^\dagger, \quad F_{\mu\nu} \longrightarrow U F_{\mu\nu} U^\dagger. \tag{14.8}$$

If we want to apply the Hamiltonian or canonical formalism to this problem, then we must fix the gauge, i.e. fix the symmetry under the above gauge transformations. The Hamiltonian is the Legendre transformation of the Lagrangian. We then need to compute the conjugate momenta given by the derivatives of the Lagrangian with respect to the time derivatives of the fields. The first observation is that the time derivative $\partial_0 A_0$ does not appear in the Lagrangian density and hence the conjugate momenta associated with A_0 vanishes identically. This means that A_0 is actually a Lagrange multiplier, i.e. it is not an independent dynamical variable, and hence it should be eliminated. The corresponding equation of motion (the 0-component of equation (14.5)) can be rewritten as $[D_i,\, E^i] = j^0$, and since it does not involve a time derivative, i.e. it is strictly speaking a constraint equation (Gauss's law) not an equation of motion. Eliminating A_0 can be done explicitly by choosing the Weyl or temporal gauge given by

$$A_0 = 0. \tag{14.9}$$

This fixes the gauge, i.e. it eliminates A_0, as we can check. This gauge is unchanged under time-independent gauge transformations which constitute the residual gauge symmetry in this case. These residual time-independent gauge symmetries are generated infinitesimally by the Gauss operator $Q = [D_i,\, E^i]$ which also commutes with the Hamiltonian. The fact that states which differ by time-independent gauge transformations are gauge equivalent translates to the requirement that the physical states (the gauge-invariant states) of the theory must satisfy Gauss's law, viz

$$[D_i,\, E^i]|\psi\rangle = j^0|\psi\rangle. \tag{14.10}$$

After gauge fixing, and the implementation of the Gauss's law as a constraint, only a global gauge symmetry group remains which we will also denote by $G = SU(N)$.

The conjugate momenta associated with the fields A_i and ϕ are given, respectively, by

$$\frac{\delta \mathcal{L}_{GG}}{\delta(\partial_0 A_i^a)} = -F^{0ia} = E^{ia}, \quad \frac{\delta \mathcal{L}_{GG}}{\delta(\partial_0 \phi^a)} = (D_0\phi)^a = \pi^a. \tag{14.11}$$

The Hamiltonian density is then found to be given by

$$\mathcal{H} = \frac{1}{2}(\vec{E}^{a2} + \vec{B}^{a2}) + \frac{1}{2}\pi^a\pi^a + \frac{1}{2}((D_i\phi)^a)^2 + V(\phi). \tag{14.12}$$

14.1.2 Ground state configurations

We will consider the gauge group $G = SU(2)$ for simplicity.

We note that the Hamiltonian density is a sum of positive definite terms. This means that the energy density of the field configurations with the lowest energy vanishes identically. The vanishing of $\vec{E}$ and π signifies that the lowest energy field configurations are in fact static. They must also have vanishing magnetic field $\vec{B}$ and minimize the potential. The vanishing of the magnetic field can be solved by $\vec{A} = 0$ while the vanishing of the potential can be solved by a constant ϕ which satisfies $V(\phi) = 0$. We have then the lowest energy field configurations

$$\vec{A} = 0, \quad \phi = \phi_0, \quad V(\phi_0) = 0. \tag{14.13}$$

These are static solutions of the Euler–Lagrange equations. There is, in this, case a vacuum degeneracy, i.e. there exists an infinite number of field configurations which solve these equations and thus minimize the energy density. Indeed, we have

$$V(\phi) = 0 \Rightarrow \phi^a\phi^a = F^2. \tag{14.14}$$

Thus, ϕ^a lives on a sphere $\mathbf{S}^2$ of radius F. This is the manifold of zeros of the potential energy. In other words, the vanishing of the potential V completely determines the modulus F of the Higgs field ϕ^a and only leaves its orientation n^a in color space arbitrary, i.e. $\phi^a = \phi_0^a = Fn^a$.

The Higgs field exhibits, therefore, a spontaneous orientation in color space which breaks the remaining global symmetry group $G = SU(2)$ down to the stability (little) isotropy group of transformations which leaves ϕ_0 invariant. Obviously, these residual symmetries form a subgroup H of the group of global rotations in color space $G = SU(2)$ which rotate the Higgs field ϕ_0 around its axis n^a. Since the sphere is a homogeneous space, any point on it can be reached starting from ϕ_0 by the application of a rotation $g \in G$. In other words, the action of $G = SU(2)$ leaves the gauge field and the modulus of the Higgs field invariant, whereas the action of H leaves the orientation of the Higgs field invariant as well. The stability group is given explicitly by

$$H = \{h \in G: h\phi_0 = \phi_0\} = U(1) \otimes \mathbf{Z}_2. \tag{14.15}$$

Z_2 is the center element of $G = SU(2)$ consisting of the two elements $\mathbf{1}$ and $-\mathbf{1}$. This division by the center is due to the fact that rotations on the sphere actually form the group $SO(3)$ (and not $SU(2)$) which is not simply connected, i.e. $SO(3) = SU(2)/\mathbf{Z}_2$.

The space of zeros of V given by the sphere $\mathbf{S}^2$ and the coset space G/H are homeomorphic to each other. We write $G/H = SU(2)/(U(1) \otimes \mathbf{Z}_2) \sim \mathbf{S}^2$. This homeomorphism is given explicitly by

$$\begin{aligned} f\colon\ & G/H \longrightarrow \mathbf{S}^2 \\ & \tilde{g} \longrightarrow f(\tilde{g}) = g\phi_0 = \phi. \end{aligned} \tag{14.16}$$

The group element g is a representative of the coset $\tilde{g}$. The above mapping is bijective (injective or one-to-one and subjective or onto), continuous, and invertible, and hence it is a homeomorphism.

The above spontaneous symmetry breaking of $SU(2)$ down to $U(1) \otimes \mathbf{Z}_2$ is, therefore, a consequence of vacuum degeneracy. It can be shown (in the unitary gauge) that the gauge field acquires a mass which is determined by the value of the Higgs field and as a consequence we obtain short-range gauge interactions. This is the Higgs phase to be contrasted with the more familiar Coulomb phase.

14.1.3 't Hooft–Polyakov monopole

Non-trivial configurations

The above trivial field configurations are characterized by lowest energy and thus they define the ground state. The topologically non-trivial field configurations are characterized by finite energy (among other things). These finite energy configurations can be obtained from equation (14.12) if, as before, $\vec{E} = \pi = 0$ (static configurations), and $\vec{B}$ and $(D_i\phi)^a$ vanish asymptotically, and furthermore, the potential is also minimized asymptotically. We write the requirement that the potential energy is asymptotically minimized as

$$\phi(\vec{x}) \longrightarrow Fn^a(\vec{x}), \quad |\vec{x}| \longrightarrow \infty. \tag{14.17}$$

Now, the unit vector n^a (which characterizes the orientation of the Higgs field) is a function on the sphere $\mathbf{S}^2$ since $|\vec{x}| \longrightarrow \infty$. Therefore, $\phi(\vec{x})$ is a mapping from the sphere which bounds space at infinity to the sphere of vacuum field configurations (which are elements of the algebra $G = SU(2)$ with fixed length F). In other words,

$$\phi:\ \mathbf{S}^2 \longrightarrow \mathbf{S}^2 \sim G/H. \tag{14.18}$$

This mapping is a topological mapping which will be characterized by a winding number (degree of the mapping) which is an integer. Indeed, the winding number is the number of times the mapping ϕ winds around its sphere (the sphere of vacuum configurations) as we wind around the boundary sphere once. We will also show that this degree, or winding number, is equal the magnetic charge of the monopole.

Recall that ϕ is actually an element of the coset space $G/H = SU(2)/(U(1) \otimes \mathbf{Z}_2)$. Thus, mathematically, the winding number (the degree or the magnetic charge) is related to the second homotopy group of $G/H = SU(2)/(U(1) \otimes \mathbf{Z}_2)$. This second homotopy group is equal to the first homotopy group of $U(1)$ where this $U(1)$ is the connected component of $H = U(1) \otimes \mathbf{Z}_2$ to the identity element. The second homotopy group is the group of mappings from the sphere into the manifold, whereas the first homotopy group is the group of mappings from the circle into the manifold. The above mapping is then characterized by the second homotopy group given explicitly by

$$\pi_2(SU(2)/(U(1) \otimes \mathbf{Z}_2)) = \pi_1(U(1)) = \mathbf{Z}. \tag{14.19}$$

Thus, the above mapping is characterized by a single topological object given by a monopole carrying a quantized (since it is proportional to the winding number) magnetic charge g_m under the unbroken abelian subgroup $U(1)$. The non-triviality of this homotopy group is what guarantees the stability of the corresponding monopole configurations.

The other requirements for finite energy configurations (in addition to equation (14.17)) are

$$\vec{B} \longrightarrow 0, \quad |\vec{x}| \longrightarrow \infty \tag{14.20}$$

$$(D_i\phi)^a(x) \longrightarrow 0, \quad |\vec{x}| \longrightarrow \infty. \tag{14.21}$$

In the last equation we see that the gauge and the Higgs fields on the sphere which bounds space at infinity are correlated such that

$$(D_i\phi)^a(x) = \partial_i\phi^a(x) - g\epsilon^{abc}A_i^b(x)\phi^c(x) = 0, \quad |\vec{x}| \longrightarrow \infty. \tag{14.22}$$

We decompose the gauge field $A_i^a(x)$ in color space (index a) into parallel and perpendicular components with respect to the direction of the Higgs field ϕ. A solution of the above equation is given by

$$A_i^a(x) = \frac{1}{F}\phi^a(x)\mathcal{A}_i(x) + \frac{1}{gF^2}\epsilon^{abc}\phi^b(x)\partial_i\phi^c(x), \quad |\vec{x}| \longrightarrow \infty. \tag{14.23}$$

The first term is the parallel component while the second term is the perpendicular component. The field $\mathcal{A}_i$ is the abelian gauge field corresponding to the remaining unbroken $U(1)$ group living in the monopole background. Indeed, we calculate the field strength

$$\begin{aligned} F_{ij}^a(x) = &\frac{1}{F}\phi^a(x)F_{ij}(x), \quad F_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i \\ &- \frac{1}{gF^3}\epsilon^{abc}\phi^a\partial_i\phi^b\partial_j\phi^c, \quad \partial^i F_{ij} = 0. \end{aligned} \tag{14.24}$$

We compute the magnetic charge q_m as the flux of the magnetic field through the boundary sphere, viz (with $n^a = \phi^a/F$)

$$\begin{aligned} q_m &= \int_{S^2} \vec{B} \cdot d\vec{\sigma} \\ &= -\int_{S^2} (\vec{\nabla} \times \vec{\mathcal{A}}) \cdot d\vec{\sigma} + \frac{1}{2g} \int_{S^2} \epsilon^{ijk} \epsilon^{abc} n^a \partial^j n^b \partial^k n^c d\sigma^i \\ &= \frac{4\pi n}{g}. \end{aligned} \tag{14.25}$$

The first term in the second line in the above equation vanishes by using Stokes' theorem and noting that the sphere has no boundary, whereas in the second term we have used the fact that the winding number of the mapping $n^a = \phi^a/F$, which by construction is equal to an integer n, is given explicitly by the relation

$$n = \frac{1}{8\pi} \int_{S^2} \epsilon^{ijk} \epsilon^{abc} n^a \partial^j n^b \partial^k n^c d\sigma^i. \tag{14.26}$$

Hence, we conclude that the magnetic charge is quantized.

't Hooft–Polyakov monopole

An extremely important monopole solution is the 't Hooft–Polyakov monopole which is a hedgehog-like spherically-symmetric regular solution of the form

$$\phi^a(\vec{x}) = F\frac{x^a}{r} = Fn^a, \ \sum_{a=1}^{3} n^a n^a = 1. \tag{14.27}$$

The coordinates x^a on the asymptotic sphere are given by the coordinates σ^i. This is a mapping with winding number equal to 1, i.e. this solution winds around the sphere of vacuum solutions once as we wind around the asymptotic sphere once. This can be calculated as follows (where we use $\epsilon^{ijk} d\sigma^i = d\sigma^j \wedge d\sigma^k$)

$$\begin{aligned} n &= \frac{1}{8\pi} \int_{S^2} \epsilon^{ijk} \epsilon^{abc} n^a \partial^j n^b \partial^k n^c d\sigma^i \\ &= \frac{1}{8\pi} \int_{S^2} \epsilon^{abc} n^a dn^b \wedge dn^c \\ &= \frac{1}{8\pi} \int_{S^2} (-2d \cos\theta \wedge d\phi) \\ &= 1. \end{aligned} \tag{14.28}$$

The magnetic charge of this monopole is, therefore, given by

$$q_m = \int_{S^2} \vec{B} \cdot d\vec{\sigma} = \frac{4\pi}{g}. \tag{14.29}$$

The corresponding field strength tensor is given by (we use $\partial^i \phi^a = F(\delta^{ia} - n^i n^a)/r$ and setting $\mathcal{A}_i = 0$)

$$F^{ij} = -\frac{1}{gr^2} n^a \epsilon^{aij}. \tag{14.30}$$

We obtain the magnetic field

$$B^i = -\frac{1}{2}\epsilon^{ijk}F^{jk} = +\frac{1}{gr^2}n^i = \frac{q_m}{4\pi r^2}n^i. \tag{14.31}$$

Thus, for an observer on the asymptotic sphere it looks like we have a magnetic field generated by a magnetic monopole with charge q_m sitting at the origin. The corresponding gauge potential can be found from the condition (14.22). We have

$$\partial_i \phi^a = g\epsilon^{abc}A_i^{\ b}\phi^c \Rightarrow \delta^{ai} - n^a n^i = g\epsilon^{abc}A_i^{\ b}x^c. \tag{14.32}$$

We get

$$A_i^{\ a} = -\frac{1}{gr^2}\epsilon^{aij}x^j. \tag{14.33}$$

14.1.4 Dirac quantization condition and Wu–Yang potential

We could try also to find the gauge potential associated with the magnetic field

$$\vec{B} = \frac{g_m}{r^2}\hat{r} \tag{14.34}$$

by the usual expression $\vec{B} = \vec{\nabla} \times \vec{A}$ (with $g_m = q_m/4\pi$). However, this expression leads to $\vec{\nabla}\vec{B} = 0$ which is contradictory to the fact that we have a magnetic charge at the origin. We discuss this crucial point in some more detail next.

Since the magnetic field is spherically symmetric, i.e. $\vec{B}$ is in the direction of $\hat{r}$, and since we are assuming the relation $\vec{B} = \vec{\nabla} \times \vec{A}$ anyway, the corresponding gauge field $\vec{A}$ must be circular, i.e.

$$\vec{A} = -g_m\frac{1}{r}A(\theta)\hat{\phi}. \tag{14.35}$$

We must then have

$$\begin{aligned}\frac{g_m}{r^2}\hat{r} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi \sin\theta)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta} \Rightarrow A(\theta) \\ &= \frac{1+\cos\theta}{\sin\theta}.\end{aligned} \tag{14.36}$$

We get the solution

$$\begin{aligned}\vec{A} &= g_m\frac{1}{r}\left(\frac{1+\cos\theta}{\sin\theta}\sin\phi, -\frac{1+\cos\theta}{\sin\theta}\cos\phi, 0\right) \\ &= g_m\frac{1}{r}\frac{\vec{r}\times\hat{z}}{r - \vec{r}\hat{z}}.\end{aligned} \tag{14.37}$$

This is the Dirac monopole gauge potential [13]. This is obviously singular along the line $\theta = 0$, i.e. along the positive z-axis. This is the famous Dirac string. Its appearance is only due to the fact that we are forcing the use of the law $\vec{B} = \vec{\nabla} \times \vec{A}$ in spite of the presence of a monopole at the origin. Indeed, the magnetic field associated with the above gauge potential is not equation (14.31) but it is given by [7]

$$B^i = g_m\left(\frac{n^i}{r^2} - 4\pi k^i \theta(z)\delta(x)\delta(y)\right). \tag{14.38}$$

We can verify that the flux of the Coulomb field (first term) cancels exactly the flux of the Dirac string (step function term) and hence the Dirac equation $\vec{\nabla} \cdot \vec{B} = 0$ is maintained.

Although the existence of the Dirac string is gauge invariant, its position is gauge dependent. To see this we consider a $U(1)$ gauge transformation (with $U = \exp(ie\Lambda)$)

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} - \frac{i}{e}U^{-1}\vec{\nabla}U = \vec{A} + \vec{\nabla}\Lambda. \tag{14.39}$$

The gauge function Λ is typically taken to be single-valued. However, it can also be taken to be a multivalued function thus giving rise to extra singular terms in the transformed gauge potential $\vec{A}'$ and its associated magnetic field $\vec{B}'$. Indeed, we can check that the magnetic flux $\Phi = \int_\Sigma \vec{B}d\vec{S}$ is constant only if the gauge function satisfies $\Lambda(\phi + 2\pi) = \Lambda(\phi)$, i.e. it is single-valued, otherwise $\Delta\Phi = \int_\Sigma (\vec{B}' - \vec{B})d\vec{S} \neq 0$ [7].

Let us then consider a multivalued gauge function $\Lambda = 2g_m\phi$. We will denote $\vec{A} = \vec{A}^S$ and $\vec{A}' = \vec{A}^N$. Then we calculate

$$\begin{aligned}\vec{A}^N &= \vec{A}^S + 2g_m\vec{\nabla}\phi = -\frac{g_m}{r}\frac{1+\cos\theta}{\sin\theta}\hat{\phi} + 2g_m\frac{\hat{\phi}}{r\sin\theta} \\ &= \frac{g_m}{r}\frac{1-\cos\theta}{\sin\theta}\hat{\phi}.\end{aligned} \tag{14.40}$$

This potential is now singular along the line $\theta = \pi$. The Dirac string is now sitting along the negative z-axis. In other words, the position of the Dirac string is gauge dependent.

In conclusion, the monopole gauge potential cannot be smoothly defined everywhere in space although the corresponding field strength is globally defined. However, a non-singular monopole gauge potential can still be constructed following Wu and Yang [14] which provides one of the most interesting examples of the profound connection existing between physics and topology. This potential is obtained by first re-parametrizing the space $\mathbf{R}^3/\{0\}$ surrounding the monopole by two overlapping coordinate patches (not a single coordinate patch as usual) covering $z > -\epsilon$ (north hemisphere $\mathbf{R}^N$) and $z < +\epsilon$ (south hemisphere $\mathbf{R}^S$), then taking the gauge potentials in these two regions to be $\vec{A}^N$ and $\vec{A}^S$ respectively, viz

$$\begin{aligned}\vec{A}^N &= +\frac{g_m}{r}\frac{1-\cos\theta}{\sin\theta}\hat{\phi},\ 0 \leqslant \theta \leqslant \frac{\pi}{2}+\frac{\epsilon}{2}:\ \mathbf{R}^N \\ \vec{A}^S &= -\frac{g_m}{r}\frac{1+\cos\theta}{\sin\theta}\hat{\phi},\ \frac{\pi}{2}-\frac{\epsilon}{2} \leqslant \theta \leqslant 0:\ \mathbf{R}^S.\end{aligned} \tag{14.41}$$

These potentials are now free from singularities since their domain of definition is restricted to the regions where there are no Dirac strings. In the overlap region $\mathbf{R}^N \cap \mathbf{R}^S$ these two potentials are well defined and they are related by the above constructed singular gauge transformation (14.40).

We consider now the dynamics of an electric charge e, of mass m and described by a position vector $\vec{r}$, in the above monopole gauge field $\vec{A}$. The Lagrangian of the electron–monopole system is of the form

$$L = \frac{1}{2}m\dot{\vec{r}} + e\dot{\vec{r}}\vec{A}. \tag{14.42}$$

Obviously, the second term is the interaction term, i.e. the interaction action is given by (with $\vec{r}(T) = \vec{r}(0)$)

$$S_{\text{int}} = e\int_0^T \dot{\vec{r}}\vec{A}\,dt = e\oint \vec{dr}\cdot\vec{A}. \tag{14.43}$$

The wavefunction ψ of the electric charge can be represented as

$$\psi(\vec{r}) = \psi_0(\vec{r})\exp\left(ie\int_0^T \dot{\vec{r}}\vec{A}\,dt\right), \tag{14.44}$$

where ψ_0 is the free wavefunction, i.e. the interaction action is just a phase.

Under singular gauge transformations the interaction action S_{int} is not necessarily invariant, i.e. it may pick up additional phase factors, but the quantum-mechanical transition amplitude given by the wavefunction must always remain invariant. In other words, the wavefunction is always a continuous function of $\vec{r}$ although its phase can be discontinuous. This leads immediately to Dirac's charge quantization condition. This is the condition that the change of the phase factor along a closed path must be a multiple of 2π, viz

$$e\oint \vec{dr}\cdot\vec{A} = 2\pi n. \tag{14.45}$$

We do this extremely important calculation explicitly. Let l be a closed path in the overlap region $\mathbf{R}^N \cap \mathbf{R}^S$. When the electric charge e is transported along this closed path once its wavefunction picks up precisely the phase factor (14.43) with $\vec{A}$ being either $\vec{A}^N$ and $\vec{A}^S$ both of which are well defined in the overlap region. By employing Stokes's theorem in the northern hemisphere we have

$$\begin{aligned}e\oint_l \vec{dr}\cdot\vec{A}^N &= e\int_{\mathbf{R}^N} \vec{dS}\cdot\vec{B} \\ &= e\int_{\mathbf{R}^N} r^2 d\Omega\frac{g_m}{r^2} = 2\pi e g_m.\end{aligned} \tag{14.46}$$

But by employing Stokes's theorem in the southern hemisphere we obtain an extra minus sign due to the opposite orientation of the element of surface $\vec{dS}$. We have

$$\begin{aligned} e\oint_l \vec{dr}\cdot\vec{A}^S &= -e\int_{\mathbf{R}^S}\vec{dS}\cdot\vec{B} \\ &= -e\int_{\mathbf{R}^S} r^2 d\Omega\frac{g_m}{r^2} = -2\pi e g_m. \end{aligned} \tag{14.47}$$

Hence, the variation of the interaction action under the singular gauge transformation which takes us from $\vec{A}^N$ to $\vec{A}^S$ is

$$\Delta S_{\text{int}} = e\int_{\mathbf{R}^N\bigcup\mathbf{R}^S}\vec{dS}\cdot\vec{B} = 4\pi e g_m. \tag{14.48}$$

This is not physically observable and hence we must have the quantization condition

$$eq_m = 2\pi n. \tag{14.49}$$

This is identical to the topological quantization condition (14.25) with the identification $g \leftrightarrow 2e$. This can also be put into the form

$$g_m = \frac{n}{g}. \tag{14.50}$$

The above Dirac–Wu–Yang abelian magnetic monopole is embedded into $SU(2)$ as $A_\mu \equiv A_\mu^{\text{Dirac}}\sigma_3/3$, i.e. $A_\mu^{\text{Dirac}} = A_\mu^3$. This will allow us to relate the singular Dirac–Wu–Yang abelian magnetic monopole with the regular 't Hooft–Polyakov non-abelian monopole via a singular gauge transformation. In other words, the string singularity of the Dirac monopole cancels exactly the singularity of the gauge transformation (provided the charge quantization condition is satisfied). We have explicitly the Dirac–Wu–Yang abelian magnetic monopole

$$\begin{aligned} &A_\mu^1 = A_\mu^2 = 0 \\ &A_0^3 = A_r^3 = A_\theta^3 = 0 \\ &A_\phi^3 = -\frac{g_m}{r}\frac{1+\cos\theta}{\sin\theta}. \end{aligned} \tag{14.51}$$

This configuration can be rewritten in a compact form as

$$A_\mu^a = -g_m\delta^{a3}(1+\cos\theta)\partial_\mu\phi. \tag{14.52}$$

The four-vector $\partial_\mu\phi$ is singular along the entire z-axis and it is given explicitly by (think about the gradient in spherical coordinates)

$$\partial_\mu\phi = \frac{1}{r\sin\theta}(0, -\sin\phi, \cos\phi, 0). \tag{14.53}$$

Under an $SU(2)$ gauge rotation U (characterized by three angles θ, ϕ and α on $\mathbf{S}^3$) this transforms as

$$A_\mu \longrightarrow A'_\mu = UA_\mu U^{-1} - \frac{i}{g} U\partial_\mu U^{-1}. \tag{14.54}$$

We consider the gauge transformation corresponding to the rotation of the unit vector on the sphere $\mathbf{S}^2$ to the third axis in color space, i.e. θ is the polar angle and ϕ is the azimuthal angle on the ordinary sphere $\mathbf{S}^2$ and $\alpha = 0$. This rotation is given explicitly by

$$\begin{aligned} U &= \exp\left(i\sigma_3\frac{\phi}{2}\right)\exp\left(i\sigma_2\frac{\theta}{2}\right)\exp\left(-i\sigma_3\frac{\phi}{2}\right) \\ &= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\exp(-i\phi) \\ \sin\frac{\theta}{2}\exp(i\phi) & \cos\frac{\theta}{2} \end{pmatrix}. \end{aligned} \tag{14.55}$$

After some calculation we arrive at the result [1, 15, 16]

$$A_i^{\ a'} = \epsilon_{iaj}\frac{x_j}{gr^2}. \tag{14.56}$$

This is the hedgehog 't Hooft–Polyakov non-abelian monopole (14.33).

14.1.5 BPS states, Bogomol'nyi bound and duality

BPS states and Bogomol'nyi bound

We have seen that the behavior of the Higgs and gauge fields on the asymptotic sphere which bounds space is given by

$$\phi^a(\vec{x}) = F\frac{x^a}{r}\ ,\ A_i^{\ a} = -\frac{1}{gr^2}\epsilon^{aij}x^j,\ \ |\vec{x}| \longrightarrow \infty. \tag{14.57}$$

Inside the monopole it is natural to write down the spherically-symmetric ansatz

$$\phi^a(\vec{x}) = F\frac{x^a}{r}H(Fgr),\ A_i^{\ a} = -\frac{1}{gr^2}\epsilon^{aij}x^j[1 - K(agr)]. \tag{14.58}$$

Clearly the boundary conditions are

$$H(\infty) = 1,\ \ K(\infty) = 1. \tag{14.59}$$

But also, in order for the energy to be finite in the core of the monopole, we must require

$$H(0) = 0,\ \ K(0) = 1. \tag{14.60}$$

This monopole solution is a local minimum of the energy and also it solves the field equations of motion. Thus, the functions H and K can be determined from the field equations or from the requirement that the energy must be minimal. The resulting system of non-linear coupled differential equations has no analytic solution except in the celebrated limiting case known as the Bogomol'nyi–Prasad–Sommerfield (BPS) limit.

Let us start by writing down the energy functional for static configurations as (using the equation of motion $\vec{D}\vec{B} = 0$, Stokes' theorem, the magnetic charge (14.25), and recalling that the covariant derivative acts in the adjoint representation, i.e. via commutators)

$$\begin{aligned}
E &= \int d^3\vec{x}\left[\frac{1}{2}\vec{B}^{a2} + \frac{1}{2}\vec{D}\phi^{a2} + V(\phi)\right] \\
&= \int d^3\vec{x}\left[\frac{1}{2}(\vec{B}^a \pm (\vec{D}\phi)^a)^2 + V(\phi) \mp \vec{B}^a(\vec{D}\phi)^a\right] \\
&= \int d^3\vec{x}\left[\frac{1}{2}(\vec{B}^a \pm (\vec{D}\phi)^a)^2 + V(\phi) \mp 2\mathrm{tr}\vec{B}(\vec{D}\phi)\right] \\
&= \int d^3\vec{x}\left[\frac{1}{2}(\vec{B}^a \pm (\vec{D}\phi)^a)^2 + V(\phi) \mp 2\mathrm{tr}\vec{D}(\vec{B}\phi)\right] \\
&= \int d^3\vec{x}\left[\frac{1}{2}(\vec{B}^a \pm (\vec{D}\phi)^a)^2 + V(\phi) \mp 2\mathrm{tr}\vec{\partial}(\vec{B}\phi)\right] \\
&= \int d^3\vec{x}\left[\frac{1}{2}(\vec{B}^a \pm (\vec{D}\phi)^a)^2 + V(\phi)\right] \mp \int_{\mathrm{S}^2} d\vec{\sigma}\vec{B}^a\phi^a \\
&= \int d^3\vec{x}\left[\frac{1}{2}(\vec{B}^a \pm (\vec{D}\phi)^a)^2 + V(\phi)\right] \mp \int_{\mathrm{S}^2} d\vec{\sigma}\left(\frac{\phi^a}{F}B^i\right)\phi^a \\
&= \int d^3\vec{x}\left[\frac{1}{2}(\vec{B}^a \pm (\vec{D}\phi)^a)^2 + V(\phi)\right] \mp Fq_m.
\end{aligned} \tag{14.61}$$

We choose the sign according to the sign of the magnetic charge or winding number to obtain (of course $n = 1$ in this particular case)

$$E = \int d^3\vec{x}\left[\frac{1}{2}(\vec{B}^a \pm (\vec{D}\phi)^a)^2 + V(\phi)\right] + \frac{4\pi F}{g}|n|. \tag{14.62}$$

The energy (a sum of positive terms) is thus bound by the third term which is proportional to the winding number, viz

$$E \geqslant \frac{4\pi F}{g}|n| = \frac{4\pi}{g^2}M_W|n|. \tag{14.63}$$

In other words, the energy of the monopole is bounded from below by the mass of the charged vector particle (or W-boson) $M_W = gF$ corresponding to the broken generators of the gauge group. This is the famous Bogomol'nyi bound. This system in this limit is therefore characterized by one single length scale which is given by the wavelength of the vector boson $1/M_W$ as opposed to the generic case where two different length scales characterize the changes in the Higgs and gauge fields.

This bound is saturated for vanishing Higgs potential

$$V = 0, \leftrightarrow \lambda = 0. \tag{14.64}$$

However, as we take the limit $\lambda \longrightarrow 0$, the asymptotic of the Higgs scalar field must remain the same. The vanishing of the Higgs potential occurs naturally in supersymmetric gauge theories with two and four supercharges.

The saturation of the Bogomol'nyi bound requires also field satisfying the so-called BPS monopole equations given by the equation

$$\vec{B}^a \pm (\vec{D}\phi)^a = 0. \tag{14.65}$$

This is a system of first-order differential equations for H and K with solution given by the Prasad–Sommerfield monopole

$$H(x) = x \coth x - 1, \;\; K(x) = \frac{x}{\sinh x}. \tag{14.66}$$

These functions reach their asymptotic values very fast. This is in fact a general result.

The region where the asymptotic behavior is obtained defines the Higgs vacuum where we have spontaneous symmetry breaking of $SU(2)$, there is no difference between the 't Hooft–Polyakov and the Dirac monopoles, and there is one single length scale given by the mass of the vector boson M_W (the asymptotic scalar field does not depend on λ). Inside the core of the monopole, which is of size $R_c = 1/M_W$, the original $SU(2)$ symmetry is restored and we obtain the perturbative phase of the Georgi–Glashow model with two independent length scales (related to g and λ).

Duality

The spectrum of the non-abelian Georgi–Glashow model with $SU(2)$ gauge group consists, therefore, of the following particles:

- A Higgs scalar particle of mass $m = 0$ and electric/magnetic charges $(q_e, q_m) = (0, 0)$.
- A vector particle (photon) of mass $m = 0$ and electric/magnetic charges $(q_e, q_m) = (0, 0)$.
- Two charged vector particles $A_\pm$ of mass $M_W = gF$ and electric/magnetic charges $(q_e, q_m) = (g, 0)$.
- A magnetic monopole, which is a spin 0 particle, of mass $M = 4\pi F/g$ and electric/magnetic charges $(q_e, q_m) = (0, 4\pi/g)$.

All these states saturate the Bogomol'nyi bound with mass

$$M = F\sqrt{q_e^2 + q_m^2}. \tag{14.67}$$

It is then obvious that in the weak coupling regime $g \longrightarrow 0$ (Coulomb phase) the magnetic monopole is very heavy, whereas the gauge and Higgs fields are very light. In the strong coupling regime $g \longrightarrow 0$ (Higgs phase) the magnetic monopole is light, whereas the gauge and Higgs fields are very heavy. The monopole is thus a non-perturbative topological configuration of the field, whereas the gauge and

Higgs fields are perturbative excitations over the vacuum only in the weak coupling regime.

The electric/magnetic Montonen–Olive duality [17] is defined by the exchange of the electric and magnetic fields and charges as $\vec{E} \longrightarrow \vec{B}$, $\vec{B} \longrightarrow -\vec{E}$ and

$$q_e = g \leftrightarrow q_m = \frac{4\pi}{g}. \tag{14.68}$$

This clearly exchanges the non-perturbative topological sector with the perturbative Noether sector, i.e. the massive vector bosons are mapped to the monopoles and vice versa. This is an example of a strong–weak coupling duality, i.e. an S-duality. Hence, under this duality, monopoles become perturbative fluctuations of some dual gauge theory, whereas the Higgs and gauge particles will constitute the non-perturbative sector of this dual theory. The Montonen–Olive conjecture states that in the strong coupling regime the theory is described by a spontaneously broken 'magnetic' gauge theory with some dual gauge group $G^\star$ (as opposed to the 'electric' gauge theory which describes the weak coupling regime). See [7, 18] for a brief and pedagogical discussion.

Also, it is important to note that the above Montonen–Olive duality transformation is actually only a particular $SO(2)$ rotation in the complex plane $q_e + iq_m$ by an angle $\theta = \pi/2$. The Montonen–Olive duality can also be extended to a full $SL(2, Z)$ duality by also including the θ parameter. The Georgi–Glashow model is self-dual under the above duality transformation since the Lagrangian and the gauge group remain invariant, i.e. $G^\star = SU(2)$. However, the invariance under duality is different from the invariance under symmetry since it actually relates two different theories describing weak and strong coupling regimes.

14.1.6 Other results

Moduli space and zero modes

The moduli space is the space of solutions $\mathcal{M} = \{\vec{A}, \phi\}$ modulo gauge transformations. It consists of disconnected components labeled by the winding number n, viz $\mathcal{M}_n$. These spaces are described by collective coordinates. Translational invariance, i.e. the fact that the monopole can be located anywhere in space, leads immediately to zero modes. The dimensionality of the moduli space in each topological sector, i.e. of $\mathcal{M}_n$, is exactly given by the number of square-integrable zero modes. This can be computed by calculating the index of the Dirac operator in the monopole background. The answer is very simply given by [19, 20].

$$\mathcal{M}_n = 4|n|. \tag{14.69}$$

The factor of four corresponds to three translations plus one phase associated with the unbroken $U(1)$ gauge rotations which is the number of collective coordinates of the fundamental monopole with $n = 1$. The $n = 1$ moduli space is $\mathbf{R}^3 \times \mathbf{S}^1$ [21].

Coupling to fermions
We can introduce massless fermions in the background monopole gauge field by a coupling theme through the usual minimal and Yukawa interactions. The 't Hooft–Polyakov monopole will then induce n fermionic zero modes where n is the winding number associated with the magnetic charge q_m as given by equation (14.25).

The spin/isospin coupling
The group of invariance of the energy functional with static fields is given by $SO(3) \times SO(3)$ where the first $SO(3)$ is spatial rotations (spin), whereas the second $SO(3)$ is color rotations (isospin). However, the static 't Hooft–Polyakov hedgehog monopole solution enjoys only a diagonal $SO(3)$ in which the space and color indices are mixed. For example, note that the hedgehog solution (14.33) does not transform covariantly under spatial rotations generated by $\vec{L}$. Thus, the monopole is actually invariant only under the combined rotations generated by the

$$\vec{K} = \vec{J} + \vec{I}, \tag{14.70}$$

where $\vec{J} = \vec{L} + \vec{S}$ is the total angular momentum and $\vec{I}$ corresponds to isospin or color rotations. The correct angular momentum operator should then be identified with $\vec{K}$, i.e.

$$K = |J - I|, \ldots, J + I - 1, J + I. \tag{14.71}$$

On the other hand, electric charges in the Georgi–Glashow model arise from $SU(2)$ representations. We have already seen that the minimal $U(1)$ electric charge q_e (written e in equation (14.42)) must be equal to $g/2$ where g is the $SU(2)$ gauge coupling constant. The Dirac quantization condition $gq_m = 4\pi$ becomes then $q_e q_m = 2\pi$. This minimal charge must transform in the fundamental representation under $SU(2)$ isospin/color rotations, viz

$$I = \frac{1}{2}, \ q_e = \frac{1}{2}g. \tag{14.72}$$

This generalizes to particles in other $SU(2)$ representations as

$$\begin{aligned} I &= n + \frac{1}{2}, \ q_e = \left(n + \frac{1}{2}\right)g \\ I &= n, \ q_e = ng. \end{aligned} \tag{14.73}$$

This is consistent with Dirac quantization condition $q_e q_m = 2\pi n$.

We consider now a charged particle with isospin $I = 1/2$ in the monopole background. This particle binds to the magnetic monopole in such a way that the total angular momentum of the bound state is $S \pm 1/2$ where S is the spin of the particle (the spin of the monopole is 0 and we are assuming zero orbital angular momenta). If the particle is a fermion then the bound state is a boson, whereas if the particle is a boson the bound state is a fermion, i.e. we can get fermions from bosons and bosons from fermions in the background of a 't Hooft–Polyakov magnetic field.

This is an example of the anomalous spin addition theorem for the charge-monopole bound state. This conversion of spin into isospin can be traced to the existence of Dirac strings. The original literature on this topic is [22–24].

Condition for the existence of monopoles
This is given by the requirement that the gauge group G must be a simply connected group such that the unbroken subgroup H contains $U(1)$ as a factor. In this case the second homotopy group is

$$\begin{aligned}\pi_2(G/(\tilde{H} \otimes U(1))) &= \pi_1(\tilde{H}) \otimes \pi_1(U(1)) \\ &= \pi_1(\tilde{H}) \otimes \mathbf{Z}^{N-1}.\end{aligned} \tag{14.74}$$

For example, magnetic monopoles fail to appear in the standard model because $SU(2) \times U(1)$ is not a simply connected group, i.e. since $\pi_2(SU(2)) = 0$.

Dyons
A magnetic monopole with an added electric charge is called a dyon [25].

Generalization to $SU(N)$
We have found that for $SU(2)$ the gauge symmetry is broken spontaneously to $U(1)$. As it turns out, $U(1)$ is the maximally abelian subgroup of $SU(2)$. Indeed, we can always choose the asymptotic solution $\phi \longrightarrow Fn^a$, which lies on a sphere $\mathbf{S}^2$, along the third axis and hence $n^1 = n^2 = 0$ and $n^3 = 1$, i.e. $\phi = (\phi_1, \phi_2) = F\sigma^3/2$ where $\phi_1 + \phi_2 = 0$ and $\phi_1 = F/2$.

For $SU(N)$ gauge group the symmetry will also be spontaneously broken down to the maximally abelian subgroup which is, in this case, given by the torus $U(1)^{N-1}$. The asymptotic solution lies on a sphere $\mathbf{S}^{N^2-2}$. In particular, in the BPS limit $\lambda \longrightarrow 0$, there is no Higgs potential and as such we can use gauge symmetry to rotate the Higgs field to its diagonal form. The asymptotic solution will take then the form

$$\phi = \text{diag}(\phi_1, \ldots, \phi_N) = \phi_a H^a, \ \sum_a \phi_a = 0. \tag{14.75}$$

The N matrices H^a are the generators of the Cartan subalgebra of $U(N)$ while ϕ^a are the components of a (root) vector in the space of the Cartan subalgebra. With the requirement $\phi_a \neq \phi_b$ for $a \neq b$ the gauge group $SU(N)$ breaks down to $U(1)^{N-1}$ which is the group which leaves the above configuration invariant. The spectrum consists, therefore, of $N-1$ massless photons, $N-1$ massless scalars, and $N(N-1)/2$ massive gauge bosons with mass $M_W = |\phi_a - \phi_b|$.

The topological classification of monopole solutions is then connected with the maps from the boundary sphere $\mathbf{S}^2$ onto the coset space $SU(N)/U(1)^{N-1}$, i.e. they are field configurations winding non-trivially around the asymptotic $\mathbf{S}^2$. Hence, we must consider the maps

$$\mathbf{S}^2 \longrightarrow SU(N)/U(1)^{N-1}. \tag{14.76}$$

As a consequence, the topological charge (winding number or magnetic charge) of the monopole is given by the elements of the second homotopy group

$$\pi_2(SU(N)/U(1)^{N-1}) = \pi_1(U(1)^{N-1}) = \mathbf{Z}^{N-1}. \tag{14.77}$$

Thus, this $SU(N)$ magnetic monopole carries a magnetic charge under each of the $N-1$ unbroken abelian gauge fields. This is also Dirac's charge quantization condition for the $SU(N)$ case in mathematical terms.

Vortices

These are the analogous topological excitations in $1+2$ dimensions relevant for superconductivity in the same way that monopoles are the topological excitations in $1+3$ dimensions relevant for confinement. These configurations appear in the abelian Georgi–Glashow–Higgs model with a $U(1)$ gauge group. This abelian group becomes spontaneously broken and the photon acquires a mass in the Higgs phase which determines the London penetration depth of the magnetic field. Indeed, the magnetic field is excluded from the superconducting region (Meissner effect).

The manifold of vacuum solutions in this case is also non-trivial given by a circle $\mathbf{S}^1$. Topological solutions are static configurations determined by the requirement of finite energy and their flux is quantized proportional to the winding number n of the mapping θ: $\mathbf{S}^1 \longrightarrow \mathbf{S}^1_\infty$ where θ is the phase of the Higgs field and $\mathbf{S}^1_\infty$ is the asymptotic circle bounding the plane. Also, the energy is determined to satisfy the Bogomol'nyi bound $E \geqslant 2\pi|n|$. This energy is saturated in some BPS-like limit in which the energy is given by the magnetic charge and the fields satisfy the Euler–Lagrange equations of motion.

14.2 Instantons

14.2.1 The theta term

The Yang–Mills action with the topological theta term is given by

$$S = -\frac{1}{2g^2}\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{16\pi^2}\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{14.78}$$

Recall that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2$ and $D_\alpha = \partial_\alpha - i[A_\alpha, \ldots]$. The path integral of interest is

$$Z = \int DA_\mu \exp(iS). \tag{14.79}$$

This is invariant under the finite gauge transformations $A_\mu \longrightarrow U^{-1}A_\mu U + iU^{-1}\partial_\mu U$ with $U = \exp(i\Lambda) \in G$ (we will consider mostly $SU(N)$).

We Wick rotate to Euclidean signature as $x^0 \longrightarrow x^4 = ix^0$ and as a consequence $d^4x \longrightarrow d^4_E x = id^4x$, $\partial_0 \longrightarrow \partial_4 = -i\partial_0$ and $A_0 \longrightarrow A_4 = -iA_0$. We compute $F_{\mu\nu}F^{\mu\nu} \longrightarrow (F^2_{\mu\nu})_E$ and $F_{\mu\nu}\tilde{F}^{\mu\nu} \longrightarrow i(F_{\mu\nu}\tilde{F}_{\mu\nu})_E$. We get then

$$Z_E = \int DA_\mu \exp(-S_E) \tag{14.80}$$

$$\begin{aligned} S_E = & \frac{1}{2g^2} \int (d^4x)_E \operatorname{Tr}\left(F_{\mu\nu}^2\right)_E \\ & + \frac{i\theta}{16\pi^2} \int (d^4x)_E \operatorname{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})_E . \end{aligned} \tag{14.81}$$

In the following we will drop the subscript E for simplicity. Note that the theta term is imaginary and, as we will show shortly, this term is in fact a surface term and thus it is actually proportional to an integer number k.

The path integral/partition function is of the form

$$\int DA_\mu \exp\left(-\frac{1}{2g^2} \int d^4x \operatorname{Tr} F_{\mu\nu}^2 - ik\theta\right). \tag{14.82}$$

In this form, it is clear that the semi-classical approximation $\hbar \longrightarrow 0$ is given by $g^2 \longrightarrow 0$. In this limit the path integral is clearly dominated by the trivial gauge configuration $A = 0$ with zero Euclidean action, i.e. the global minimum.

However, it is also clear that the above path integral will receive very important contributions from other minima, i.e. gauge configurations characterized by finite Euclidean action which are known as instantons. The trivial instanton is precisely the perturbative vacuum and the expansion around it is the perturbative loop expansion defined by the usual Feynman diagrams. The goal of instanton calculus is, among other things, to develop a systematic expansion around each instanton configuration of finite action $S_{\text{cl}} = \hat{S}_{\text{cl}}/g^2$. Thus, the leading contribution of an instanton configuration is proportional to $\exp(-\hat{S}_{\text{cl}}/g^2)$ which leads to corrections which are further suppressed by powers of g^2 (to be contrasted with the perturbative expansion around the trivial sector where the exponential factor is 1 and cannot be observed).

For finite theories such as the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory instantons seem to be the dominant configurations and the semi-classical approximation is valid since the coupling constant g^2 does not run[1]. For asymptotically free theories[2], the coupling constant g^2 runs with energy and in particular it becomes large in the infrared and as a consequence instantons do not dominate the path integral. In the presence of scalar fields the gauge group can become spontaneously broken and thus the running of the coupling constant g^2 may be appropriately cut off and the semi-classical approximation, and thus instantons, can still play a crucial

[1] Another crucial example in which instantons give the complete solution in the semi-classical approximation is compact QED and abelian gauge theories in general.

[2] For example, four-dimensional pure Yang–Mills gauge theories and (the very similar) two-dimensional principal chiral theories are given by the actions

$$S = \frac{1}{2g^2} \int d^2x \operatorname{Tr} \partial_\mu U^+ \partial_\mu U, \quad UU^\dagger = 1, \quad U \in G. \tag{14.83}$$

role. In three dimensions it seems that instantons play a decisive role, whereas in four dimensions, they play a major role.

Now we find the equations of motion in the trivial sector $\theta = 0$. We have

$$\begin{aligned}\delta S_E &= \frac{1}{g^2}\int d^4x \ \mathrm{Tr}\ F_{\mu\nu}\delta F_{\mu\nu} \\ &= \frac{2}{g^2}\int d^4x \ \mathrm{Tr}\ F_{\mu\nu}D_\mu \delta A_\nu \\ &= -\frac{2}{g^2}\int d^4x \ \mathrm{Tr}\ D_\mu F_{\mu\nu}\cdot \delta A_\nu + \frac{2}{g^2}\int d^4x \ \mathrm{Tr}\ D_\mu(F_{\mu\nu}\delta A_\nu) \\ &= -\frac{2}{g^2}\int d^4x \ \mathrm{Tr}\ D_\mu F_{\mu\nu}\cdot \delta A_\nu + \frac{2}{g^2}\int d^4x \ \mathrm{Tr}\ \partial_\mu(F_{\mu\nu}\delta A_\nu).\end{aligned} \tag{14.84}$$

The equations of motion for variations of the gauge field which vanish at infinity are, therefore, given by

$$D_\mu F_{\mu\nu} = 0. \tag{14.85}$$

Next, we consider the general sector $\theta \neq 0$. First we show that the second term in the action S_E does not affect the equations of motion. In other words, the theta term is only a surface term. We define

$$\mathcal{L}_\theta = \frac{1}{16\pi^2}\,\mathrm{Tr}\ F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{14.86}$$

We compute the variation

$$\begin{aligned}\delta\mathcal{L}_\theta &= \frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\ \mathrm{Tr}\ F_{\mu\nu}\delta F_{\alpha\beta} \\ &= \frac{1}{8\pi^2}\epsilon_{\mu\nu\alpha\beta}\ \mathrm{Tr}\ F_{\mu\nu}D_\alpha\delta A_\beta.\end{aligned} \tag{14.87}$$

We use the Jacobi identity

$$\begin{aligned}\epsilon_{\mu\nu\alpha\beta}D_\alpha F_{\mu\nu} &= \epsilon_{\mu\nu\alpha\beta}\big(\partial_\alpha F_{\mu\nu} - i[A_\alpha, F_{\mu\nu}]\big) \\ &= -\epsilon_{\mu\nu\alpha\beta}[A_\alpha, [A_\mu, A_\nu]] \\ &= 0.\end{aligned} \tag{14.88}$$

Thus

$$\begin{aligned}\delta\mathcal{L}_\theta &= \frac{1}{8\pi^2}\epsilon_{\mu\nu\alpha\beta}\ \mathrm{Tr}\ D_\alpha(F_{\mu\nu}\delta A_\beta) \\ &= \frac{1}{8\pi^2}\epsilon_{\mu\nu\alpha\beta}\ \mathrm{Tr}\left(\partial_\alpha(F_{\mu\nu}\delta A_\beta) - i[A_\alpha, F_{\mu\nu}\delta A_\beta]\right) \\ &= \partial_\alpha \delta\mathcal{K}_\alpha,\end{aligned} \tag{14.89}$$

where

$$\delta\mathcal{K}_\alpha = \frac{1}{8\pi^2}\epsilon_{\alpha\mu\nu\beta}\,\mathrm{Tr}\,F_{\mu\nu}\delta A_\beta. \tag{14.90}$$

This shows explicitly that the theta term will not contribute to the equations of motion for variations of the gauge field which vanish at infinity.

In order to find the current $\mathcal{K}_\alpha$ itself we consider a one-parameter family of gauge fields $A_\mu(x,\tau) = \tau A_\mu(x)$ with $0 \leqslant \tau \leqslant 1$ [26]. By using the above result we have immediately

$$\begin{aligned}\frac{\partial}{\partial\tau}\mathcal{K}_\alpha &= \frac{1}{8\pi^2}\epsilon_{\alpha\mu\nu\beta}\,\mathrm{Tr}\,F_{\mu\nu}(x,\tau)\frac{\partial}{\partial\tau}A_\beta \\ &= \frac{1}{8\pi^2}\epsilon_{\alpha\mu\nu\beta}\,\mathrm{Tr}\left(\tau\partial_\mu A_\nu - \tau\partial_\nu A_\mu - i\tau^2[A_\mu, A_\nu]\right)\cdot A_\beta(x).\end{aligned} \tag{14.91}$$

By integrating both sides with respect to τ between $\tau = 0$ and $\tau = 1$ and setting $\mathcal{K}_\alpha(x,1) = \mathcal{K}_\alpha(x)$ and $\mathcal{K}_\alpha(x,0) = 0$ we get the so-called Chern–Simons current

$$\begin{aligned}\mathcal{K}_\alpha &= \frac{1}{8\pi^2}\epsilon_{\alpha\mu\nu\beta}\,\mathrm{Tr}\left(\frac{1}{2}\partial_\mu A_\nu - \frac{1}{2}\partial_\nu A_\mu - \frac{i}{3}[A_\mu, A_\nu]\right)\cdot A_\beta(x) \\ &= \frac{1}{8\pi^2}\,\mathrm{Tr}\left(\tilde{F}_{\alpha\beta}A_\beta + \frac{i}{3}\epsilon_{\alpha\mu\nu\beta}A_\mu A_\nu A_\beta\right).\end{aligned} \tag{14.92}$$

14.2.2 Vacuum degeneracy and Bogomol'nyi bound

Vacuum degeneracy

The existence of instantons, like the existence of monopoles, can be traced down to the property of vacuum degeneracy. Instantons are static solutions of the classical equations of motion of Yang–Mills gauge theory which are also configurations of finite energy [9, 27, 28]. We will employ again the Weyl or temporal gauge $A_0 = 0$. The static solution will correspond as before to $\vec{E}^a = 0$ while vanishing energy corresponds to $\vec{B}^a = 0$. The vanishing of the magnetic field can be solved by $\vec{A}^a = 0$. The Weyl gauge is unchanged under time-independent gauge transformations which changes $\vec{A}^a = 0$ to the pure gauges

$$\vec{A} = iU^{-1}(\vec{x})\vec{\nabla}U(\vec{x}). \tag{14.93}$$

Thus, the manifold of zero-energy vacuum solutions is $U \in SU(2) \sim \mathbf{S}^3$. Hence, a given solution $U(\vec{x}) \in SU(2)$ defines a map from $\mathbf{R}^3$ to $\mathbf{S}^3$. As usual, we will impose the condition that at infinity the group element $U(\vec{x})$ approaches the identity in the group, viz

$$U(\vec{x}) \longrightarrow \mathbf{1},\ |\vec{x}| \longrightarrow \infty. \tag{14.94}$$

This effectively compactifies the space $\mathbf{R}^3$, i.e. identify the points at infinity, which produces the sphere $\mathbf{S}^3_\infty$ which is actually the boundary of $\mathbf{R}^4$ (we are implicitly

assuming Euclidean signature). In other words, $U(\vec{x})$ is a map from a 3-sphere to a 3-sphere, namely

$$U(\vec{x}):\ \mathbf{S}^3_\infty \longrightarrow SU(N)|_{N=2} \sim \mathbf{S}^3. \tag{14.95}$$

These maps can be classified by the third homotopy group. They can then be attributed a winding number known as the Pontryagin number or second Chern class given by

$$k \in \pi_3(SU(N)). \tag{14.96}$$

This number counts how many times we cover the group 3-sphere if $\vec{x}$ covers the space 3-sphere once, i.e. how many times the group wraps itself around spatial infinity $\mathbf{S}^3_\infty$.

The above winding number can be expressed in terms of the gauge field as follows [29, 30]

$$\begin{aligned} k &= \frac{1}{16\pi^2}\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}\tilde{F}_{\mu\nu} \\ &= \int d^4x \partial_\alpha \mathcal{K}_\alpha \\ &= \int_{\mathbf{S}^3_\infty} d^3\sigma_\alpha \mathcal{K}_\alpha \\ &= \frac{1}{16\pi^2}\epsilon_{\alpha\mu\nu\beta}\int_{\mathbf{S}^3_\infty} d^3\sigma_\alpha\left(\partial_\mu A^a_\nu A^a_\beta + \frac{1}{3}f_{abc}A^a_\mu A^b_\nu A^c_\beta\right). \end{aligned} \tag{14.97}$$

The first line is precisely the theta term. The last line, as we will show explicitly, gives an integer only of A_μ is a pure gauge. The winding number k can also be rewritten in terms of the gauge element U as

$$k = \frac{1}{24\pi^2}\epsilon_{\alpha\mu\nu\beta}\int_{\mathbf{S}^3_\infty} d^3\sigma_\alpha\ \mathrm{Tr}\left(U^{-1}\partial_\mu U\right)\left(U^{-1}\partial_\nu U\right)\left(U^{-1}\partial_\beta U\right). \tag{14.98}$$

We have, therefore, an infinite number of vacuum solutions with vanishing energy characterized by an integer number k. These topological sectors are connected by large gauge transformations which can change the winding number k. We note that 'large gauge transformations', in contrast to the usual transformations which are also termed 'small gauge transformations', are not continuously connected to the identity of the gauge group.

In order to get finite, not vanishing, energy we will require the group elements $U(\vec{x})$ to only approach the pure gauges (14.93) as we approach the spatial infinity $\mathbf{S}^3_\infty = \partial\mathbf{R}^4$. We will have then an infinite number of vacuum solutions with finite energy characterized by the winding number k.

Bogomol'nyi bound

We go back to our original action

$$\begin{aligned} S &= \frac{1}{2g^2}\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}^2 + \frac{i\theta}{16\pi^2}\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}\tilde{F}_{\mu\nu} \\ &= \frac{1}{2g^2}\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}^2 + i\theta k. \end{aligned} \tag{14.99}$$

Using $\tilde{F}_{\mu\nu}^2 = \epsilon_{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}/4 = F_{\mu\nu}^2/2$ we have

$$\begin{aligned} \frac{1}{2g^2}\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}^2 &= \frac{1}{4g^2}\int d^4x\ \mathrm{Tr}(F_{\mu\nu} \mp \tilde{F}_{\mu\nu})^2 \\ &\quad \pm \frac{1}{2g^2}\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}\tilde{F}_{\mu\nu} \\ &= \frac{1}{4g^2}\int d^4x\ \mathrm{Tr}(F_{\mu\nu} \mp \tilde{F}_{\mu\nu})^2 \pm \frac{8\pi^2 k}{g^2}. \end{aligned} \tag{14.100}$$

We note that for $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$ the action takes the value $\pm 8\pi^2 k/g^2$. Thus, for $F_{\mu\nu} = +\tilde{F}_{\mu\nu}$ and $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$ we must have $k > 0$ and $k < 0$, respectively, since the action is positive definite. We conclude, therefore, that the Yang–Mills action in an instanton configuration with topological charge k is bounded by

$$\frac{1}{2g^2}\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}^2 \geqslant \frac{8\pi^2|k|}{g^2}. \tag{14.101}$$

This is the Bogomol'nyi bound. The equality holds if and only if the so-called duality equations (instanton equations) are satisfied, viz

$$F_{\mu\nu} = \tilde{F}_{\mu\nu},\ k > 0,\ \text{instantons} \tag{14.102}$$

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu},\ k < 0,\ \text{anti-instantons}. \tag{14.103}$$

We will focus on the self-dual solution corresponding to $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ since parity maps positive k to negative k and vice versa. As we already noted the self-dual and anti-self-dual solutions minimize the action in a given instanton solution (i.e. for a fixed k) but also they actually solve the equations of motion.

Indeed, it is obvious that since $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$ we have $D_\mu F_{\mu\nu} = D_\mu\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}D_\mu F_{\alpha\beta}/2 = 0$ which is the Bianchi identity. In some sense the duality equations, which are first-order, are the square root of the full equations of motion. A solution of the equations of motion does not necessarily solve the duality equations. The values of the Yang–Mills action in the self-dual and anti-self-dual solutions are

$$S = \frac{8\pi^2|k|}{g^2} + i\theta k. \tag{14.104}$$

Alternatively,

$$S = 2\pi i k\tau^*,\ k > 0,\ S = 2\pi' k\tau,\ k < 0. \tag{14.105}$$

The complex coupling is

$$\tau = \frac{4i\pi}{g^2} + \frac{\theta}{2\pi}. \tag{14.106}$$

14.2.3 The topological charge or instanton number

Before we find the explicit one-instanton solution due to Belavin, Polyakov, Schwartz and Tyupkin [9] we discuss a little further the topological charge, i.e. the integral over spacetime of the theta term given by

$$\begin{aligned} k &= \frac{1}{16\pi^2}\int d^4x \ \mathrm{Tr}\ F_{\mu\nu}\tilde{F}_{\mu\nu} \\ &= \frac{1}{16\pi^2}\epsilon_{\alpha\mu\nu\beta}\int_{\mathbf{S}^3_\infty} d^3\sigma_\alpha\left(\partial_\mu A_\nu^a A_\beta^a + \frac{1}{3}f_{abc}A_\mu^a A_\nu^b A_\beta^c\right). \end{aligned} \tag{14.107}$$

A Yang–Mills instanton is a solution of the equations of motion which has finite action. In order to have a finite action the field strength $F_{\mu\nu}$ must approach 0 at infinity faster than $1/x^2$, viz[3]

$$F^I_{\mu\nu}(x) = O\left(\frac{1}{|\vec{x}|^2}\right), \ |\vec{x}|^2 \longrightarrow \infty. \tag{14.108}$$

We can immediately deduce that the gauge field must approach a pure gauge at infinity, viz

$$A^I_\mu(x) = iU^{-1}\partial_\mu U + O\left(\frac{1}{|\vec{x}|}\right), \ |\vec{x}| \longrightarrow \infty. \tag{14.109}$$

This can be checked by simple substitution in $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. As we have discussed, we can imagine that the four-dimensional Euclidean spacetime $\mathbf{R}^4$ is bounded by a large three-sphere $\mathbf{S}^3_\infty$ in the same way that we can imagine that the plane is bounded by a large $\mathbf{S}^1_\infty$, viz

$$\partial\mathbf{R}^4 = \mathbf{S}^3_\infty. \tag{14.110}$$

Now a gauge configuration $A^I_\mu(x)$ at infinity (on the sphere $\mathbf{S}^3_\infty$) defines a group element U which satisfies (from the above asymptotic behavior) the equation $\partial_\mu U^{-1} = iA^I_\mu U^{-1}$ or equivalently

$$\frac{d}{ds}U^{-1}(x(s), x_0) = i\frac{dx^\mu}{ds}A^I_\mu(x(s))U^{-1}(x(s), x_0). \tag{14.111}$$

The solution is given by the path-ordered Wilson line

[3] The requirement of finite action can be neatly satisfied if we compactify $\mathbf{R}^4$ by adding one point (at) ∞ to obtain the four-sphere $\mathbf{S}^4$.

$$U^{-1}(x, x_0) = \mathcal{P}\exp\left(i\int_0^1 ds\frac{dy^\mu}{ds}A_\mu^I(y(s))\right). \tag{14.112}$$

The path is labeled by the parameter s which runs from $s = 0$ ($y = x_0$) to $s = 1$ ($y = x$) and the path-ordering operator $\mathcal{P}$ is defined such that terms with higher values of s are always put on the left in every order in the Taylor expansion of the exponential.

In the above formula for U^{-1} the points x and x_0 are both at infinity, i.e. on the sphere $\mathbf{S}^3_\infty$. In other words, gauge configurations $A_\mu^I(x)$ with finite action (instanton configurations) define a map from $\mathbf{S}^3_\infty$ into G, viz

$$U^{-1}\colon \mathbf{S}^3_\infty \longrightarrow G. \tag{14.113}$$

These maps are classified by homotopy theory. As an example, we concentrate on the group $G = SU(2)$. The group $SU(2)$ is topologically a three-sphere since any element $U \in SU(2)$ can be expanded (in the fundamental representation) as

$$U = n_4 + i\vec{n}\vec{\tau} \tag{14.114}$$

and as a consequence the unitarity condition $U^+U = 1$ becomes $n_4^2 + \vec{n}^2 = 1$. In this case we have, therefore, maps from the three-sphere to the three-sphere, viz

$$U^{-1}\colon \mathbf{S}^3_\infty \longrightarrow SU(2) = \mathbf{S}^3. \tag{14.115}$$

These maps are characterized by an integer k (known variously as the Pontryagin class, the winding number, the instanton number and the topological charge) defined by

$$k = \frac{1}{16\pi^2}\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{14.116}$$

This number measures how many times the second $\mathbf{S}^3$ (group) wraps (covers) the first sphere $\mathbf{S}^3_\infty$ (space). In fact this is the underlying reason why k must be quantized. In other words, k is an element of the third homotopy group $\pi_3(\mathbf{S}^3)$, viz[4]

$$k \in \pi_3(SU(2)) = \pi_3(\mathbf{S}^3) = \mathbf{Z}. \tag{14.117}$$

To see this more explicitly we write equation (14.116) as

$$\begin{aligned} k &= \int_{\partial\mathbf{R}^4=S^3_\infty} d^3\sigma_\alpha \mathcal{K}_\alpha \\ &= \frac{i}{24\pi^2}\epsilon_{\alpha\mu\nu\beta}\int_{\partial\mathbf{R}^4=\mathbf{S}^3_\infty} d^3\sigma_\alpha\ \mathrm{Tr}\ A_\mu A_\nu A_\beta. \end{aligned} \tag{14.118}$$

In the above equation we have used the current (14.92) and the fact that on $\mathbf{S}^3_\infty$ we have $F_{\mu\nu} = 0$. We can also use the fact that $A_\mu = A_\mu^I = iU^{-1}\partial_\mu U \equiv iL_\mu$ on $\mathbf{S}^3_\infty$ and also (where τ_a are Pauli matrices)

[4] In general $\pi_n(\mathbf{S}^n) = \mathbf{Z}$. It is obvious that $\pi_1(\mathbf{S}^1) = \pi_2(\mathbf{S}^2) = \mathbf{Z}$.

$$U^{-1} = n_4 - in_a\tau_a = -in_\mu\tau_\mu,\ \ \tau_\mu = (\tau_a,\, i) \tag{14.119}$$

$$U = n_4 + in_a\tau_a = in_\mu\bar{\tau}_\mu,\ \ \bar{\tau}_\mu = (\tau_a,\, -i). \tag{14.120}$$

Thus, by using the fact that $U^{-1}\cdot\partial_\mu U = -\partial_\mu U^{-1}\cdot U$ we have $L_\mu = n_\alpha\partial_\mu n_\beta\tau_{\alpha\beta}$ where

$$\tau_{\alpha\beta} = \frac{1}{2}(\tau_\alpha\bar{\tau}_\beta - \tau_\beta\bar{\tau}_\alpha) = -\tau_{\beta\alpha}. \tag{14.121}$$

The spinor representation $\tau_{\alpha\beta}$ of $SU(2)$ is related to the vector representation τ_a of $SU(2)$ by $\tau_{\mu\nu} = i\bar{\eta}^a_{\mu\nu}\tau_a$ where $\bar{\eta}^a_{\mu\nu}$ are the 't Hooft symbols (more on this below). We conclude that $\mathrm{Tr}\,\tau_{\mu\nu} = 0$. We will also use

$$[\tau_{AB},\, \tau_{CD}] = -2(\delta_{AC}\tau_{BD} + \delta_{BD}\tau_{AC} - \delta_{AD}\tau_{BC} - \delta_{BC}\tau_{AD}) \tag{14.122}$$

$$\{\tau_{AB},\, \tau_{CD}\} = -2(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} - \epsilon_{ABCD}). \tag{14.123}$$

We compute

$$\begin{aligned}
\epsilon_{\alpha\mu\nu\beta}\,\mathrm{Tr}\,A_\mu A_\nu A_\beta &= -i\epsilon_{\alpha\mu\nu\beta}\,\mathrm{Tr}\,L_\mu L_\nu L_\beta \\
&= -\frac{i}{2}\epsilon_{\alpha\mu\nu\beta}(n_A\partial_\mu n_B)(n_C\partial_\nu n_D)(n_E\partial_\beta n_F)\mathrm{Tr}[\tau_{AB},\, \tau_{CD}]\tau_{EF} \\
&= -\frac{i}{2}\epsilon_{\alpha\mu\nu\beta}(n_A\partial_\mu n_B)(n_C\partial_\nu n_D)(n_E\partial_\beta n_F) \\
&\quad\times \mathrm{Tr}(-\delta_{AC}\{\tau_{BD},\, \tau_{EF}\} - \delta_{BD}\{\tau_{AC},\, \tau_{EF}\} \\
&\quad + \delta_{AD}\{\tau_{BC},\, \tau_{EF}\} + \delta_{BC}\{\tau_{AD},\, \tau_{EF}\}).
\end{aligned} \tag{14.124}$$

By inspection all the terms in the above equation are zero because of either $n_A\partial_\mu n_A = 0$ (since $n_A^2 = 1$) or $\epsilon_{\alpha\mu\nu\beta}\partial_\mu n_A\partial_\nu n_A = 0$. The only non-zero contribution comes from the term proportional to the epsilon tensor in the first anticommutator which gives

$$\begin{aligned}
\epsilon_{\alpha\mu\nu\beta}\,\mathrm{Tr}\,A_\mu A_\nu A_\beta &= -\frac{i}{2}\epsilon_{\alpha\mu\nu\beta}(n_A\partial_\mu n_B)(n_C\partial_\nu n_D)(n_E\partial_\beta n_F)\,\mathrm{Tr}(-2\delta_{AC}\epsilon_{BDEF}) \\
&= 2i\epsilon_{\alpha\mu\nu\beta}\epsilon_{ABCD}n_A\partial_\mu n_B\partial_\nu n_C\partial_\beta n_D.
\end{aligned} \tag{14.125}$$

We get then

$$k = -\frac{1}{12\pi^2}\epsilon_{\alpha\mu\nu\beta}\int_{\partial\mathbf{R}^4=\mathbf{S}^3_\infty} d^3\sigma_\alpha\epsilon_{ABCD}n_A\partial_\mu n_B\partial_\nu n_C\partial_\beta n_D. \tag{14.126}$$

Let us consider the so-called fundamental map given by

$$n_A = x_A/r,\ r = \sqrt{x_A^2}. \tag{14.127}$$

This corresponds to $k = -1$, i.e. the $\mathbf{S}^3$ (group) wraps the $\mathbf{S}^3_\infty$ (space) only once in the negative direction. We compute $\partial_\mu n_B = (\delta_{\mu B} - n_\mu n_B)/r$ and as a consequence

$$\epsilon_{\alpha\mu\nu\beta}\epsilon_{ABCD}n_A\partial_\mu n_B\partial_\nu n_C\partial_\beta n_D = \epsilon_{\alpha\mu\nu\beta}\epsilon_{A\mu\nu\beta}n_A/r^3 = 6n_\alpha/r^3. \tag{14.128}$$

We get then

$$\begin{aligned} -k &= \frac{1}{2\pi^2}\int_{\partial\mathbf{R}^4=S^3_\infty} d^3\sigma_\alpha \frac{x_\alpha}{r^4} \\ &= \frac{1}{2\pi^2 r_\infty^4}\int_{\partial\mathbf{R}^4=S^3_\infty} d^3\sigma_\alpha x_\alpha \\ &= \frac{1}{2\pi^2 r_\infty^4}\int d^4x\,\partial_\alpha x_\alpha \\ &= \frac{2}{\pi^2 r_\infty^4}\int d^4x \\ &= \frac{2}{\pi^2 r_\infty^4}\frac{1}{2}\pi^2 r_\infty^4 \\ &= +1. \end{aligned} \tag{14.129}$$

The topological charge $k = +1$ can be obtained by making the replacements $U \longrightarrow U^{-1}$, $U^{-1} \longrightarrow U$. Equivalently we should consider the anti-fundamental map $U = n_4 - in_a\tau_a = -in_\mu\tau_\mu$, $U^{-1} = n_4 + in_a\tau_a = in_\mu\bar{\tau}_\mu$ and as a consequence $L_\mu = n_\alpha\partial_\mu n_\beta\bar{\tau}_{\alpha\beta}$ where $\bar{\tau}_{\mu\nu} = (\bar{\tau}_\mu\tau_\nu - \bar{\tau}_\nu\tau_\mu)/2 = i\eta^a_{\mu\nu}\tau_a$ is the other spinor representation of $SU(2)$ and $\eta^a_{\mu\nu}$ are also 't Hooft symbols (see below). We have thus $\mathrm{Tr}\,\bar{\tau}_{\mu\nu} = 0$ and the commutators and anticommutators of $\bar{\tau}_{\mu\nu}$ are again given by equations (14.122) and (14.123) with the replacements $\tau_{\mu\nu} \longrightarrow \bar{\tau}_{\mu\nu}$ and $\epsilon_{ABCD} \longrightarrow -\epsilon_{ABCD}$. It is obvious that this last point will give rise to a the value $k = +1$.

The equation (14.126) can be rewritten as

$$k = -\frac{1}{12\pi^2}\epsilon_{abc}\int_{\partial\mathbf{R}^4=S^3_\infty} d^3\xi\epsilon_{ABCD}n_A\partial_a n_B\partial_b n_C\partial_c n_D. \tag{14.130}$$

This can be shown as follows. By introducing a local parametrization $\xi_a = \xi_a(x)$ of the $SU(2)$ group elements we can rewrite k as

$$k = \frac{1}{24\pi^2}\epsilon_{\alpha\mu\nu\beta}\int_{\partial\mathbf{R}^4=\mathbf{S}^3_\infty} d^3\sigma_\alpha \frac{\partial\xi_a}{\partial x_\mu}\frac{\partial\xi_b}{\partial x_\nu}\frac{\partial\xi_c}{\partial x_\beta}\,\mathrm{Tr}\,L_aL_bL_c,\quad L_a = U^{-1}\partial_a U. \tag{14.131}$$

By using $d^3\sigma_\alpha = (\epsilon_{\alpha\mu\nu\beta}dx_\mu \wedge dx_\nu \wedge dx_\beta)/6$ and $\epsilon_{\alpha\mu\nu\beta}\epsilon_{\alpha\mu'\nu'\beta'} = \delta^{\mu'\nu'\beta'}_{[\mu\nu\beta]} = \delta_\mu^{\mu'}(\delta_\nu^{\nu'}\delta_\beta^{\beta'} - \delta_\beta^{\nu'}\delta_\nu^{\beta'}) + \delta_\nu^{\mu'}(\delta_\beta^{\nu'}\delta_\mu^{\beta'} - \delta_\mu^{\nu'}\delta_\beta^{\beta'}) + \delta_\beta^{\mu'}(\delta_\mu^{\nu'}\delta_\nu^{\beta'} - \delta_\nu^{\nu'}\delta_\mu^{\beta'})$ we get

$$\begin{aligned} k &= \frac{1}{24\pi^2}\frac{1}{6}\delta^{\mu'\nu'\beta'}_{[\mu\nu\beta]}\int_{\partial\mathbf{R}^4=\mathbf{S}^3_\infty} dx_{\mu'} \wedge dx_{\nu'} \wedge dx_{\beta'} \\ &\quad\times \frac{\partial\xi_a}{\partial x_\mu}\frac{\partial\xi_b}{\partial x_\nu}\frac{\partial\xi_c}{\partial x_\beta}\,\mathrm{Tr}\,L_aL_bL_c \\ &= \frac{1}{24\pi^2}\int_{\partial\mathbf{R}^4=\mathbf{S}^3_\infty} d\xi_a \wedge d\xi_b \wedge d\xi_c\,\mathrm{Tr}\,L_aL_bL_c \\ &= \frac{1}{24\pi^2}\int_{\partial\mathbf{R}^4=\mathbf{S}^3_\infty} d^3\xi\epsilon_{abc}\,\mathrm{Tr}\,L_aL_bL_c. \end{aligned} \tag{14.132}$$

This then leads to equation (14.130) by using the result

$$-\epsilon_{abc}\ \mathrm{Tr}\ L_a L_b L_c = 2\epsilon_{abc}\epsilon_{ABCD} n_A \partial_a n_B \partial_b n_C \partial_c n_D. \tag{14.133}$$

The result (14.132) is precisely equation (14.98) if we also employ the replacement $\epsilon_{\alpha\mu\nu\beta} d^3\sigma_\alpha \longrightarrow \epsilon_{abc} d^3\xi$ one more time.

14.2.4 BPST instanton

't Hooft symbols

The Euclidean Lorentz group is $SO(4) = SU(2)_L \times SU(2)_R$. The vector representation of $SU(2)$ is generated by Pauli matrices τ_a. We will work with $\tau_\mu = (\tau_a, i)$, $\bar{\tau}_\mu = (\tau_a, -i)$. These satisfy

$$\bar{\tau}_\alpha \tau_\beta + \bar{\tau}_\beta \tau_\alpha = 2\delta_{\alpha\beta},\ \ \tau_\alpha \bar{\tau}_\beta + \tau_\beta \bar{\tau}_\alpha = 2\delta_{\alpha\beta}. \tag{14.134}$$

There are two inequivalent spinor representations of $SU(2)$ given by

$$\tau_{\alpha\beta} = \frac{1}{2}(\tau_\alpha \bar{\tau}_\beta - \tau_\beta \bar{\tau}_\alpha) = -\tau_{\beta\alpha},\ \ \bar{\tau}_{\alpha\beta} = \frac{1}{2}(\bar{\tau}_\alpha \tau_\beta - \bar{\tau}_\beta \tau_\alpha) = -\bar{\tau}_{\beta\alpha}. \tag{14.135}$$

These are anti-hermitian matrices. The vector and spinor representations are related by 't Hooft symbols, viz

$$\tau_{\mu\nu} = i\bar{\eta}^a_{\mu\nu}\tau_a,\ \ \bar{\tau}_{\mu\nu} = i\eta^a_{\mu\nu}\tau_a. \tag{14.136}$$

Explicitly we compute

$$\bar{\eta}^c_{ab} = \epsilon_{abc},\ \ \bar{\eta}^b_{4a} = \delta^b_a = -\bar{\eta}^b_{a4} \tag{14.137}$$

$$\eta^c_{ab} = \epsilon_{abc},\ \ \eta^b_{4a} = -\delta^b_a = -\eta^b_{a4}. \tag{14.138}$$

We have $\tau_{ab} = i\epsilon_{abc}\tau_c$, $\tau_{4a} = -\tau_{a4} = i\tau_a$. We compute $[\tau_{ab}, \tau_{c4}] = -2(\delta_{ac}\tau_{b4} - \delta_{bc}\tau_{a4})$ and $[\tau_{ab}, \tau_{cd}] = -2i(\epsilon_{cda}\tau_b - \epsilon_{cdb}\tau_a)$ which collectively can be rewritten as

$$[\tau_{AB}, \tau_{CD}] = -2(\delta_{AC}\tau_{BD} + \delta_{BD}\tau_{AC} - \delta_{AD}\tau_{BC} - \delta_{BC}\tau_{AD}). \tag{14.139}$$

We also compute $\{\tau_{ab}, \tau_{c4}\} = 2\epsilon_{abc}$ and $\{\tau_{ab}, \tau_{cd}\} = -2\epsilon_{abi}\epsilon_{cdi}$ which can be rewritten collectively as

$$\{\tau_{AB}, \tau_{CD}\} = -2(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} - \epsilon_{ABCD}). \tag{14.140}$$

Similarly, we compute

$$[\bar{\tau}_{AB}, \bar{\tau}_{CD}] = -2(\delta_{AC}\bar{\tau}_{BD} + \delta_{BD}\bar{\tau}_{AC} - \delta_{AD}\bar{\tau}_{BC} - \delta_{BC}\bar{\tau}_{AD}) \tag{14.141}$$

$$\{\bar{\tau}_{AB}, \bar{\tau}_{CD}\} = -2(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + \epsilon_{ABCD}). \tag{14.142}$$

By using $\tau_{ab} = i\epsilon_{abc}\tau_c = \bar{\tau}_{ab}$, $\tau_{4a} = -\tau_{a4} = i\tau_a$, $\bar{\tau}_{4a} = -\bar{\tau}_{a4} = -i\tau_a$ and $\epsilon_{abc4} = \epsilon_{abc} = -\epsilon_{4abc}$ we can check that $\tau_{\mu\nu}$ is anti-self-dual and $\bar{\tau}_{\mu\nu}$ is self-dual, viz

$$\tau_{\mu\nu} = -\tilde{\tau}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\tau_{\alpha\beta} \tag{14.143}$$

$$\bar{\tau}_{\mu\nu} = \tilde{\bar{\tau}}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{\tau}_{\alpha\beta}. \tag{14.144}$$

We can also compute

$$\begin{aligned}\epsilon_{\mu\nu\alpha\beta}\tau_{\beta\rho} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\epsilon_{\rho\sigma\gamma\beta}\tau_{\sigma\gamma} \\ &= \frac{1}{2}\delta^{\rho\sigma\gamma}_{[\mu\nu\alpha]}\tau_{\sigma\gamma} \\ &= \delta_{\mu\rho}\tau_{\nu\alpha} + \delta_{\nu\rho}\tau_{\alpha\mu} + \delta_{\alpha\rho}\tau_{\mu\nu}.\end{aligned} \tag{14.145}$$

Similarly, we compute

$$\epsilon_{\mu\nu\alpha\beta}\bar{\tau}_{\beta\rho} = -\delta_{\mu\rho}\bar{\tau}_{\nu\alpha} - \delta_{\nu\rho}\bar{\tau}_{\alpha\mu} - \delta_{\alpha\rho}\bar{\tau}_{\mu\nu}. \tag{14.146}$$

BPST instanton

Next we find explicit solutions for the duality equations for $SU(2)$ gauge theory with $k = \pm 1$. First, we recall that the fundamental map $U = n_4 + in_a\tau_a = in_\mu\bar{\tau}_\mu$ corresponds to $k = -1$. The corresponding anti-instanton gauge field is $A^I_\mu = iU^{-1}\partial_\mu U$. Explicitly, we compute (using $\bar{\eta}^a_{\alpha\beta} = -\bar{\eta}^a_{\beta\alpha}$, i.e. the 't Hooft symbol is antisymmetric and $\tau_{\mu\nu} = i\bar{\eta}^a_{\mu\nu}\tau_a$)

$$\begin{aligned}A^I_\mu &= in_\alpha\partial_\mu n_\beta\tau_{\alpha\beta} \\ &= \frac{i}{r}n_\alpha(\delta_{\mu\beta} - n_\mu n_\beta)\tau_{\alpha\beta} \\ &= -\frac{1}{r}n_\alpha(\delta_{\mu\beta} - n_\mu n_\beta)\bar{\eta}^a_{\alpha\beta}\tau_a \\ &= \frac{1}{r}\bar{\eta}^a_{\mu\alpha}n_\alpha\tau_a \\ &= -\frac{i}{r}\tau_{\mu\alpha}n_\alpha, \ (k = -1, \text{ anti-instanton}).\end{aligned} \tag{14.147}$$

For $k = +1$ we should consider the anti-fundamental map $U = n_4 - in_a\tau_a = -in_\mu\tau_\mu$ and we compute the gauge field

$$\begin{aligned}A^I_\mu &= \frac{1}{r}\eta^a_{\mu\alpha}n_\alpha\tau_a \\ &= -\frac{i}{r}\bar{\tau}_{\mu\alpha}n_\alpha, \ (k = +1, \text{ instanton}).\end{aligned} \tag{14.148}$$

The first observation is the fact that this configuration is not invariant under the rotational symmetry group $SU(2)_L$ of $SO(4)$ unless we also rotate the configuration

under the gauge (isospin) symmetry group $SU(2)$. The instanton configuration breaks the product of the two groups $SU(2)_L$ and $SU(2)$ down to a diagonal subgroup. Indeed, we have the symmetry breaking pattern

$$SU(2)_L \times SU(2) \longrightarrow SU(2)_{\text{diagonal}}. \tag{14.149}$$

The second observation concerns the behavior of the above instanton configuration when approaching the origin which is obviously very singular. The $k = +1$ (instanton) and $k = -1$ (anti-instanton) configurations can also be rewritten as $A_\mu^I = -i\bar{\tau}_{\mu\alpha}\partial_\alpha \ln r$ and $A_\mu^I = -i\tau_{\mu\alpha}\partial_\alpha \ln r$ respectively. The anti-self-duality of the anti-instanton configuration ($k = -1$) and the self-duality of the instanton configuration ($k = +1$), respectively, cannot be expected to hold since $A_\mu^I = iU^{-1}\partial_\mu U$ is only an asymptotic behavior.

A systematic method of finding instanton and anti-instanton configurations is as follows. We concentrate on the instanton and leave the anti-instanton as an exercise. We try the more general ansatz

$$-A_\mu^{In} = -i\alpha\tau_{\mu\alpha}\partial_\alpha \ln \phi(r). \tag{14.150}$$

We compute

$$\begin{aligned} -F_{\alpha\beta}^{In} = & -i\alpha\tau_{\beta\mu}\partial_\alpha\partial_\mu \ln\phi + i\alpha\tau_{\alpha\mu}\partial_\beta\partial_\mu \ln\phi \\ & - 2i\alpha^2\tau_{\mu\beta}\partial_\mu \ln\phi\partial_\alpha \ln\phi + 2i\alpha^2\tau_{\mu\alpha}\partial_\mu \ln\phi\partial_\beta \ln\phi \\ & + 2i\alpha^2\tau_{\alpha\beta}(\partial_\mu \ln\phi)^2. \end{aligned} \tag{14.151}$$

By using the properties (14.143) and (14.145) of 't Hooft symbols we compute the dual field strength

$$\begin{aligned} -\tilde{F}_{\alpha\beta}^{In} = & -i\alpha\tau_{\beta\mu}\partial_\mu\partial_\alpha \ln\phi + i\alpha\tau_{\alpha\mu}\partial_\mu\partial_\beta \ln\phi - i\alpha\tau_{\alpha\beta}\partial_\mu^2 \ln\phi \\ & + 2i\alpha^2\tau_{\beta\mu}\partial_\alpha \ln\phi\partial_\mu \ln\phi \\ & - 2i\alpha^2\tau_{\alpha\mu}\partial_\beta \ln\phi\partial_\mu \ln\phi. \end{aligned} \tag{14.152}$$

The solution of the self-duality equation $-F_{\alpha\beta}^{In} = -\tilde{F}_{\alpha\beta}^{In}$ satisfies the equation

$$-F_{\alpha\beta}^{In} = -\tilde{F}_{\alpha\beta}^{In} \Rightarrow 2\alpha(\partial_\mu \ln\phi)^2 = -\partial_\mu^2 \ln\phi. \tag{14.153}$$

Without any loss of generality we can choose $\alpha = 1/2$ because we can always perform the rescaling $\phi \longrightarrow \phi \longrightarrow \phi^{1/2\alpha}$. As a consequence we get the equation

$$\frac{\partial_\mu^2\phi}{\phi} = 0. \tag{14.154}$$

A solution is given by $\phi = 1/r^2$. It is straightforward to check that for $x \neq 0$ we have $\partial_\mu^2(1/r^2) = 0$, whereas for $x = 0$ we must use $\partial_\mu^2(1/r^2) = -4\pi^2\delta^4(x)$ which also leads to $\partial_\mu^2\phi/\phi = 0$. The generalization of the solution $\phi = 1/r^2$ is

$$\phi = 1 + \frac{\rho^2}{(x-a)^2}. \tag{14.155}$$

The additive constant has been set equal to one so that for large r the instanton gauge field approaches 0. The instanton gauge field takes then the form

$$\begin{aligned} A_\mu^{In} &= \frac{i}{2}\tau_{\mu\alpha}\partial_\alpha \ln\left(1 + \frac{\rho^2}{(x-a)^2}\right) \\ &= \frac{i}{2}\tau_{\mu\alpha}\frac{-2\rho^2(x-a)_\alpha}{(x-a)^2[(x-a)^2+\rho^2]} \\ &= -i\tau_{\mu\alpha}\frac{\rho^2(x-a)_\alpha}{(x-a)^2[(x-a)^2+\rho^2]},\ k=1 \text{ (singular)}. \end{aligned} \tag{14.156}$$

This is singular at $x = a$. Near $x = a$ we have

$$A_\mu^{In} = -i\tau_{\mu\alpha}\frac{(x-a)_\alpha}{(x-a)^2},\ x \longrightarrow a. \tag{14.157}$$

This is a pure gauge field configuration, viz

$$A_\mu^{In} = iU^{-1}\partial_\mu U,\ U = \frac{i(x-a)_\mu}{|x-a|}\bar{\tau}_\mu,\ x \longrightarrow a. \tag{14.158}$$

The generic configuration A_μ^{In} can then be put in the form

$$A_\mu^{In} = \left(iU^{-1}\partial_\mu U\right)\frac{\rho^2}{(x-a)^2+\rho^2},\ k=1 \text{ (singular)}. \tag{14.159}$$

The singular behavior at $x = a$ can be removed by a singular gauge transformation (given precisely by U^{-1}) which maps the singularity to $x \longrightarrow \infty$. Indeed, we compute

$$\begin{aligned} A_\mu^{In'} &= U\left(A_\mu^{In} + i\partial_\mu\right)U^{-1} \\ &= \left(iU\partial_\mu U^{-1}\right)\frac{(x-a)^2}{(x-a)^2+\rho^2},\ k=1 \text{ (regular)}. \end{aligned} \tag{14.160}$$

This have the correct asymptotic behavior, viz

$$\begin{aligned} A_\mu^{In'} &= iU\partial_\mu U^{-1} \\ &= iU'^{-1}\partial_\mu U',\ x \longrightarrow \infty. \end{aligned} \tag{14.161}$$

The group element U' is given by

$$U' = -\frac{i(x-a)_\mu}{|x-a|}\tau_\mu \longrightarrow -in_\mu\tau_\mu,\ x \longrightarrow \infty. \tag{14.162}$$

This is precisely the behavior which leads to the $k = +1$ configuration (14.148). We check now explicitly that the field strengths are self-dual. We will assume that $a = 0$ for simplicity. We have then the regular gauge field

$$A_\mu^{In'} = \left(iU\partial_\mu U^{-1}\right)\phi, \;\; \phi = 1 - \frac{\rho^2}{r^2+\rho^2}. \tag{14.163}$$

We compute

$$\begin{aligned} F_{\mu\nu}^{In'} &= \partial_\mu A_\nu^{In'} - \partial_\nu A_\mu^{In'} - i\left[A_\mu^{In'}, A_\nu^{In'}\right] \\ &= i\phi(1-\phi)\left[\partial_\mu U\partial_\nu U^{-1} - \partial_\nu U\partial_\mu U^{-1}\right] + i\partial_r\phi\left[n_\mu U\partial_\nu U^{-1} - n_\nu U\partial_\mu U^{-1}\right] \\ &= \frac{2ir^2\rho^2}{(r^2+\rho^2)^2}\partial_\mu n_\alpha \partial_\nu n_\beta \bar{\tau}_{\alpha\beta} + \frac{2ir\rho^2}{(r^2+\rho^2)^2} n_\alpha(n_\mu\partial_\nu n_\beta - n_\nu\partial_\mu n_\beta)\bar{\tau}_{\alpha\beta} \\ &= \frac{2i\rho^2}{(r^2+\rho^2)^2}\bar{\tau}_{\mu\nu}, \;\; k = 1 \text{ (regular)}. \end{aligned} \tag{14.164}$$

This is clearly self-dual. From this result we can deduce immediately the field strength of the singular gauge field A_μ^{In} by simply applying the singular gauge transformation g as follows

$$\begin{aligned} F_{\mu\nu}^{In} &= U^{-1}F_{\mu\nu}^{In'}U \\ &= \frac{(x-a)_\alpha}{|x-a|}\tau_\alpha \cdot \frac{2i\rho^2\bar{\tau}_{\mu\nu}}{(r^2+\rho^2)^2} \cdot \frac{(x-a)_\beta}{|x-a|}\bar{\tau}_\beta, \;\; k = 1 \text{ (singular)}. \end{aligned} \tag{14.165}$$

For the anti-instanton configuration with topological charge $k = -1$ we proceed in the same way. We start from the ansatz

$$A_\mu^{a-In} = -i\beta\bar{\tau}_{\mu\alpha}\partial_\alpha \ln\phi(r). \tag{14.166}$$

We impose the anti-self-duality condition $F_{\alpha\beta}^{a-In} = -\tilde{F}_{\alpha\beta}^{a-In}$ which allows us to determine both the constant β and the function ϕ. This leads immediately to the singular anti-instanton gauge field. Next we determine the singular gauge transformation U^{-1} which moves the singularity at $x = a$ to $x \longrightarrow \infty$. This singular gauge transformation can be used to construct the regular anti-instanton gauge configuration as $A_\mu^{a-In'} = U(A_\mu^{a-In} + i\partial_\mu)U^{-1}$. This will have the correct asymptotic behavior which we have discussed at the beginning of this section and, as a consequence, it will have a topological charge $k = -1$. The field strengths for the regular and singular anti-instanton gauge fields can then be found following the method used for the instanton fields.

14.2.5 Tunneling and large gauge transformations

Instantons, which strictly live in Euclidean space, can be also viewed as mediating quantum tunneling between different vacua in Minkowski spacetime. We go back then to Minkowski signature.

Recall also that in the temporal gauge $A_0 = 0$ the surviving gauge invariance of a Yang–Mills action is given by the time-independent unitary matrices

$$\Omega = \Omega(\vec{x}). \tag{14.167}$$

Clearly these transformations commute with the Hamiltonian, i.e. $[H, \Omega] = 0$, and hence they can be diagonalized simultaneously, viz

$$H|E\rangle = E|E\rangle, \;\; \Omega(\vec{x})|E\rangle = \omega(\vec{x})|E\rangle. \tag{14.168}$$

By Lorentz invariance the eigenvalues $\omega(\vec{x})$ must also be constant in space, i.e. $\omega(\vec{x}) = \omega$. They originate from two different classes of gauge transformations given by:

- **Small gauge transformations:** These are the transformations which are homotopic, i.e. connected, to the identity and which form a closed subgroup of the full gauge group. They are, in fact, the gauge transformations which are targeted by the gauge fixing procedure and they form the physical equivalency class. They must have no winding or else they would be singular. Since they are connected to the identity and they have no winding we must therefore have $\omega = 1$.
- **Large gauge transformations:** These transformations are not connected to the identity, they are not gauge-fixed, and they also change the winding number. They are, in fact, the gauge transformations corresponding to instanton configurations. Indeed, if we assume that $\Omega(\vec{x})$ takes the same limit when going to spatial infinity in any direction, then it can be viewed as a function on $\mathbf{S}^3 = \mathbf{R}^3 \cup \{\infty\}$. Hence, for the gauge group $SU(2)$ the large gauge transformations $\Omega(\vec{x})$ are maps from $\mathbf{S}^3$ (space) to $\mathbf{S}^3$ (group) which are classified by the third homotopy group $\pi_3(\mathbf{S}^3) = \mathbf{Z}$.

Note that this $\mathbf{S}^3$ (space) is not the boundary of $\mathbf{R}^4$ as before but it is a spacelike surface corresponding to some fixed time t. Indeed, we will consider a foliation of Minkowski spacetime in which all spatial surfaces at fixed t are compactified to spheres $\mathbf{S}^3$ by assuming that $\Omega(x)$ takes the same limit at spatial infinity in any direction on each surface. We can thus make the identification $\Omega(x)|_{t=\text{fixed}} = \Omega(\vec{x})$.

The one-to-one map $\Omega_1(x)$ is characterized by winding number 1 while the one-to-n map $\Omega_n(x) = (\Omega_1(x))^n$ is characterized by winding number n. We have explicitly (back in Euclidean signature)

$$\Omega_1(x) = \frac{x_4 \mathbf{1}_2 + i x_i \tau_i}{|x|}. \tag{14.169}$$

Their eigenvalues are characterized by an angle $\theta \in [0, 2\pi[$ as

$$\Omega_1(x)|E\rangle = e^{i\theta}|E\rangle, \;\; \Omega_n(x)|E\rangle = e^{in\theta}|E\rangle. \tag{14.170}$$

The angle θ is a Lorentz invariant number which provides another fundamental parameter of the theory called the instanton angle (think of it as spin).

We now show that large gauge transformations change the winding number.

We consider initial and final configurations at the infinite past $t = -T \longrightarrow -\infty$ and the infinite future $t = +T \longrightarrow +\infty$ which describe vacua of the theory, i.e. they have vanishing energies and definite winding numbers. They are, therefore, pure gauges. The winding number in Minkowski spacetime is defined in the same way as in the Euclidean space because it is an affine quantity which does not depend on the metric.

We consider a continuous line of gauge fields $A_i(\lambda, \vec{x})$, parameterized by a real number $\lambda \in [0, 1]$, which connects the two vacua at $t = -T$ and $t = +T$. In other words, $A_i(0, \vec{x})$ is the gauge field configuration at $t = -T$ and $A_i(1, \vec{x})$ is the gauge field configuration at $t = +T$. We can think of λ as a function of t which for $\lambda(t) = t$ gives the configurations $A_i(t, \vec{x})$ but different $\lambda(t)$ yield paths which run at different speeds through the same sequence of configurations $A_i(t, \vec{x})$ [10].

We will further assume, for simplicity, that these two configurations are related by the one-to-one map $\Omega_1(T, \vec{x})$, viz

$$A_i(1, \vec{x}) = \Omega_1^{-1} A_i(0, \vec{x}) \Omega_1 + i\Omega_1^{-1} \partial_i \Omega_1. \tag{14.171}$$

They describe, therefore, the same physical situation.

The winding numbers at the surfaces $t = -T$ and $t = +T$ are integers (0 and +1 respectively) since the gauge fields there are pure gauges everywhere on these spacelike surfaces, whereas for any $0 < \lambda < 1$ the gauge field is not pure gauge everywhere on the corresponding spacelike surface and hence the winding number is not an integer. More precisely, the winding number can be re-defined by integrating from $t = -T$ to only up to a time t, and thus the continuous winding number will be seen to interpolate between the values of the winding number at the surfaces $t = -T$ and $t = +T$. Moreover, the gauge field $A_i(\lambda, \vec{x})$ will also be found to be closely related to the instanton with a topological charge given by the difference between the winding numbers at the surfaces $t = -T$ and $t = +T$.

Indeed, if we assume that the configuration for $\lambda = 0$ has vanishing energy then, by the residual gauge symmetry (14.167), we can choose $A_i(0, \vec{x}) = 0$ and as a consequence $A_i(1, \vec{x})$ is effectively the limit of the instanton configuration with topological charge $n = +1$ when $t \longrightarrow +T$. Furthermore, by requiring that the paths from $t = -T$ to $t = +T$ have finite energies the gauge fields $A_i(\lambda, \vec{x})$ for $0 < \lambda < 1$ must vanish only for $|\vec{x}| \longrightarrow \infty$ and as a consequence they are not pure gauges everywhere, i.e. they must have continuous winding numbers. We write

$$A_i \longrightarrow i\Omega^{-1}(x) \partial_i \Omega(x), \text{ on boundary}, \tag{14.172}$$

where

$$\begin{aligned} &\Omega(x) = \Omega_1(x), \ t \longrightarrow +T \\ &\Omega(x) = \mathbf{1}, \text{ elsewhere on boundary}. \end{aligned} \tag{14.173}$$

The winding number can be written as an integral over the boundary of the box (figure 14.1). The contribution to the winding number coming from the sides vanishes because of the gauge condition $A_0 = 0$, whereas the contribution from the bottom of the box vanishes since $A_i = 0$ there. Hence all contributions to the

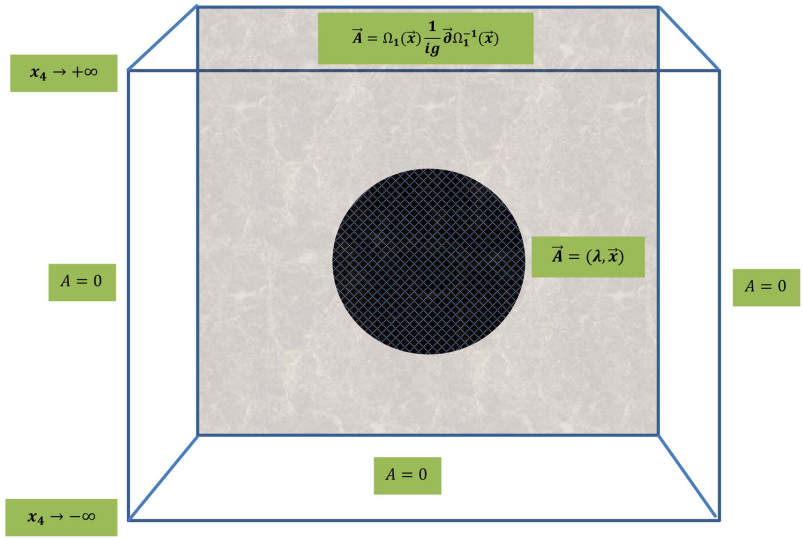

Figure 14.1. Tunneling through instantons.

winding number come from the top of the box which is obviously equal to $n = 1$. Thus, the winding number can be changed continuously from 0 at $t = -T$ to +1 at $t = +T$ by means of the large gauge transformation $\Omega_1(x)$.

The winding number is a gauge-invariant quantity and in general it will be equal to the difference between the winding numbers of the spacelike surfaces at the top and the bottom of the box, viz

$$k = \Delta k = n|_{t=+T} - n|_{t=-T}. \tag{14.174}$$

On the other hand, since the gauge configurations for $\lambda = 0$ and $\lambda = 1$ are vacua of the Yang–Mills system, i.e. with vanishing energies, the energy for $0 < \lambda < 1$ must be necessarily higher. Alternatively, for $0 < \lambda < 1$ the fields $A_i(\lambda, \vec{x})$ cannot be solutions of the equation of motion otherwise we cannot change the winding, i.e. these paths have positive energy at some intermediate times. In other words, there is a potential barrier in-between the vacua $\lambda = 0$ and $\lambda = 1$ and as a consequence the system can tunnel through this barrier via the large gauge transformation (instanton) $\Omega_1(x)$.

By substituting the paths $A_i(\lambda, \vec{x})$ in the Yang–Mills Lagrangian we obtain the Lagrangian for a point particle $\lambda = \lambda(t)$ given explicitly by [10]

$$L = \frac{1}{2}m(\lambda)\dot{\lambda}^2 - V(\lambda). \tag{14.175}$$

The position-dependent mass $m(\lambda)$ and the potential barrier $V(\lambda)$ are both positive definite. In particular $V(\lambda)$ depends only on the magnetic field. The corresponding Hamiltonian is obviously (with $p(\lambda) = m(\lambda)\dot{\lambda}$)

$$H = \frac{1}{2m(\lambda)} p(\lambda)^2 + V(\lambda). \tag{14.176}$$

This is vanishing at $t = -T$ and $t = +T$ and thus as we go from the vacuum at $t = -T$ with winding number zero to the vacuum at $t = +T$ with a non-trivial winding number the energy must necessarily become positive at some intermediate times and tunneling is then required.

In the quantum theory we should solve the Schrödinger equation $H|\psi\rangle = E|\psi\rangle$. However, semi-classical treatment of tunneling is generically sufficient. We have $p = \sqrt{2m(E - V)}$ and $p = -i\partial/\partial\lambda$ and hence

$$p|\psi\rangle = -i\frac{\partial}{\partial\lambda}|\psi\rangle \Rightarrow \psi \propto \exp\left(i \int \sqrt{2m(E - V)}\, d\lambda\right). \tag{14.177}$$

This gives the transition amplitude in the allowed region $E > V$. In the forbidden region $E < V$ there is still a possibility for transition given by the tunneling wavefunction

$$\psi \propto \exp\left(-\int \sqrt{2m(V - E)}\, d\lambda\right). \tag{14.178}$$

Since we are considering tunneling from one vacuum with $E = 0$ to another vacuum with $E = 0$ we can see that

$$\int \sqrt{2m(E - V)}\, d\lambda = \int p\dot{\lambda} dt = \int L dt = S. \tag{14.179}$$

Note that the sign switch $V - E \longrightarrow E - V$ is equivalent to $p \longrightarrow ip$ which, since $p = m\dot{\lambda}$, is equivalent to $t \longrightarrow x_4 = it$. In other words, we are working in Euclidean space and $S = iS_E$ where S_E is the Euclidean action. Hence, in the semi-classical approximation the dominant contribution to the tunneling rate is given by the minima of the Euclidean action, i.e. by instantons. The tunneling rate is given explicitly by

$$R = \exp(-2S_E), \;\; S_E = \int_{t=-T}^{t+T} \sqrt{2m(V - E)}\, d\lambda. \tag{14.180}$$

The tunneling rate is bounded from above by

$$R = \exp(-2S_E) \leqslant \exp\left(-\frac{16\pi^2}{g^2}|n|\right). \tag{14.181}$$

This is due to the Bogomol'nyi bound which also holds in Minkowski spacetime for the paths $A_i(\lambda, \vec{x})$ [10]. As it turns out, the paths with the same winding number k and with the smallest action S_E are those with parallel electric and magnetic field. These are the most probable paths which connect the two vacua at $t = -T$ and $t = +T$ with different winding numbers. Among these paths there is one particular path that, when viewed in Euclidean space, is precisely an instanton, i.e. a gauge field

configuration $A_i(\lambda, \vec{x})$ satisfying the duality equation[5]. See [10] and references therein for a very careful treatment and more detail.

Thus, the vacuum with vanishing energy $E = 0$ consists of several topological sectors with winding numbers n, and instantons with topological charges k mediate vacuum tunneling between these vacua provided $k = \Delta n$. The true θ-vacuum is then a linear combination of the form

$$|\theta\rangle = \sum_n \exp(-in\theta)|n\rangle. \tag{14.182}$$

It is almost obvious that the vacuum-to-vacuum amplitudes should be given in terms of the theta term as follows

$$\langle\theta'|\exp(-itH)|\theta\rangle = \delta(\theta' - \theta)\sum_n \int [dA]_n \exp\left(-i\int d^4x\mathcal{L}_{\text{eff}}\right) \tag{14.183}$$

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{YM}} + \frac{\theta}{16\pi^2}\text{tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right). \tag{14.184}$$

14.2.6 The moduli space and ADHM construction

In this section we will follow [6, 10, 11].

Collective coordinates, zero modes and moduli space

We have found that the Yang–Mills action in an instanton configuration is bound from below by the Bogomol'nyi bound given by

$$\frac{1}{2g^2}\int d^4x\ \text{Tr}\ F_{\mu\nu}^2 \geqslant \frac{8\pi^2|k|}{g^2}. \tag{14.185}$$

The topological charge k is given by the theta term as

$$k = \frac{1}{16\pi^2}\int d^4x\ \text{Tr}\ F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{14.186}$$

A Yang–Mills instanton is a solution of the equations of motion, i.e. it minimizes the action, which has either a self-dual or anti-self-dual field strength. Indeed, the Bogomol'nyi bound is saturated (in fact by minimum action solution) if and only if the so-called duality equations (instanton equations) are satisfied, viz

$$F_{\mu\nu} = \tilde{F}_{\mu\nu},\ k > 0,\ \text{instantons} \tag{14.187}$$

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu},\ k < 0,\ \text{anti–instantons}. \tag{14.188}$$

These equations are properly understood in Euclidean space.

[5] Note that this path in Minkowski spacetime does not solve the field equation, in order to be able to change the winding number, but when viewed in Euclidean space it is an instanton which solves the duality equation.

A finite action solution of the Euclidean classical equations of motion or equivalently of the above duality equations for $SU(2)$ gauge group with topological charge $k = +1$, i.e. a regular one-instanton solution, was already constructed in the previous section. It is given explicitly by equation (14.160) or equivalently

$$A_\mu^{In'} = -i\frac{(x-a)_\alpha \bar{\tau}_{\mu\alpha}}{(x-a)^2 + \rho^2}. \tag{14.189}$$

The $SU(2)$ spinor representation $\bar{\tau}_{\mu\alpha}$ is an anti-hermitian matrix given in terms of 't Hooft symbols $\eta^a_{\mu\alpha}$ by $\bar{\tau}_{\mu\alpha} = i\eta^a_{\mu\alpha}\tau_a$.

The parameters a_μ and ρ correspond to the position and the size of the instanton and they are called collective coordinates. Thus, there are five collective coordinates (four translations and one scale). However, there are three extra collective coordinates corresponding to the gauge orientation given by a global $SU(2)$ rotation U, viz

$$A_\mu^{In'}(x;\, a,\, \rho,\, \theta) = U^{-1}(\theta) A_\mu^{In'}(x;\, a,\, \rho) U(\theta). \tag{14.190}$$

In total we have, therefore, eight collective coordinates for the $SU(2)$ instanton with $k = +1$ charge. For a general gauge group $SU(N)$ we will have instead $4N$ collective coordinates for the one-instanton solution, whereas for higher topological charge k in $SU(2)$ we will have $8k$ collective coordinates. In general, the number of collective coordinates for a generic instanton with topological charge k in an $SU(N)$ gauge group is given by the formula

$$\dim \mathcal{I}_{k,N} = 4kN. \tag{14.191}$$

The charge k instanton can be viewed in many circumstances as k charge 1 (well separated) instantons where each one of them is characterized by a position and a size and a gauge orientation. This is the origin of the factor k. On the other hand, the origin of the factor N comes down to the fact that the one-instanton $SU(2)$ solution (14.189) suitably embedded in $SU(N)$ is the most general one-instanton solution in an $SU(N)$ gauge theory.

The space of all solutions of the self-duality equation $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ modulo gauge transformations in a given topological sector k in an $SU(N)$ gauge theory is called the moduli space and it is denoted by $\mathcal{I}_{k,N}$. The dimension of this space is precisely $4kN$. The coordinates on the moduli space $\mathcal{I}_{k,N}$ are given precisely by the collective coordinates. Equivalently the coordinate on the moduli space $\mathcal{I}_{k,N}$ are given by the so-called zero modes defined as follows. Let $A_\mu = A_\mu(x_\mu, X^\alpha)$, where X^α are the collective coordinates, be a solution of $F_{\mu\nu} = \tilde{F}_{\mu\nu}$. If we require that the perturbed configuration $A'_\mu = A_\mu + \delta A_\mu$ is still a solution of $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ then the perturbation δA_μ must solve the linearized self-duality equation

$$\mathcal{D}_\mu \delta A_\nu - \mathcal{D}_\nu \delta A_\mu = \epsilon_{\mu\nu\alpha\beta} \mathcal{D}^\alpha \delta A^\beta. \tag{14.192}$$

The covariant derivative $\mathcal{D}_\mu$ is evaluated on the background solution A_μ. The solutions of this equation are called the zero modes. Obviously each collective

coordinate X^α is associated with a zero mode which satisfies equation (14.192) given by

$$\delta_\alpha A_\mu = \frac{\partial A_\mu}{\partial X^\alpha}. \tag{14.193}$$

It is also obvious that any local gauge transformation is a solution to the linearized self-duality equation and thus the general solution is

$$\delta_\alpha A_\mu = \frac{\partial A_\mu}{\partial X^\alpha} + \mathcal{D}_\mu \Omega_\alpha. \tag{14.194}$$

Note that each zero mode is associated with a different infinitesimal gauge transformation labeled by α. We gauge fix by choosing Ω_α such that the zero mode $\delta_\alpha A_\mu$ is orthogonal to any other gauge transformation, viz

$$\int d^4x \ \mathrm{Tr}\ \delta_\alpha A_\mu \mathcal{D}_\mu \eta = 0, \ \ \forall \eta. \tag{14.195}$$

This leads to the gauge fixing condition

$$\mathcal{D}_\mu \delta_\alpha A_\mu = 0. \tag{14.196}$$

The Atiyah–Singer index theorem states that the number of zero modes, i.e. solutions of equation (14.192), subjected to the gauge fixing condition (14.196), is exactly given by $4kN$, i.e. equal to the dimension of the moduli space $\mathcal{I}_{k,N}$.

The moduli space $\mathcal{I}_{k,N}$ is a smooth manifold except at possible localized points corresponding to small instantons $\rho \longrightarrow 0$. This manifold is actually a hyper-Kahler manifold with a reduced holonomy group $Sp(kN) \in SO(4kN)$ and three complex structures J^i satisfying $J^i J^j = -\delta^{ij} + \epsilon^{ijk} J^k$. The metric is given by the overlap of the zero modes, viz

$$g_{\alpha\beta} = \frac{1}{2g^2} \int d^4x \ \mathrm{Tr}\ \delta_\alpha A_\mu \delta_\beta A_\mu. \tag{14.197}$$

As a simple example it can be shown that the moduli space $\mathcal{I}_{1,2}$ of a single instanton in an $SU(2)$ gauge theory is given by the hyper-Kahler manifold

$$\mathcal{I}_{1,\,2} = \mathbf{R}^4 \times \mathbf{R}^4/\mathbf{Z}_2. \tag{14.198}$$

If fermions, such as for example adjoint Weyl spinors λ and $\bar{\lambda}$ with kinetic term $i\ \mathrm{Tr}\ \bar{\lambda}\bar{\sigma}_\mu \mathcal{D}^\mu \lambda$, are coupled to the gauge field then in the background of an instanton $F = \tilde{F}$ the spinor λ will contribute fermion zero modes while the spinor $\bar{\lambda}$ will not lead to any further zero modes. In the background of an anti-instanton $F = -\tilde{F}$ it is $\bar{\lambda}$ that picks up fermion zero modes while λ does not pick up any zero modes. Thus, fermions lead to extra collective coordinates.

ADHM construction

The so-called ADHM (Atiyah, Drinfeld, Hitchin and Manin) construction allows us to produce all solutions of the self-dual Yang–Mills equation $F = \tilde{F}$ using twistor

space [31]. A more physical approach based on type II string theory and D-brane physics can be found in [32]. Here we will adopt the pedagogical streamlined version of the ADHM construction found in [11] which does not rely on string theory which is beyond the scope of this chapter.

The spacetime coordinate will be denoted by x_n. A quaternionic structure of spacetime is introduced by viewing x_n as a (2, 2) representation of the group $SU(2)_L \times SU(2)_R$ (which is the covering group of $SO(4)$) with components $x_{\alpha\dot{\alpha}}$ or $\bar{x}^{\dot{\alpha}\alpha}$ given, respectively, by

$$x_{\alpha\dot{\alpha}} = x_n \sigma_{n\alpha\dot{\alpha}}, \;\; \bar{x}^{\dot{\alpha}\alpha} = x_n \bar{\sigma}_n^{\dot{\alpha}\alpha}. \tag{14.199}$$

The indices α and $\dot{\alpha}$ take the two values 1 and 2 and they are the spinor indices of $SU(2)_L$ and $SU(2)_R$ respectively. The matrices $\sigma_{n\alpha\dot{\alpha}}$ and $\bar{\sigma}_n^{\dot{\alpha}\alpha}$ are given by $\sigma_n = (i\vec{\tau}, \mathbf{1})$ and $\bar{\sigma}_n = (-i\vec{\tau}, \mathbf{1})$ where τ^c are Pauli matrices. Indices are raised and lowered with the ϵ-tensor, viz $\bar{x}^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} x_{\beta\dot{\beta}}$.

We introduce now an $(N + 2k) \times 2k$ complex-valued matrix $\Delta_{\lambda i\dot{\alpha}}$ which is linear in x by the relation

$$\Delta_{\lambda i\dot{\alpha}}(x) = a_{\lambda i\dot{\alpha}} + b^{\alpha}_{\lambda i} x_{\alpha\dot{\alpha}}. \tag{14.200}$$

The conjugate is the matrix

$$\bar{\Delta}_i^{\dot{\alpha}\lambda}(x) \;=\; (\Delta_{\lambda i\dot{\alpha}}(x))^* = \bar{a}_i^{\dot{\alpha}\lambda} + \bar{x}^{\dot{\alpha}\alpha} \bar{b}_{i\alpha}^{\lambda}. \tag{14.201}$$

The index λ is called the ADHM index and runs from 1 to $N + 2k$, whereas the index i is called the instanton index and runs from 1 to k. The complex-valued matrices a and b provide an over-complete set of collective coordinates on the moduli space satisfying the so-called ADHM constraints.

The matrix $\Delta_{\dot{\alpha}}(x)$ can also be viewed as a map $\Delta_{\dot{\alpha}}(x)$: $\mathbf{C}^k \longrightarrow C^{N+2k}$ which is assumed to be injective, whereas the matrix $\bar{\Delta}^{\dot{\alpha}}(x)$ should be viewed as a map $\bar{\Delta}^{\dot{\alpha}}(x)$: $\mathbf{C}^{N+2k} \longrightarrow C^k$ which is assumed to be surjective. These are the so-called non-degeneracy conditions.

The matrix $\bar{\Delta}(x)$ is $2k \times (N + 2k)$ so it has $2k$ rows and $N + 2k$ columns, i.e. it has N fewer rows than columns. As a consequence the matrix $\bar{\Delta}(x)$ will have N null eigenvectors denoted by the $(N + 2k) \times N$-dimensional complex-valued matrix $U_{\lambda u}$ where $u = 1, 2, \ldots, N$. Explicitly we have

$$\bar{\Delta}_i^{\dot{\alpha}\lambda} U_{\lambda u} = 0. \tag{14.202}$$

The hermitian conjugate of this equation is given by

$$\bar{U}_u^{\lambda} \Delta_{\lambda i\dot{\alpha}} = 0. \tag{14.203}$$

The eigenvectors U are orthonormalized according to

$$\bar{U}_u^{\lambda} U_{\lambda v} = \delta_{uv}. \tag{14.204}$$

The ADHM ansatz construct the $SU(N)$ gauge field A_n in the instanton sector k from the eigenvectors U as formally given by the pure gauge

$$(A_n)_{uv} = \frac{1}{g}\bar{U}_u^{\lambda}\partial_n U_{\lambda v}. \tag{14.205}$$

This configuration solves the self-duality equation provided the matrices Δ and $\bar{\Delta}$ satisfy the requirement

$$\bar{\Delta}_i^{\dot{\alpha}\lambda}\Delta_{\lambda j\dot{\beta}} = \delta_{\dot{\beta}}^{\dot{\alpha}}(f^{-1})_{ij}. \tag{14.206}$$

The function f is a hermitian $k \times k$ matrix which is x-dependent. Obviously, $f(x) \longrightarrow \mathbf{1}_k/x^2$ when $|x| \longrightarrow \infty$. Its inverse f^{-1} exists by virtue of the non-degeneracy conditions.

Equations (14.202), (14.203) and (14.206) allow us to derive the projector or completeness relation

$$\mathcal{P}_{\lambda}^{\mu} = U_{\lambda u}\bar{U}_u^{\mu} = \delta_{\lambda}^{\ \mu} - \Delta_{\lambda i\dot{\alpha}} f_{ij}\,\bar{\Delta}_j^{\dot{\alpha}\mu}. \tag{14.207}$$

This identity will allow us to show that the field strength F_{mn} associated with the above ADHM ansatz is indeed self-dual, viz

$$\begin{aligned} F_{mn} &= \partial_m A_n - \partial_n A_m + g[A_m, A_n] \\ &= \frac{4}{g}\bar{U}b\sigma_{mn}f\bar{b}U. \end{aligned} \tag{14.208}$$

The σ_{mn} are the Lorentz generators defined by $\sigma_{mn} = (\sigma_m\bar{\sigma}_n - \sigma_n\bar{\sigma}_m)/4$ which are by construction self-dual and as a consequence F_{mn} is self-dual.

The instanton number associated with the ADHM can also be checked to be equal to k by using Osborn's formula [33]. We have

$$-\frac{g^2}{16\pi^2}\int d^4x \,\mathrm{tr}_N F_{mn}^2 = \frac{1}{16\pi^2}\int d^4x \Box^2 \,\mathrm{tr}_k \log f. \tag{14.209}$$

14.2.7 The 1-loop calculation

The 1-loop calculation around the instanton solution involves the reduction of the path integral measure over the instanton to an integral over the moduli space of collective coordinates. This in turn requires the determination of the exact normalization of the zero modes and their measure. We consider for simplicity only bosonic collective coordinates and follow closely [10]. See also references therein.

First, we start by decomposing the field into a background and a fluctuation in the usual way, viz

$$A_\mu = A_\mu^{\mathrm{cl}}(\gamma) + A_\mu^{\mathrm{qu}}, \tag{14.210}$$

where γ_i, $i = 1, \ldots, 4kN$, are the collective coordinates. Then we add gauge fixing and ghost terms where we choose the gauge condition

$$D_\mu^{\rm cl} A_\mu^{\rm qu} = 0. \tag{14.211}$$

The expanded quadratic action is given by

$$S = S_{\rm cl} + \frac{1}{g^2}\int d^4x\ {\rm tr}\Big(A_\mu^{\rm qu} M_{\mu\nu} A_\nu^{\rm qu} + b M_{\rm gh} c\Big),\ \ S_{\rm cl} = \frac{8\pi^2}{g^2}|k|. \tag{14.212}$$

The operators $M_{\mu\nu}$ and $M_{\rm gh}$ are given by (by dropping also the subscript cl)

$$M_{\mu\nu} = D^2\delta_{\mu\nu} + 2F_{\mu\nu},\ \ M_{\rm gh} = D^2. \tag{14.213}$$

The zero modes are precisely the normalizable eigenfunctions of the operator $M_{\mu\nu}$ with zero eigenvalues. They are given explicitly by the definition

$$Z_\mu^{(i)} = \frac{\partial A_\mu^{\rm cl}}{\partial\gamma_i} + D_\mu^{\rm cl}\Lambda^i. \tag{14.214}$$

They must satisfy the gauge condition

$$D_\mu^{\rm cl} Z_\mu^{(i)} = 0. \tag{14.215}$$

Compare equations (14.214) and (14.215) with equations (14.194) and (14.196). The inner product of zero modes is defined by the relation

$$\begin{aligned} U^{ij} = \langle Z^{(i)}|Z^{(j)}\rangle &= -\frac{2}{g^2}\int d^4x\ {\rm tr}\ Z_\mu^{(i)} Z^{\mu(j)} \\ &= \frac{1}{g^2}\int d^4x Z_\mu^{(i)a} Z^{\mu(j)a}. \end{aligned} \tag{14.216}$$

Let us consider for simplicity the case of the anti-instanton with topological charge $k = -1$ and $SU(2)$ gauge group. In this case we have $4|k|N = 4.1.2 = 8$ zero modes. There are four translational zero modes, one dilatational zero mode, three gauge modes. If we generalize to $SU(N)$ then the number of zero modes becomes $4N$: four translational, one dilatational and $4N-5$ gauge orientations. Recall that the $SU(N)$ solution is obtained by embedding the $SU(2)$ solution in an $N\times N$ matrix and then rotating by a unitary matrix U as

$$A_\mu^{SU(N)} = U\begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix} U^\dagger. \tag{14.217}$$

The stability group is obviously $SU(N-2)\times U(1)$ and thus the $4N-5$ gauge orientations find their origin in the set of all independent unitary matrices U which is given by the coset or Grassmannian

$$SU(N)/(SU(N-2)\times U(1)). \tag{14.218}$$

The four translational zero modes and the one dilatational zero mode and their

normalization are given respectively by (with collective coordinates denoted by x_0^μ (below we should always shift $x \longrightarrow x - x_0$) and ρ respectively)

$$Z_\mu^{(\nu)} = F_{\mu\nu}^{\rm cl} \rightarrow U^{\mu\nu} = S_{\rm cl}\delta^{\mu\nu} \tag{14.219}$$

$$Z_\mu^{(\rho)} = -2\frac{\rho\bar{\sigma}_{\mu\nu}x_\nu}{(x^2+\rho^2)^2} \rightarrow U^{\mu\nu} = 2S_{\rm cl}. \tag{14.220}$$

The gauge zero modes are obtained from equation (14.190). The gauge orientations are encoded in the matrix $U(\theta)$ which is an element of the Grassmannian (14.218). We take $U(\theta) = \exp(\theta^a T_a)$. The group vielbein $e_\alpha^a(\theta)$, where the index α is curved in contrast to the index a which is flat, is defined by $U^{-1}\partial U/\partial\theta^\alpha = e_\alpha^a T_a$. The vielbein is essentially the square root of the group metric, viz $g_{\alpha\beta} = e_\alpha^a e_\beta^a$. The Haar measure and the group volume are then given by

$$\mu(\theta) = \det e_\alpha^a d^3\theta = \sqrt{\det g_{\alpha\beta}}\, d^3\theta, \quad V = \int \mu(\theta). \tag{14.221}$$

After some calculation we find that the gauge zero modes and their normalization are given by [10]

$$Z_\mu^{(\alpha)} = U^{-1}\left[D_\mu(A(\theta=0))\left(\frac{x^2}{x^2+\rho^2}\partial^\alpha U\cdot U^{-1}\right)\right]U \rightarrow U^{\alpha\beta} = g_{\alpha\beta}\left(\frac{1}{2}\rho^2 S_{\rm cl}\right). \tag{14.222}$$

The metric U^{ij} in the space of zero modes is therefore an 8×8 diagonal matrix given by

$$U^{ij} = U\begin{pmatrix} S_{\rm cl}\delta^{\mu\nu} & 0 & 0 \\ 0 & 2S_{\rm cl} & 0 \\ 0 & 0 & g_{\alpha\beta}\left(\frac{1}{2}\rho^2 S_{\rm cl}\right)\end{pmatrix}U^\dagger. \tag{14.223}$$

We compute the measure

$$\sqrt{\det U}\, d^4x_0 d\rho d^3\theta = \frac{2^{11}\pi^8\rho^3}{g^8}\sqrt{\det g_{\alpha\beta}(\theta)}\, d^4x_0 d\rho d^3\theta. \tag{14.224}$$

Generalization to $SU(N)$ reads

$$\sqrt{\det U}\, d^4x_0 d\rho d^{4N-5}\theta = \frac{2^{2N+7}\pi^{4N}\rho^{4N-5}}{g^{4N}}\sqrt{\det g_{\alpha\beta}(\theta)}\, d^4x_0 d\rho d^{4N-5}\theta. \tag{14.225}$$

We see from the action (14.212) that the integration over the fluctuation field $A_\mu^{\rm qu}$ is Gaussian yielding, therefore, an amputated determinant $(\det{}' M)^{-1/2}$, i.e. the determinant of M leaving out its zero eigenvalues corresponding to the zero modes. The remaining path integral over the zero modes can then be converted to an integral over the collective coordinates as

$$\int \mathcal{D}A_\mu \ldots \exp(-S \ldots) = \sum_{k=-\infty}^{+\infty} \int \prod_{i=1}^{4N} \frac{d\gamma_i}{\sqrt{2\pi}} \sqrt{\det U} \times \exp(-S_{\rm cl})(\det{}' M)^{-1/2} \ldots \tag{14.226}$$

Obviously, the dots stand for terms depending on the ghost fields.

For the case of the anti-instanton with topological charge $k = -1$ we have $\prod_{i=1}^{4N} d\gamma_i = d^4x_0 d\rho d^{4N-5}\theta$ and hence we obtain the contribution

$$\int \mathcal{D}A_\mu \ldots \exp(-S \ldots) = \frac{1}{(2\pi)^{2N}} \frac{2^{2N+7}\pi^{4N}}{g^{4N}} \int d^4x_0 \int \rho^{4N-5} d\rho \times \int d^{4N-5}\theta \sqrt{\det g}\, \exp(-S_{\rm cl})(\det{}' M)^{-1/2} \ldots \tag{14.227}$$

If we are only interested in calculating gauge-invariant correlators then we can integrate out gauge orientations yielding the volume of the Grassmannian (14.218), viz

$$\int d^{4N-5}\theta \sqrt{\det g} = \text{Volume}\left(\frac{SU(N)}{SU(N-2) \times U(1)}\right) = \frac{2^{4N-5}\pi^{2N-2}}{(N-1)!(N-2)!}. \tag{14.228}$$

Hence, the contribution of the anti-instanton with $k = -1$ to the path integral reduces to

$$\int \mathcal{D}A_\mu \ldots \exp(-S \ldots) = \frac{1}{(N-1)!(N-2)!} \frac{2^{4N+2}\pi^{4N-2}}{g^{4N}} \int d^4x_0 \times \int \rho^{4N-5} d\rho\, \exp(-S_{\rm cl})(\det{}' M)^{-1/2} \ldots \tag{14.229}$$

If fermions are added to the theory then there will be extra zero modes and collective coordinates which should be treated in the same way.

The 1-loop correction due to the non-zero modes is captured by the product of the determinants corresponding to the gauge and the ghost fields which are given, respectively, by

$$(\det{}' M)^{-1/2} = (\det{}' \Delta_-)^{-1}, \;\; \Delta_- = -D^2 - \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu} \tag{14.230}$$

$$\det{}' M_{\rm gf} = (\det \Delta_+)^{+1/2}, \;\; \Delta_+ = -D^2. \tag{14.231}$$

If n real adjoint scalars and $\mathcal{N}$ Majorana spinors are included in the action an extra product of three determinants is reproduced which is given by

$$(\det \Delta_+)^{-n/4}(\det{}' \Delta_-)^{\mathcal{N}/4}(\det \Delta_+)^{+\mathcal{N}/4}. \tag{14.232}$$

The complete 1-loop correction due to the non-zero modes is then

$$(\det{}' \Delta_-)^{-1+\mathcal{N}/4}(\det \Delta_+)^{(2+\mathcal{N}-n)/4}. \tag{14.233}$$

This result reduces to the ratio of the two determinants Δ_+ and Δ_- (and hence equals 1 at least formally) for the values of n and $\mathcal{N}$ such that $\mathcal{N} = 1 + n/2$ which correspond to supersymmetric Yang–Mills gauge theories where $\mathcal{N}$ is exactly the number of supersymmetries. See the next chapter for the construction of these remarkable theories.

14.3 Exercises

Exercise 1:

- Rewrite the Yang–Mills action in terms of the electric and magnetic fields (equation (14.2)).
- Derive the field equations of motion (14.5) and (14.6).
- Show explicitly that A_0 is not an independent dynamical variables.
- Derive the Hamiltonian density (14.12).

Exercise 2:

Show that the mapping (14.16) is a homeomorphism.

Exercise 3:

- Show that the solution of the monopole equation

$$\partial_i \phi^a(x) = g\epsilon^{abc} A_i^{\ b}(x)\phi^c(x) \tag{14.234}$$

  is given by the monopole field (16.366). Compute the corresponding field strength. Extract the abelian gauge field corresponding to the unbroken $U(1)$ group.
- Verify that the flux of the Coulomb field in (14.38) cancels exactly the flux of the Dirac string (step function term) and hence the Dirac equation $\vec{\nabla} \cdot \vec{B} = 0$ is maintained.
- Show that for a multivalued gauge function Λ the magnetic flux $\Phi = \int_\Sigma \vec{B}\vec{dS}$ is not constant.

Exercise 4:

- Show that the wavefunctions in the northern and southern hemispheres are connected via the singular gauge transformation $U = \exp(2ieg_m\phi)$ (see equation (14.40)) by the relation

$$\psi^N = U\psi^S. \tag{14.235}$$

- Construct explicitly the singular gauge rotation in $SU(2)$ that transforms the singular Dirac–Wu–Yang abelian magnetic monopole into the regular

't Hooft–Polyakov non-abelian monopole. Show in particular that the Dirac string singularity cancels under this gauge transformation.

Exercise 5:

Verify the results

$$\partial_\mu^2(1/r^2) = -4\pi^2\delta^4(x) \tag{14.236}$$

$$-i\tau_{\mu\alpha}\frac{(x-a)_\alpha}{(x-a)^2} = iU^{-1}\partial_\mu U, \;\; U = \frac{i(x-a)_\mu}{|x-a|}\bar{\tau}_\mu. \tag{14.237}$$

Exercise 6:

Solve the anti-self-duality condition $F_{\alpha\beta}^{a-In} = -\tilde{F}_{\alpha\beta}^{a-In}$ for the anti-instanton ansatz

$$A_\mu^{\;a-In} = -i\beta\bar{\tau}_{\mu\alpha}\partial_\alpha \ln\phi(r). \tag{14.238}$$

Determine the constant β and the function ϕ. Determine the singular gauge transformation U^{-1} which moves the singularity at $x = a$ to $x \longrightarrow \infty$. Construct the regular anti-instanton gauge configuration as $A_\mu^{\;a-In'} = U(A_\mu^{\;a-In} + i\partial_\mu)U^{-1}$. Determine the asymptotic behavior and verify explicitly that the topological charge is indeed $k = -1$. Verify the anti-self-duality condition for the regular configuration.

Exercise 7:

Show that instantons do not curve Euclidean spacetime because of their topological nature. Hint: calculate the energy–momentum tensor for a self-dual field configuration.

Exercise 8:

Show that the moduli space $\mathcal{I}_{1,2}$ of a single instanton in an $SU(2)$ gauge theory is given by the hyper-Kahler manifold

$$\mathcal{I}_{1,2} = \mathbf{R}^4 \times \mathbf{R}^4/\mathbf{Z}_2. \tag{14.239}$$

Exercise 9:

Show explicitly that the field strength F_{mn} associated with the ADHM ansatz is self-dual by showing that

$$\begin{aligned} F_{mn} &= \partial_m A_n - \partial_n A_m + g[A_m, A_n] \\ &= \frac{4}{g}\bar{U}b\sigma_{mn}f\bar{b}U. \end{aligned} \tag{14.240}$$

Exercise 10:

- Compute the expanded quadratic action (14.212).
- Show explicitly that $Z_\mu^{(i)}$ given by (14.214) is an eigenfunctions of the operator $M_{\mu\nu}$ with zero eigenvalue.
- Show that the four translational zero modes and the one dilatational zero mode and their normalization are given, respectively, by

$$Z_\mu^{(\nu)} = F_{\mu\nu}^{\rm cl} \to U^{\mu\nu} = S_{\rm cl}\delta^{\mu\nu} \tag{14.241}$$

$$Z_\mu^{(\rho)} = -2\frac{\rho\bar{\sigma}_{\mu\nu}x_\nu}{(x^2+\rho^2)^2} \to U^{\mu\nu} = 2S_{\rm cl}. \tag{14.242}$$

- Show that the gauge zero modes and their normalization are given by

$$Z_\mu^{(\alpha)} = U^{-1}\left[D_\mu(A(\theta=0))\left(\frac{x^2}{x^2+\rho^2}\partial^\alpha U\cdot U^{-1}\right)\right]U \to U^{\alpha\beta} = g_{\alpha\beta}\left(\frac{1}{2}\rho^2 S_{\rm cl}\right). \tag{14.243}$$

Exercise 11:

There is a connection between instantons and tunneling trajectories in quantum mechanics. Consider for example the quantum-mechanical ϕ^4 model

$$S = \int dt\left(\frac{1}{2}\dot{\phi}^2 - \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\right). \tag{14.244}$$

The trivial minimum is given by $\phi_\pm = \pm\mu/\sqrt{\lambda}$. The one-instanton solutions parametrized by the initial time t_0 (here we are really dealing with kinks) is

$$\phi = \frac{\mu}{\sqrt{\lambda}}\tanh\frac{\mu(t-t_0)}{\sqrt{2}}. \tag{14.245}$$

These interpolate between ϕ_+ at $t=+\infty$ and ϕ_- at $t=-\infty$. In other words these solutions allow tunneling between the two wells of the potential.

Show that their contribution to the path integral is

$$Z = e^{-\frac{2\sqrt{2}\mu^3}{3\lambda}}\int dt_0. \tag{14.246}$$

This is very large in the large time limit but negligible in the limit of small time in accord with the fact that these solutions are tunneling trajectories.

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Advanced topics
Badis Ydri

Chapter 15

Introducing supersymmetry

In this chapter we introduce supersymmetry following mostly [1]. In particular, we will emphasize the formal quantum field theory aspects of the formalism of global $N = 1$ supersymmetry with a detailed calculation of the corresponding F- and d-terms following also [2]. A brief description of $N = 2$ is also given. The classic text on supersymmetry of Wess and Bagger [3] remains, in our view, one of the best books on quantum field theory. We have also benefited from [4, 5].

15.1 Lorentz symmetry revisited

15.1.1 Lorentz/Poincaré invariance

The spacetime metric is taken to be

$$g_{\mu\nu} = (+1, -1, -1, -1). \tag{15.1}$$

We will use the Weyl or chiral representation of Dirac matrices given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \ \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{15.2}$$

where we have introduced the spin matrices

$$\sigma^\mu = (1, \sigma^i) = (1, -\sigma_i) = \bar{\sigma}_\mu, \ \bar{\sigma}^\mu = (1, -\sigma^i) = (1, \sigma_i) = \sigma_\mu. \tag{15.3}$$

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15.4}$$

There are six generators of the Lorentz group $SO(1, 3)$. The rotation generators are J_i and the boost generators are K_i, $i = 1, 2, 3$. The commutation relations are

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \ [J_i, K_j] = i\epsilon_{ijk}K_k, \ [K_i, K_j] = i\epsilon_{ijk}K_k. \tag{15.5}$$

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This Lie algebra is a complexified $SU(2) \times SU(2)$. Indeed, by defining the complex generators $J_i^{\pm} = (J_i \pm iK_i)/2$ we can replace the above commutation relations by

$$\left[J_i^{\pm}, J_j^{\pm}\right] = i\epsilon_{ijk}J_k^{\pm}, \ \left[J_i^{\pm}, J_j^{\mp}\right] = 0. \tag{15.6}$$

This is the group $SL(2, C)$. The group $SL(2, C)$ is the universal cover of the Lorentz group $SO(1, 3)$ in the same way that $SU(2)$ is the universal cover of the rotation group $SO(3)$.

The Poincaré group is a semi-direct product of the Lorentz group and the group of translations. The generators of translations are the components of the energy–momentum 4-vector operator denoted $P_\mu = (P_0, P_i)$. The energy operator P_0 is a scalar while the momentum operator P_i is an ordinary vector and hence we must have the commutation relations

$$[J_i, P_j] = i\epsilon_{ijk}P_k, \ [J_i, P_0] = 0, \ [K_i, P_j] = -iP_0, \ [K_i, P_0] = -iP_j. \tag{15.7}$$

We must also have

$$[P_\mu, P_\nu] = 0. \tag{15.8}$$

The generators of the Lorentz group can also be defined by $M_{0i} = -M_{i0} = K_i$ and $M_{ij} = \epsilon_{ijk}J_k$ such that the Poincaré algebra becomes

$$[P_\mu, P_\nu] = 0 \tag{15.9}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\nu\rho}M_{\mu\sigma} - ig_{\mu\rho}M_{\nu\sigma} - ig_{\nu\sigma}M_{\mu\rho} + ig_{\mu\sigma}M_{\nu\rho} \tag{15.10}$$

$$[M_{\mu\nu}, P_\rho] = -ig_{\rho\mu}P_\nu + ig_{\rho\nu}P_\mu. \tag{15.11}$$

The Dirac fermion is a 4-(complex)component spinor which transforms reducibly under Lorentz transformations. It transforms as the reducible $(1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group.

The chiral Dirac (Weyl) fermion is a 2-component complex spinor which transforms irreducibly under Lorentz transformations.

A left-handed Weyl spinor ψ transforms as the irreducible $(1/2, 0)$ representation of the Lorentz group, whereas a right-handed Weyl spinor $\bar{\lambda}$ transforms as the irreducible $(0, 1/2)$ representation of the Lorentz group. The Dirac spinor is a direct sum of left-handed and right-handed Weyl spinors, viz

$$\psi_D = \begin{pmatrix} \psi \\ \bar{\lambda} \end{pmatrix}. \tag{15.12}$$

A Majorana (real) fermion is a Dirac fermion ψ_D with $\lambda = \psi$. Spacetime dimension four is special in the sense that Majorana (real) and Weyl (chiral) fermions are equivalent.

The left-handed 2-component Weyl spinor ψ transforms under the action of an element $\mathcal{M}$ of $SL(2, C) \simeq SO(1, 3)$ as

$$\psi_\alpha \longrightarrow \psi'_\alpha = \mathcal{M}_\alpha{}^\beta \psi_\beta. \tag{15.13}$$

A general element of $SL(2, C)$ is of the form

$$\mathcal{M} = \exp(iz_i\sigma_i), \;\; z_i \in C. \tag{15.14}$$

The right-handed 2-component Weyl spinor $\bar{\lambda}$ transforms under the action of $\mathcal{M}$ as

$$\bar{\lambda}_{\dot{\alpha}} \longrightarrow \bar{\lambda}'_{\dot{\alpha}} = (\mathcal{M}^*)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\lambda}_{\dot{\beta}}. \tag{15.15}$$

Thus, the representation $\mathcal{M}$ and its complex conjugate $\mathcal{M}^*$ are not equivalent (in contrast with $SU(2)$).

We introduce the Levi-Civita antisymmetric tensors

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \;\; \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{15.16}$$

Then we will raise and lower indices as follows

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \;\; \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \tag{15.17}$$

$$\bar{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}}, \;\; \bar{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\beta}}. \tag{15.18}$$

By using the facts $\epsilon^{\alpha\beta} = (i\sigma^2)^{\alpha\beta}$, $\epsilon_{\alpha\beta} = (-i\sigma^2)_{\alpha\beta}$, $\sigma_2\sigma_i^T\sigma_2 = -\sigma_i$ and also the fact that $\mathcal{M} = a\mathbf{1} + b_i\sigma_i$ we can show that

$$\psi'^\alpha = \psi^\beta(\sigma_2\mathcal{M}^T\sigma_2)_\beta{}^\alpha = \psi^\beta(\mathcal{M}^{-1})_\beta{}^\alpha. \tag{15.19}$$

Similarly, we can show that

$$\bar{\lambda}'^{\dot{\alpha}} = \bar{\lambda}^{\dot{\beta}}(\mathcal{M}^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}. \tag{15.20}$$

Explicitly the Dirac spinor is given by

$$(\psi_D)_{\hat{\alpha}} = \begin{pmatrix} \psi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix}. \tag{15.21}$$

The undotted index $\alpha = 1, 2$ is a chiral index, the dotted index $\dot{\alpha} = \dot{1}, \dot{2}$ is an anti-chiral index, whereas the hatted index $\hat{\alpha} = 1, 2, \dot{1}, \dot{2}$ is a Dirac index.

The Dirac conjugate of the Dirac spinor ψ_D is

$$(\bar{\psi}_D)^{\hat{\alpha}} = (\lambda^\alpha \bar{\psi}_{\dot{\alpha}}). \tag{15.22}$$

The dotted and undotted indices are related by hermitian conjugation. Indeed we can see from the transformation laws that

$$(\psi_\alpha)^+ = \bar{\psi}_{\dot{\alpha}}, \;\; (\lambda^\alpha)^+ = \bar{\lambda}^{\dot{\alpha}}. \tag{15.23}$$

The spin matrices σ^μ and $\bar{\sigma}^\mu$ have undotted–dotted and dotted–undotted indices respectively. In other words we have $(\sigma^\mu)_{\alpha\dot{\beta}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$.

The Lorentz generators in the spinor representation are given by

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}. \tag{15.24}$$

The $SL(2, C)$ generators for ψ_α and $\bar{\lambda}^{\dot{\alpha}}$ are $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ respectively. They are given explicitly by

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha{}^\beta &= \frac{i}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}\bar{\sigma}^{\nu\dot{\gamma}\beta} - \sigma^\nu_{\alpha\dot{\gamma}}\bar{\sigma}^{\mu\dot{\gamma}\beta}\right) \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &= \frac{i}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\gamma}\sigma^\nu_{\gamma\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\gamma}\sigma^\mu_{\gamma\dot{\beta}}\right). \end{aligned} \tag{15.25}$$

There are two Casimir operators of the Poincaré group. These are $P^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$ where W^μ is the Pauli–Lubanski vector defined by

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}. \tag{15.26}$$

Therefore, irreducible representations of the Poincaré group are characterized by two numbers.

For massive particles we can go to the rest frame where $P_\mu = (m, 0, 0, 0)$ and thus $P^2 = m^2$ and $W^2 = -m^2 s(s+1)$ where s is the spin of the particle. The irreducible representations of the Poincaré group are then characterized by the mass m and the spin s.

For massless particles there is no rest frame and usually we work in the reference frame where $P_\mu = (E, 0, 0, E)$ and thus $P^\mu P_\mu = 0$ and $P^\mu W_\mu = 0$. We can also verify that $W^\mu W_\mu = 0$ and thus we must have $W^\mu = hP^\mu$ where h is called the helicity, which is the projection of angular momentum along the direction of motion. Thus irreducible representations of the Poincaré group are characterized by a single number which is the helicity.

15.1.2 Spinor calculus

We can suppress spinor indices consistently by adopting the summation convention

$$\psi^\alpha{}_\alpha \text{ and } \bar{\chi}_{\dot{\alpha}}{}^{\dot{\alpha}}. \tag{15.27}$$

For example

$$\psi\chi = \psi^\alpha\chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \tag{15.28}$$

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}\bar{\chi}^{\dot{\alpha}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\alpha}}\bar{\chi}^{\dot{\beta}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} \tag{15.29}$$

$$(\psi\chi)^+ = \left(\psi^\alpha\chi_\alpha\right)^+ = (\chi_\alpha)^+(\psi^\alpha)^+ = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}. \tag{15.30}$$

Also (by using $(\sigma^\mu)^*_{\alpha\dot{\beta}} = (\sigma^\mu)_{\beta\dot{\alpha}}$),

$$(\chi\sigma^{\mu}\bar{\psi})^{+} = \left(\chi^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\right)^{+} = (\bar{\psi}^{\dot{\beta}})^{+}(\sigma^{\mu})^{*}_{\alpha\dot{\beta}}(\chi^{\alpha})^{+} = \psi^{\beta}(\sigma^{\mu})_{\beta\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \psi\sigma^{\mu}\bar{\chi}\,. \tag{15.31}$$

By using $(\sigma^{\mu})^{*}_{\alpha\dot{\beta}} = (\sigma^{\mu})_{\beta\dot{\alpha}}$ and $(\bar{\sigma}^{\mu})^{*\dot{\alpha}\beta} = (\bar{\sigma}^{\mu})^{\beta\dot{\alpha}}$ we can show that $(\sigma^{\mu\nu})^{*}_{\alpha}{}^{\beta} = (\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}$ and thus

$$(\xi\sigma^{\mu\nu}\theta)^{+} = \bar{\theta}\bar{\sigma}^{\mu\nu}\bar{\xi}. \tag{15.32}$$

Some of the Fierz identities we will use extensively are

$$\theta^{\alpha}\theta^{\beta} = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta,\ \ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \tag{15.33}$$

$$\phi\sigma^{\mu}\bar{\chi} = -\bar{\chi}\bar{\sigma}^{\mu}\phi \tag{15.34}$$

$$(\theta\phi)(\theta\psi) = -\frac{1}{2}(\theta\theta)(\phi\psi) \tag{15.35}$$

$$(\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) = -\frac{1}{2}(\bar{\phi}\bar{\psi})(\bar{\theta}\bar{\theta}) \tag{15.36}$$

$$(\theta\phi)(\chi\sigma^{\mu}\bar{\eta}) = -\frac{1}{2}[(\theta\sigma^{\mu}\bar{\eta})(\chi\phi) - 2i(\theta\sigma_{\nu}\bar{\eta})(\chi\sigma^{\mu\nu}\phi)] \tag{15.37}$$

$$(\theta\phi)(\bar{\chi}\bar{\eta}) = -\frac{1}{2}(\theta\sigma^{\mu}\bar{\eta})(\bar{\chi}\bar{\sigma}_{\mu}\phi) \tag{15.38}$$

$$(\theta\sigma^{\mu}\bar{\theta})(\sigma^{\nu}\bar{\theta})_{\alpha} = \frac{1}{2}\eta^{\mu\nu}\theta_{\alpha}(\bar{\theta}\bar{\theta}) + i(\sigma^{\mu\nu}\theta)_{\alpha}(\bar{\theta}\bar{\theta}) \tag{15.39}$$

$$(\theta\sigma^{\mu}\bar{\theta})(\theta\sigma^{\nu})_{\dot{\alpha}} = \frac{1}{2}\eta^{\mu\nu}\bar{\theta}_{\dot{\alpha}}(\theta\theta) - i(\bar{\theta}\bar{\sigma}^{\mu\nu})_{\alpha}(\theta\theta) \tag{15.40}$$

$$(\theta\sigma^{\mu}\bar{\theta})(\theta\sigma^{\nu}\bar{\theta}) = \frac{1}{2}\eta^{\mu\nu}(\bar{\theta}\bar{\theta})(\theta\theta). \tag{15.41}$$

15.2 Supersymmetry algebra and representations

It is well established (the Coleman–Mandula theorem [6]) that the only symmetries of the S-matrix are:

- Poincaré invariance;
- The discrete symmetries C, P, T;
- Internal symmetries.

A central assumption underlying the Coleman–Mandula theorem is the requirement that the symmetry Lie algebra involves only commutators. It can be shown (the

Haag–Lopuszanski–Sohnius theorem [7]) that supersymmetry which involves anti-commuting generators as well as commuting generators is the only possible extension of Poincaré symmetry.

The new supersymmetry generators will be denoted by $Q_\alpha^{\ A}$, $A = 1, \ldots, N$. They must be anticommuting and thus it is natural to assume that they transform under the Lorentz group as the undotted spinors ψ_α and ψ^α, i.e. as the (1/2, 0) representation. The adjoint of these generators are $\bar{Q}_{\dot{\alpha}A}$ which transform under the Lorentz group as the dotted spinors $\bar{\lambda}^{\dot{\alpha}}$ and $\bar{\lambda}_{\dot{\alpha}}$, i.e. as the (0, 1/2) representation. Explicitly we must have

$$\left[M_{\mu\nu},\, Q_\alpha^{\ A}\right] = -(\sigma_{\mu\nu})_\alpha^{\ \ \beta} Q_\beta^{\ A},\ \left[M_{\mu\nu},\, \bar{Q}_A^{\ \dot{\alpha}}\right] = -(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\ \ \dot{\beta}} Q_A^{\ \dot{\beta}}. \tag{15.42}$$

The anticommutator $\{Q_\alpha^{\ A}, \bar{Q}_{\dot{\beta}B}\}$ must transform as the tensor product (1/2, 0) $\otimes$ (0, 1/2), i.e. as the vector representation (1/2, 1/2). It must, therefore, be proportional to P^μ. We write $\{Q_\alpha^{\ A}, \bar{Q}_{\dot{\beta}B}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu C_B^{\ A}$. It is almost obvious that C is a positive definite hermitian matrix. Hence, we can choose the supersymmetry generators such as $C_B^{\ A} = \delta_B^{\ A}$. We get then

$$\left\{Q_\alpha^{\ A}, \bar{Q}_{\dot{\beta}B}\right\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta_B^{\ A}. \tag{15.43}$$

The supersymmetry generators commute with the generators of translations, viz [1]

$$\left[P_\mu,\, Q_\alpha^{\ A}\right] = \left[P_\mu,\, \bar{Q}_A^{\ \dot{\alpha}}\right] = 0. \tag{15.44}$$

Next we would like to compute the anticommutators $\{Q_\alpha^{\ A}, Q_\beta^{\ B}\}$. This must transform as the tensor product (1/2, 0) $\otimes$ (1/2, 0) = (0, 0) $\oplus$ (1, 0). Recall that $M_{\mu\nu}$ transforms in the (1, 0) representation. Thus we have in general $\{Q_\alpha^{\ A}, Q_\beta^{\ B}\} = \epsilon_{\alpha\beta} X^{AB} + \epsilon_{\beta\gamma}(\sigma^{\rho\sigma})_\alpha^{\ \ \gamma} M_{\rho\sigma} Y^{AB}$. The two Lorentz scalars matrices X^{AB} and Y^{AB} are antisymmetric and symmetric, respectively, under the exchange $A \leftrightarrow B$. Since the supersymmetry generators commute with the generators of translations we must have $\epsilon_{\beta\gamma}(\sigma^{\rho\sigma})_\alpha^{\ \ \gamma}[M_{\rho\sigma}, P_\theta] Y^{AB} = 0$ and thus $Y^{AB} = 0$. We get then

$$\left\{Q_\alpha^{\ A}, Q_\beta^{\ B}\right\} = \epsilon_{\alpha\beta} X^{AB}. \tag{15.45}$$

Similarly,

$$\left\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\right\} = \epsilon_{\dot{\alpha}\dot{\beta}} (X^{AB})^+. \tag{15.46}$$

The complex Lorentz scalars X^{AB} (which can also be checked to commute with $\bar{Q}_{\dot{\alpha}}^{\ \ A}$ and $\bar{Q}_{\dot{\beta}}^{\ \ B}$) are called central charges.

Let B_a be the generators of the internal global symmetry group, viz

$$[B_a, B_b] = i f_{ab}^{\ \ c} B_c. \tag{15.47}$$

Clearly these generators must be Lorentz scalars under Poincaré transformations, viz $[P^\mu, B_a] = [M_{\mu\nu}, B_a] = 0$. Under supersymmetry generators we must have

$$\left[Q_\alpha^{\ A}, B_a\right] = S_a^{\ A}{}_B Q_\alpha^{\ B}, \ [\bar{Q}_{\dot{\alpha}A}, B_a] = -(S_a^*)_A{}^B \bar{Q}_{\dot{\alpha}B}. \tag{15.48}$$

The central charges X^{AB} generate an abelian invariant subalgebra of the compact Lie algebra $\mathcal{A}$ generated by B_a. To see this we first use the Jacobi identity $[B_a, \{Q_\alpha^{\ A}, Q_\beta^{\ B}\}] - \{Q_\beta^{\ B}, [B_a, Q_\alpha^{\ A}]\} + \{Q_\alpha^{\ A}, [Q_\beta^{\ B}, B_a]\} = 0$ to deduce

$$[B_a, X^{AB}] + S_a^{\ B}{}_C X^{AC} - S_a^{\ A}{}_C X^{BC} = 0. \tag{15.49}$$

This means that X^{AB} (which can always be rewritten as linear combinations of the B_a) generate an invariant subalgebra of $\mathcal{A}$.

Next, it is not difficult to show that $[X^{AB}, X^{CD}] = 0$ which means that the X^{AB} generate an abelian invariant subalgebra of $\mathcal{A}$. In general, the algebra $\mathcal{A}$ is a direct sum of a semi-simple algebra $\mathcal{A}_1$ and an abelian algebra $\mathcal{A}_2$. Hence X^{AB} must be in $\mathcal{A}_2$ and as a consequence $[B_a, X^{AB}] = 0$.

As pointed out above we can write

$$X^{AB} = (a^a)^{AB} B_a, \ (a^a)^{AB} = -(a^a)^{BA}, \ (a^a)^{AB} = -(a^a)_{AB}. \tag{15.50}$$

Substituting in equation (15.49) we get

$$S_a^{\ B}{}_C (a^b)^{AC} = S_a^{\ A}{}_C (a^b)^{BC}. \tag{15.51}$$

By using the fact that S_a is hermitian we rewrite this equation as

$$S_a^{\ A}{}_C (a^b)^{CB} = -(a^b)^{AC} (S_a^*)_C{}^B. \tag{15.52}$$

The coefficients $S_a^{\ A}{}_B$ furnish a representation of the algebra $\mathcal{A}$ while the coefficients $(a^a)^{AB}$ intertwine the representation S_a with its complex conjugate $-S_a^*$.

Since $X^{AB} = (a^a)^{AB} B_a$ we have

$$(X^{AB})^+ = -(a^a)^*_{AB} B_a. \tag{15.53}$$

For $N = 1$ the central charges X^{AB} (and as a consequence the coefficients $(a^a)^{AB} = 0$) vanish, whereas the coefficients S_a are real. We have then $[Q_\alpha, B_a] = S_a Q_\alpha$ and $[\bar{Q}_{\dot{\alpha}}, B_a] = -S_a \bar{Q}_{\dot{\alpha}}$. Furthermore, from the Jacobi identity $[[Q_\alpha, B_a], B_b] + [[B_b, Q_\alpha], B_a] + [[B_a, B_b], Q_\alpha] = 0$ we conclude that we must have $f_{ab}^c = 0$ and thus the algebra $\mathcal{A}$ is abelian. We can always rescale the abelian generators B_a such that

$$[Q_\alpha, B_a] = Q_\alpha, \ [\bar{Q}_{\dot{\alpha}}, B_a] = -\bar{Q}_{\dot{\alpha}}. \tag{15.54}$$

Clearly any linear combination of the abelian generators B_a will satisfy the above equation. We can check that there is only one independent combination of the B_a (which we will call R) which acts non-trivially on the supersymmetry generators, whereas all other combinations commute with Q_α and $\bar{Q}_{\dot{\alpha}}$. We have then

$$[Q_\alpha, R] = Q_\alpha, \ [\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}}. \tag{15.55}$$

This global $U(1)$ symmetry is known as R-symmetry and obviously the supersymmetry generators Q_α and $\bar{Q}_{\dot{\alpha}}$ carry R-charges $+1$ and -1 respectively.

The R-symmetry group for $N > 1$ supersymmetry is $SU(N)_R$.

The supersymmetric Casimirs required to characterize the irreducible representations of supersymmetry are the mass operator $P^2 = P^\mu P_\mu$ (as before) and $C^2 = C_{\mu\nu}C^{\mu\nu}$, $C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu$ where B_μ is the supersymmetric Pauli–Lubanski vector defined by

$$B_\mu = W_\mu - \frac{1}{4}\bar{Q}_{\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\alpha}\beta}Q_\beta. \tag{15.56}$$

It is quite clear that finding representations of the supersymmetry algebra on single particle states depends on (1) the number of supersymmetries, (2) whether the states are massive or massless and (3) whether or not there are central charges. We will not do this lengthy exercise at this stage but we will simply quote the most relevant results. First, we note that for massive states we will work in the rest frame $P_\mu = (m, 0, 0, 0)$, whereas for massless states we will work in the light-cone frame $P_\mu = (E, 0, 0, E)$.

We have then the following possibilities:

- An $N = 1$ massless supermultiplet (no central charges) contains four states: The two states with helicities $(\lambda, \lambda + 1/2)$ and their CPT conjugates with helicities $(-\lambda - 1/2, -\lambda)$. The most important $N = 1$ massless supermultiplets are:
  - The chiral multiplet (ψ_a, ϕ) corresponding to $\lambda = 0$. The spinor ψ_a is a Weyl fermion and ϕ a complex scalar. This is a matter multiplet transforming in some representation of the gauge group.
  - The $N = 1$ gauge multiplet (A_μ, λ_a) corresponding to $\lambda = 1/2$ (or $\lambda = -1/2$). The spinor λ_α is known as a gaugino field and it is a Weyl fermion while A_μ is a massless gauge field. They must be in the adjoint representation of the gauge group.
- An $N = 2$ massless supermultiplet (no central charges) contains eight states: The four states with helicities $(\lambda, \lambda + 1/2, \lambda + 1/2, \lambda + 1)$ and their CPT conjugates with helicities $(-\lambda - 1, -\lambda - 1/2, -\lambda - 1/2, -\lambda)$. The most important $N = 2$ massless supermultiplets are:
  - The $N = 2$ gauge multiplet $(A_\mu, \lambda_a^+, \lambda_a^-, \phi)$ corresponding to $\lambda = 0$. The spinor $\lambda_\alpha^\pm$ are Weyl fermions, A_μ is a massless gauge field while ϕ is a complex scalar (gauge scalar). They must be in the adjoint representation of the gauge group. The spinors $\lambda_\alpha^\pm$ transform as a doublet under $SU(2)_R$ while A_μ and ϕ transform as singlets. The $N = 2$ gauge multiplet is equivalent to the $N = 1$ gauge multiplet plus the $N = 1$ chiral multiplet all in the adjoint representation of the gauge group.

- The hypermultiplet $(\psi_\alpha^+, H_+, H_-, \psi_\alpha^+)$ corresponding to $\lambda = -1/2$. The spinor $\psi_\alpha^\pm$ are Weyl fermions while $H^\pm$ are complex scalars[1]. This is a matter multiplet transforming in some representation of the gauge group. The scalar fields $H^\pm$ transform as a doublet under $SU(2)_R$ while $\psi_\alpha^\pm$ transform as singlets.

- An $N = 4$ massless supermultiplet (no central charges) contains 32 states: The 16 states with helicities (λ, $\lambda + 1/2$(4 times), $\lambda + 1$(6 times), $\lambda + 3/2$(4 times), $\lambda + 2$) and their CPT conjugates. The most important case corresponds to $\lambda = -1$ which is actually CPT self-conjugate and thus contains 16 states only. This is precisely the $N = 4$ gauge multiplet containing a massless gauge field, 4 Weyl fermions and 6 real scalars, viz $(A_\mu, \lambda_a^A, \phi^I)$, $A = 1, \ldots, 4$ and $I = 1, \ldots, 6$. The $N = 4$ gauge multiplet is equivalent to the $N = 2$ gauge multiplet plus the $N = 2$ hypermultiplet all in the adjoint representation of the gauge group. The fermions λ^A transform in the fundamental representation of $SU(4)_R$, the scalar fields ϕ^I transform in the **6** of $SO(6) \simeq \text{spin}(6) = SU(4)$ while A_μ is a singlet.
- It is well known that $N = 3$ supersymmetry coincides with $N = 4$ supersymmetry because of the CPT theorem.

15.3 $N = 1$ supersymmetry

15.3.1 $N = 1$ superspace

We introduce four anticommuting Grassmannian coordinates θ^α, $\bar{\theta}_{\dot{\alpha}}$ such that $\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\} = 0$. Now we will enlarge the four-dimensional spacetime labeled by x^μ to a $(4 + 4)$-dimensional superspace labeled by $(x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$ known as $N = 1$ rigid superspace. This is a coset space defined as the $N = 1$ super-Poincaré group modded out by the Lorentz group in analogy with spacetime which can be defined as the Poincaré group modded out by the Lorentz group.

Next we rewrite the $N = 1$ supersymmetry algebra as a Lie algebra. We compute

$$[\theta Q, \bar{\theta}\bar{Q}] = 2\theta^\alpha (\sigma^\mu)_\alpha{}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} P_\mu = 2\theta\sigma^\mu\bar{\theta}P_\mu \tag{15.57}$$

$$[\theta Q, \theta Q] = [\bar{\theta}\bar{Q}, \bar{\theta}\bar{Q}] = 0. \tag{15.58}$$

Note that

$$(\theta Q)^+ = (\theta^\alpha Q_\alpha)^+ = (Q_\alpha)^+ (\theta^\alpha)^+ = \bar{Q}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \bar{Q}\bar{\theta} = \bar{\theta}\bar{Q}. \tag{15.59}$$

Superspace derivatives are defined by (recall that we raise and lower indices on spinors with the symbol ϵ, viz $\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta$, $\bar{\theta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\beta}}$)

[1] The state $(-1/2, 0, 0, -1/2)$ on its own may be CPT self-conjugate in which case we call this supermultiplet a half-hypermultiplet.

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha},\ \partial^\alpha = \frac{\partial}{\partial\theta_\alpha} = -\epsilon^{\alpha\beta}\partial_\beta \tag{15.60}$$

$$\bar{\partial}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}},\ \bar{\partial}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\partial}^{\dot{\beta}}. \tag{15.61}$$

The action of these derivatives on the Grassmannian coordinates is quite obvious. We note, however, the results

$$\partial_\alpha(\theta^\beta\theta^\gamma) = \partial_\alpha\theta^\beta\cdot\theta^\gamma - \theta^\beta\cdot\partial_\alpha\theta^\gamma = \delta_\alpha^\beta\theta^\gamma - \delta_\alpha^\gamma\theta^\beta \tag{15.62}$$

$$\partial_\rho(\theta\theta) = \delta_\rho^\alpha\cdot\theta_\alpha - \theta^\alpha\cdot\partial_\rho\theta_\alpha = \theta_\rho - \theta^\alpha\epsilon_{\alpha\beta}\delta_\rho^\beta = 2\theta_\rho \tag{15.63}$$

$$\partial^2(\theta\theta) = \partial^\alpha\partial_\alpha(\theta\theta) = 2\partial^\alpha\theta_\alpha = 4 \tag{15.64}$$

$$\bar{\partial}^2(\bar{\theta}\bar{\theta}) = -4 \tag{15.65}$$

$$\partial_{\dot{\alpha}}(\bar{\theta}\bar{\theta}) = -2\bar{\theta}_{\dot{\alpha}}. \tag{15.66}$$

Superspace integration will be given by Berezin integration. A general function for a single Grassmann variable θ is $f(\theta) = f_0 + \theta f_1$. The Berezin integral in this case is defined by

$$\int d\theta\theta = 1,\ \int d\theta = 0. \tag{15.67}$$

The integral of a general function is

$$\int d\theta f(\theta) = f_1. \tag{15.68}$$

Thus integration is equivalent to differentiation since

$$\frac{df(\theta)}{d\theta} = f_1. \tag{15.69}$$

Also, we conclude that

$$\int d\theta\frac{df(\theta)}{d\theta} = 0. \tag{15.70}$$

Furthermore, the Berezin integration is translationally invariant since

$$\int d\theta f(\theta) = \int d(\theta+\xi)f(\theta+\xi). \tag{15.71}$$

A delta function can be defined by

$$\delta(\theta) = \theta. \tag{15.72}$$

We define the Berezin measure for $N = 1$ superspace by

$$\begin{aligned} d^2\theta &= -\frac{1}{4}\epsilon_{\alpha\beta}d\theta^\alpha d\theta^\beta = \frac{1}{2}d\theta^1 d\theta^2, \\ d^2\bar{\theta} &= -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}d\bar{\theta}_{\dot{\alpha}}d\bar{\theta}_{\dot{\beta}} = \frac{1}{2}d\bar{\theta}^{\dot{1}}d\bar{\theta}^{\dot{2}}. \end{aligned} \tag{15.73}$$

This is because

$$\begin{aligned} \int d^2\theta(\theta\theta) &= \int \frac{1}{2}d\theta^1 d\theta^2(2\theta^2\theta^1) = 1, \\ \int d^2\bar{\theta}(\bar{\theta}\bar{\theta}) &= \int \frac{1}{2}d\bar{\theta}^{\dot{1}}d\bar{\theta}^{\dot{2}}(2\bar{\theta}^{\dot{2}}\bar{\theta}^{\dot{1}}) = 1. \end{aligned} \tag{15.74}$$

The full measure on the $N = 1$ superspace is

$$d^4\theta = d^2\theta d^2\bar{\theta}. \tag{15.75}$$

15.3.2 Scalar superfield

A scalar superfield is the most general scalar function $\Phi(x, \theta, \bar{\theta})$ on the $N = 1$ superspace. There are finite terms in the Taylor expansion of this field in powers of θ^α and $\bar{\theta}_{\dot{\alpha}}$ given by

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) = {} & f(x) + (\theta\phi(x)) + (\bar{\theta}\bar{\chi}(x)) + (\theta\theta)m(x) \\ & + (\bar{\theta}\bar{\theta})n(x) + (\theta\sigma^\mu\bar{\theta})v_\mu(x) + (\theta\theta)(\bar{\theta}\bar{\lambda}(x)) \\ & + (\bar{\theta}\bar{\theta})(\theta\psi(x)) + (\theta\theta)(\bar{\theta}\bar{\theta})d(x). \end{aligned} \tag{15.76}$$

It is clear that $(\bar{\theta}\bar{\sigma}^\mu\theta)v'_\mu$ and $(\theta\phi)(\bar{\theta}\bar{\psi})$ are redundant while $(\theta\theta)(\theta\lambda'(x))$ and $(\bar{\theta}\bar{\theta})(\bar{\theta}\bar{\psi}'(x))$ are zero. The fields f, m, n, v_μ, d are bosonic while ϕ, $\bar{\chi}$, $\bar{\lambda}$ and ψ are fermionic.

The action of the ordinary translation generator P_μ on Φ is, as usual, given by the differential operator $i\partial_\mu$, viz

$$\left(1 + i\epsilon^\mu P_\mu\right)\Phi(x, \theta, \bar{\theta}) = \Phi(x + \epsilon, \theta, \bar{\theta}) \rightarrow P_\mu = -i\partial_\mu. \tag{15.77}$$

Next we need to find the representation of the supersymmetry generators Q^α and $\bar{Q}_{\dot{\alpha}}$ as superspace differential operators. Let ξ^α be a constant Weyl spinor and consider the generator $\delta_\xi = i\xi Q + i\bar{\xi}\bar{Q}$. We expect that the action of $i\xi Q$ on the superfield $\Phi(x, \theta, \bar{\theta})$ is to translate the spinor θ by ξ while the action of $i\bar{\xi}\bar{Q}$ is to translate the spinor $\bar{\theta}$ by $\bar{\xi}$. Since $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu$ we expect that x^μ will also be translated. As a consequence Q_α and $\bar{Q}_{\dot{\alpha}}$ must be of the form

$$Q_\alpha = -i\left(\partial_\alpha - ic(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\right). \tag{15.78}$$

By using $Q_\alpha^+ = \bar{Q}_{\dot{\alpha}}$, $\partial_\alpha^+ = \bar{\partial}_{\dot{\alpha}}$, $(i\partial_\mu)^+ = i\partial_\mu$, $(\sigma^\mu)^*_{\alpha\dot{\beta}} = (\sigma^\mu)_{\beta\dot{\alpha}}$ and $(\bar{\theta}^{\dot{\beta}})^+ = \theta^\beta$ we get

$$\bar{Q}_{\dot{\alpha}} = i\Big(\bar{\partial}_{\dot{\alpha}} - ic^*\theta^{\beta}(\sigma^{\mu})_{\beta\dot{\alpha}}\partial_{\mu}\Big). \tag{15.79}$$

A direct calculation shows that

$$\{Q_{\alpha}, \bar{Q}_{\dot{\beta}}\}\Phi = -i(c + c^*)(\sigma^{\mu})_{\alpha\dot{\beta}}\partial_{\mu}\Phi \tag{15.80}$$

$$\begin{aligned}(1 + i\xi Q + i\bar{\xi}\bar{Q})\Phi(x^{\mu}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}) = \Phi(x^{\mu} - ic\xi\sigma^{\mu}\bar{\theta} + ic^*\theta\sigma^{\mu}\bar{\xi}, \theta^{\alpha} \\ + \xi^{\alpha}, \bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}}).\end{aligned} \tag{15.81}$$

Thus we must choose $c = 1$.

Next we can compute the action of linear supersymmetry transformations on the components of the superfield Φ. The variation of the first term in the expansion of Φ is

$$\delta_{\xi} f = -i(\xi\sigma^{\mu}\bar{\theta})\partial_{\mu} f + i(\theta\sigma^{\mu}\bar{\xi})\partial_{\mu} f. \tag{15.82}$$

The variation of the linear terms in θ and $\bar{\theta}$ are given by (by using the Fierz identities (15.35), (15.36) and (15.37))

$$\begin{aligned}\delta_{\xi}(\theta\phi) &= -i(\theta\partial_{\mu}\phi)(\xi\sigma^{\mu}\bar{\theta}) + i(\theta\partial_{\mu}\phi)(\theta\sigma^{\mu}\bar{\xi}) + \xi\phi \\ &= \frac{i}{2}\Big[(\theta\sigma^{\mu}\bar{\theta})(\xi\partial_{\mu}\phi) - 2i(\theta\sigma_{\nu}\bar{\theta})\big(\xi\sigma^{\mu\nu}\partial_{\mu}\phi\big)\Big] \\ &\quad -\frac{i}{2}(\theta\theta)(\partial_{\mu}\phi\sigma^{\mu}\bar{\xi}) + \xi\phi\end{aligned} \tag{15.83}$$

$$\begin{aligned}\delta_{\xi}(\bar{\theta}\bar{\chi}) = -i(\bar{\theta}\partial_{\mu}\bar{\chi})(\xi\sigma^{\mu}\bar{\theta}) + i(\bar{\theta}\partial_{\mu}\bar{\chi})(\theta\sigma^{\mu}\bar{\xi}) + \bar{\xi}\bar{\chi} = \frac{i}{2}(\bar{\theta}\bar{\theta})(\xi\sigma^{\mu}\partial_{\mu}\bar{\chi}) \\ -\frac{i}{2}\Big[(\theta\sigma^{\mu}\bar{\theta})(\xi\partial_{\mu}\chi) - 2i(\theta\sigma_{\nu}\bar{\theta})\big(\xi\sigma^{\mu\nu}\partial_{\mu}\chi\big)\Big]^{+} + \bar{\xi}\bar{\chi}\,.\end{aligned} \tag{15.84}$$

The variation of the cubic terms in θ and $\bar{\theta}$ are given by (here we use also the Fierz identity (15.38))

$$\begin{aligned}\delta_{\xi}((\theta\theta)(\bar{\theta}\bar{\lambda})) &= -i(\theta\theta)(\bar{\theta}\partial_{\mu}\bar{\lambda})(\xi\sigma^{\mu}\bar{\theta}) + (\theta\theta)(\bar{\xi}\bar{\lambda}) + 2(\theta\xi)(\bar{\theta}\bar{\lambda}) \\ &= \frac{i}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\big(\xi\sigma^{\mu}\partial_{\mu}\bar{\lambda}\big) + (\theta\theta)(\bar{\xi}\bar{\lambda}) + (\theta\sigma^{\mu}\bar{\theta})(\xi\sigma_{\mu}\bar{\lambda})\end{aligned} \tag{15.85}$$

$$\begin{aligned}\delta_{\xi}((\bar{\theta}\bar{\theta})(\theta\psi)) &= i(\bar{\theta}\bar{\theta})(\theta\partial_{\mu}\psi)(\theta\sigma^{\mu}\bar{\xi}) + (\bar{\theta}\bar{\theta})(\xi\psi) + 2(\bar{\theta}\bar{\xi})(\theta\psi) \\ &= -\frac{i}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\big(\partial_{\mu}\psi\sigma^{\mu}\bar{\xi}\big) + (\bar{\theta}\bar{\theta})(\xi\psi) + (\theta\sigma^{\mu}\bar{\theta})(\psi\sigma_{\mu}\bar{\xi}).\end{aligned} \tag{15.86}$$

The variation of the quadratic terms in θ and $\bar{\theta}$ are given by

$$\delta_{\xi}((\theta\theta)m(x)) = -i(\theta\theta)(\xi\sigma^{\mu}\bar{\theta})\partial_{\mu}m + 2(\theta\xi)m \tag{15.87}$$

$$\delta_\xi((\bar\theta\bar\theta)n(x)) = i(\bar\theta\bar\theta)(\theta\sigma^\mu\bar\xi)\partial_\mu n + 2(\bar\theta\bar\xi)n. \tag{15.88}$$

Also, by using the Fierz identity (15.39) we compute

$$\begin{aligned}\delta_\xi((\theta\sigma^\mu\bar\theta)v_\mu(x)) &= i\partial_\nu v_\mu(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\xi) - i\partial_\nu v_\mu(\theta\sigma^\mu\bar\theta)(\xi\sigma^\nu\bar\theta)\\ &\quad + (\xi\sigma^\mu\bar\theta)v_\mu + (\theta\sigma^\mu\bar\xi)v_\mu\\ &= \left[-\frac{i}{2}\partial^\mu v_\mu(\xi\theta)(\bar\theta\bar\theta) - \partial_\mu v_\nu(\xi\sigma^{\mu\nu}\theta)(\bar\theta\bar\theta)\right]^+\\ &\quad -\frac{i}{2}\partial^\mu v_\mu(\xi\theta)(\bar\theta\bar\theta) - \partial_\mu v_\nu(\xi\sigma^{\mu\nu}\theta)(\bar\theta\bar\theta)\\ &\quad + (\xi\sigma^\mu\bar\theta)v_\mu + (\theta\sigma^\mu\bar\xi)v_\mu.\end{aligned} \tag{15.89}$$

The variation of the last term in the expansion of Φ is (by using the Fierz identities (15.34), (15.35) and (15.36))

$$\begin{aligned}\delta_\xi((\theta\theta)(\bar\theta\bar\theta)d(x)) &= 2(\theta\theta)(\bar\theta\bar\xi)d - 2i(\theta\theta)(\bar\theta\bar\xi)(\xi\sigma^\mu\bar\theta)\partial_\mu d\\ &\quad + 2(\bar\theta\bar\theta)(\theta\xi)d + 2i(\bar\theta\bar\theta)(\theta\xi)(\theta\sigma^\mu\bar\xi)\partial_\mu d.\\ &= 2(\theta\theta)(\bar\theta\bar\xi)d + 2(\bar\theta\bar\theta)(\theta\xi)d.\end{aligned} \tag{15.90}$$

By using the above results we can derive the following supersymmetry transformation laws

$$\delta_\xi f = \xi\phi + \bar\xi\bar\chi \tag{15.91}$$

$$\delta_\xi\phi_\alpha = 2\xi_\alpha m + (\sigma^\mu)_{\alpha\dot\beta}\bar\xi^{\dot\beta}(i\partial_\mu f + v_\mu) \tag{15.92}$$

$$\delta_\xi\bar\chi^{\dot\alpha} = 2\bar\xi^{\dot\alpha}n + \xi^\beta(\sigma^\mu)_{\beta\dot\gamma}\epsilon^{\dot\gamma\dot\alpha}(i\partial_\mu f - v_\mu) \tag{15.93}$$

$$\delta_\xi m = \bar\xi\bar\lambda - \frac{i}{2}\partial_\mu\phi\sigma^\mu\bar\xi \tag{15.94}$$

$$\delta_\xi n = \xi\psi + \frac{i}{2}\xi\sigma^\mu\partial_\mu\bar\chi \tag{15.95}$$

$$\begin{aligned}\delta_\xi v_\mu &= \xi\sigma_\mu\bar\lambda + \psi\sigma_\mu\bar\xi + \frac{i}{2}\xi\partial_\mu\phi - \frac{i}{2}\partial_\mu\bar\chi\,\bar\xi\\ &\quad -\xi\sigma^{\mu\nu}\partial_\nu\phi - \partial_\nu\bar\chi\,\bar\sigma^{\mu\nu}\bar\xi\end{aligned} \tag{15.96}$$

$$\delta_\xi d = \frac{i}{2}\partial_\mu(\xi\sigma^\mu\bar\lambda - \psi\sigma^\mu\bar\xi) \tag{15.97}$$

$$\delta_\xi \psi_\alpha = 2\xi_\alpha d + i(\sigma^\mu \bar{\xi})_\alpha \partial_\mu n - \frac{i}{2}\partial^\mu v_\mu \xi_\alpha - \frac{1}{2}(\partial_\mu v_\nu - \partial_\nu v_\mu)(\xi \sigma^{\mu\nu})^\beta \epsilon_{\alpha\beta} \tag{15.98}$$

$$\delta_\xi \bar{\lambda}^{\dot{\alpha}} = 2\bar{\xi}^{\dot{\alpha}} d + i\xi^\beta (\sigma^\mu)_{\beta\dot{\gamma}} \epsilon^{\dot{\gamma}\dot{\alpha}} \partial_\mu m + \frac{i}{2}\partial^\mu v_\mu \bar{\xi}^{\dot{\alpha}} - \frac{1}{2}(\partial_\mu v_\nu - \partial_\nu v_\mu)(\bar{\sigma}^{\mu\nu}\bar{\xi})^\alpha. \tag{15.99}$$

The four last transformation laws are slightly different from those obtained by Lykken.

15.3.3 Chiral superfield

Clearly the general scalar field Φ cannot correspond to an irreducible representation of the $N = 1$ supersymmetry algebra since it obviously contains too many components. The scalar superfield Φ provides a reducible representation although not a fully reducible one, i.e. it is not given by a direct sum of irreducible representations. We would like to reduce the number of components of Φ in a supersymmetric covariant fashion and thus we must impose covariant constraints on Φ.

Towards this end we introduce the so-called covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$ which anticommute with the supersymmetry generators Q_α and $\bar{Q}_{\dot{\alpha}}$, viz

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = 0 \tag{15.100}$$

$$\{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0. \tag{15.101}$$

Then if Φ is a scalar superfield then $D_\alpha \Phi$ and $\bar{D}_{\dot{\alpha}}\Phi$ are also superfields in the sense that $(1 + \delta_\xi) D_\alpha \Phi(x, \theta, \bar{\theta}) = D_\alpha \phi(x - i\xi\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi})$[2]. Furthermore, the condition $D_\alpha \Phi = 0$ is clearly covariant under supersymmetry, i.e. $D_\alpha \delta_\xi \Phi = \delta_\xi (D_\alpha \Phi) = 0$. It is obvious that D_α, $\bar{D}_{\dot{\alpha}}$ are generalizations of the derivatives ∂_α, $\bar{\partial}_{\dot{\alpha}}$. Indeed we find

$$D_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu \tag{15.102}$$

$$\bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i\theta^\beta (\sigma^\mu)_{\beta\dot{\alpha}} \partial_\mu. \tag{15.103}$$

It is clear that

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i(\sigma^\mu)_{\alpha\dot{\beta}} \partial_\mu. \tag{15.104}$$

[2] Also $\partial_\mu \Phi$ is a superfield (in an obvious way) while $\partial_\alpha \Phi$, $\partial_{\dot{\alpha}} \Phi$ are not superfields since they do not contain a d-term.

An $N = 1$ chiral superfield is a general scalar superfield fulfilling the supersymmetric covariant constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{15.105}$$

This has fewer components than the general scalar superfield and in fact provides an irreducible representation of the $N=1$ supersymmetry algebra. Writing this condition in components we get

$$\begin{aligned} &\bar{\chi}_{\dot{\alpha}} = 0,\ \ n = 0,\ \ v_\mu = i\partial_\mu f,\ \ \psi = 0, \\ &\bar{\lambda}_{\dot{\alpha}} = -\frac{i}{2}\partial_\mu \phi^\beta (\sigma^\mu)_{\beta\dot{\alpha}},\ \ d = -\frac{1}{4}\partial_\mu \partial^\mu f. \end{aligned} \tag{15.106}$$

We can check directly that these conditions are supersymmetric invariant by using the supersymmetry transformations derived in the previous section. We get

$$\begin{aligned} &\delta_\xi f = \xi\phi,\ \ \delta_\xi \phi_\alpha = 2\xi_\alpha m + 2i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\xi}^{\dot{\beta}}\partial_\mu f, \\ &\delta_\xi \bar{\chi}^{\dot{\alpha}} = 0,\ \ \delta_\xi n = 0,\ \ \delta_\xi \psi = 0. \end{aligned} \tag{15.107}$$

Equivalently

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) = f + \theta\phi + (\theta\theta)m + i(\theta\sigma^\mu\bar{\theta})\partial_\mu f \\ -\frac{i}{2}(\theta\theta)(\partial_\mu \phi \sigma^\mu \bar{\theta}) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu \partial^\mu f. \end{aligned} \tag{15.108}$$

By using

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu},\ \ \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2\eta^{\mu\nu}, \tag{15.109}$$

and

$$\hat{\epsilon}(\sigma^\mu)^T = -\bar{\sigma}^\mu \tilde{\epsilon},\ \ (\sigma^\mu)^T \hat{\epsilon} = -\tilde{\epsilon}\bar{\sigma}^\mu, \tag{15.110}$$

where $\hat{\epsilon}^{\alpha\beta} = \epsilon^{\alpha\beta}$ and $\tilde{\epsilon}_{\alpha\beta} = \epsilon_{\alpha\beta}$, we get

$$\delta_\xi d = -\frac{1}{4}\partial_\mu \partial^\mu \phi \xi,\ \ \delta\bar{\lambda}^{\dot{\alpha}} = \partial_\mu\big(i\xi^\beta (\sigma_\mu)_{\beta\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\alpha}} m - \partial_\mu f \bar{\xi}^{\dot{\alpha}}\big). \tag{15.111}$$

We also get[3]

$$\delta_\xi v^\mu = i\partial_\mu \phi \xi - \partial_\nu \phi \sigma^{\mu\nu}\xi - \xi\sigma^{\mu\nu}\partial_\nu \phi. \tag{15.112}$$

A more elegant approach is to determine the most general form of a chiral superfield by solving directly the constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$. Let us introduce the new bosonic coordinates

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta},\ \ \bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}. \tag{15.113}$$

Then

[3] First show that $(\sigma^{\mu\nu})^T = \tilde{\epsilon}\sigma^{\mu\nu}\tilde{\epsilon}$. Then show that $\xi\sigma^{\mu\nu}\phi = \xi^\alpha(\sigma^{\mu\nu})_\alpha{}^\beta \phi_\beta = \phi_\beta(\hat{\epsilon}\sigma^{\mu\nu}\tilde{\epsilon})_\alpha{}^\beta \xi^\alpha = -\theta^\beta(\sigma^{\mu\nu})_\beta{}^\alpha \xi_\alpha$.

$$\bar{D}_{\dot{\alpha}} y^{\mu} = D_{\alpha} \bar{y}^{\mu} = 0. \tag{15.114}$$

Note also that

$$\bar{D}_{\dot{\alpha}} \theta^{\beta} = D_{\alpha} \bar{\theta}^{\dot{\beta}} = 0. \tag{15.115}$$

Then clearly the most general solution of $\bar{D}_{\dot{\alpha}} \Phi = 0$ is $\Phi = \Phi(y, \theta)$. The expansion of this superfield in powers of θ is then very simply given by

$$\Phi(y, \theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{15.116}$$

This superfield consists of two complex scalar fields, A and F, and a single complex left-handed Weyl spinor and, therefore, contains $4 + 4 = 8$ real off-shell field components. This is twice the number of degrees of freedom in the on-shell fundamental $N = 1$ massive irreducible representation of $N = 1$ supersymmetry.

Similarly we can define anti-chiral superfields by the condition $D_{\alpha}\Phi = 0$ and thus $\Phi = \Phi(\bar{y}^{\mu}, \bar{\theta})$. Clearly if Φ is a chiral (left-handed) superfield then Φ^{+} is an anti-chiral (right-handed) superfield.

In terms of y the supersymmetry generators and the covariant derivatives are given by (we simply make the replacements $\partial_{\mu} \longrightarrow \partial^{y}_{\mu}$, $\partial_{\alpha} \longrightarrow \partial_{\alpha} + i(\sigma^{\mu}\bar{\theta})_{\alpha}\partial^{y}_{\mu}$, $\bar{\partial}_{\dot{\alpha}} \longrightarrow \bar{\partial}_{\dot{\alpha}} - i(\theta\sigma^{\mu})_{\dot{\alpha}}\partial^{y}_{\mu}$)

$$\begin{aligned} D_{\alpha} &= \partial_{\alpha} + 2i(\sigma^{\mu}\bar{\theta})_{\alpha}\partial^{y}_{\mu}, \;\; \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} \\ Q_{\alpha} &= -i\partial_{\alpha}, \;\; \bar{Q}_{\dot{\alpha}} = i\left(\bar{\partial}_{\dot{\alpha}} - 2i(\theta\sigma^{\mu})_{\dot{\alpha}}\partial_{\mu}\right). \end{aligned} \tag{15.117}$$

The supersymmetry transformation of a chiral superfield is

$$\delta\Phi(y^{\mu}, \theta) = i(\xi Q + \bar{\xi}\bar{Q})\Phi(y^{\mu}, \theta) \tag{15.118}$$

$$(\Phi + \delta\Phi)(y^{\mu}, \theta) = \Phi(y^{\mu} + 2i\theta\sigma^{\mu}\bar{\xi} + i\xi\sigma^{\mu}\bar{\xi}, \theta + \xi). \tag{15.119}$$

Explicitly we have the linear supersymmetry transformations

$$\delta\Phi(y^{\mu}, \theta) = 2i\partial_{\mu}A(\theta\sigma^{\mu}\bar{\xi}) - i\sqrt{2}(\theta\theta)\left(\partial_{\mu}\psi\sigma^{\mu}\bar{\xi}\right) + \sqrt{2}\xi\psi + 2(\theta\xi)F. \tag{15.120}$$

Equivalently

$$\begin{aligned} \delta_{\xi}A &= \sqrt{2}\,\xi\psi \\ \delta_{\xi}F &= -i\sqrt{2}\,\partial_{\mu}\psi\sigma^{\mu}\bar{\xi} \\ \delta_{\xi}\psi^{\alpha} &= \sqrt{2}\,\xi^{\alpha}F + \sqrt{2}\,i\partial_{\mu}A(\sigma^{\mu}\bar{\xi})^{\alpha}. \end{aligned} \tag{15.121}$$

15.3.4 Vector superfield

A vector superfield is a general scalar field which satisfies the following covariant reality condition

$$V(x, \theta, \bar{\theta}) = V^{+}(x, \theta, \bar{\theta}). \tag{15.122}$$

In terms of components this condition reads

$$f = f^+,\ \chi = \phi,\ m = n^+,\ v_\mu = v_\mu^+,\ \lambda = \psi,\ d = d^+. \tag{15.123}$$

Thus

$$\begin{aligned} V = f + \theta\chi + \bar{\theta}\bar{\chi} + (\theta\theta)n^+ + (\bar{\theta}\bar{\theta})n + (\theta\sigma^\mu\bar{\theta})v_\mu \\ + (\theta\theta)(\bar{\theta}\bar{\lambda}) + (\bar{\theta}\bar{\theta})(\theta\lambda) + (\theta\theta)(\bar{\theta}\bar{\theta})d. \end{aligned} \tag{15.124}$$

An example of a vector superfield is $\Phi + \Phi^+$ where Φ is a chiral superfield. This is given by (where we will set $f = A$, $\phi = \sqrt{2}\psi$ and $m = F$ in equation (15.108))

$$\begin{aligned} \Phi + \Phi^+ = A + A^+ + \sqrt{2}\theta\psi + \sqrt{2}\bar{\theta}\bar{\psi} + \theta\theta F + \bar{\theta}\bar{\theta}F^+ \\ + i(\theta\sigma^\mu\bar{\theta})\partial_\mu(A - A^+) \\ -\frac{i}{\sqrt{2}}(\theta\theta)\left(\partial_\mu\psi\sigma^\mu\bar{\theta}\right) + \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})\left(\theta\sigma^\mu\partial_\mu\bar{\psi}\right) \\ -\frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu(A + A^+). \end{aligned} \tag{15.125}$$

An infinitesimal abelian gauge transformation can be defined by

$$V \longrightarrow V' = V + \Phi + \Phi^+. \tag{15.126}$$

In terms of components we have

$$\begin{aligned} f &\longrightarrow f + A + A^+ \\ \chi &\longrightarrow \chi + \sqrt{2}\psi \\ n &\longrightarrow n + F^+ \\ v_\mu &\longrightarrow v_\mu + i\partial_\mu(A - A^+) \\ \lambda &\longrightarrow \lambda + \frac{i}{\sqrt{2}}\sigma^\mu\partial_\mu\bar{\psi} \\ d &\longrightarrow d - \frac{1}{4}\partial_\mu\partial^\mu(A + A^+). \end{aligned} \tag{15.127}$$

It is clear that any action which is invariant under (15.126) will not depend on many components of V. The reason is very simple. The five first components of $\Phi + \Phi^+$ are completely unconstrained and thus they can be used to gauge away the first five components of V. Indeed we can choose

$$f + A + A^+ = 0, \chi + \sqrt{2}\psi = 0,\ n + F^+ = 0. \tag{15.128}$$

Then the transformed superfield V' becomes

$$V' = (\theta\sigma^\mu\bar{\theta})v'_\mu + (\theta\theta)(\bar{\theta}\bar{\lambda}') + (\bar{\theta}\bar{\theta})(\theta\lambda') + (\theta\theta)(\bar{\theta}\bar{\theta})d', \tag{15.129}$$

where

$$v'_\mu = v_\mu + i\partial_\mu(A - A^+), \;\; \lambda' = \lambda + \frac{i}{\sqrt{2}}\sigma^\mu\partial_\mu\bar{\psi},$$
$$d' = d - \frac{1}{4}\partial_\mu\partial^\mu(A + A^+). \tag{15.130}$$

The gauge-fixed superfield V' is called the Wess–Zumino gauge-fixed superfield. We write

$$V' = V_{WZ}. \tag{15.131}$$

Thus, any vector superfield V can be written as

$$V = V_{WZ} - (\Phi + \Phi^+). \tag{15.132}$$

As it turns out, fixing the Wess–Zumino gauge does not fix the abelian gauge symmetry. Indeed we can shift v'_μ in $V' = V_{WZ}$ and $A - A^+$ in $\Phi + \Phi^+$ as $v'_\mu \longrightarrow v'_\mu - i\partial_\mu(A' - A'^+)$ and $A - A^+ \longrightarrow A - A^+ - i\partial_\mu(A' - A'^+)$ without changing V.

15.3.5 Spinor superfield

The Wess–Zumino gauge given by equation (15.128) is not preserved by supersymmetry transformations. Thus any action written in terms of the vector superfield V will necessarily contain unphysical degrees of freedom. The spinor superfield will, on the other hand, only contain the Wess–Zumino gauge-fixed fields v_μ, λ and d. It is a chiral superfield defined by

$$W_\alpha = -\frac{1}{4}(\bar{D}\bar{D})D_\alpha V. \tag{15.133}$$

This is a chiral superfield, viz $\bar{D}_{\dot{\alpha}}W_\beta = 0$. The anti-chiral spinor superfield $W_{\dot{\alpha}}$ is then obviously defined by

$$\bar{W}_{\dot{\alpha}} = -\frac{1}{4}(DD)\bar{D}_{\dot{\alpha}}V. \tag{15.134}$$

Furthermore, we can check that

$$\bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} = D^\alpha W_\alpha. \tag{15.135}$$

Thus W_α is not a general chiral superfield. More important is the fact that W_α is invariant under the infinitesimal abelian gauge transformation (15.126). Indeed we compute (by using $D_\alpha\Phi^+ = 0$, $\bar{D}_{\dot{\alpha}}\Phi = 0$, $[\bar{D}_{\dot{\alpha}}, \partial_\mu] = 0$ and $\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu$)

$$\begin{aligned} W'_\alpha &= -\frac{1}{4}(\bar{D}\bar{D})D_\alpha V' \\ &= W_\alpha - \frac{1}{4}(\bar{D}\bar{D})D_\alpha(\Phi + \Phi^+) \\ &= W_\alpha - \frac{1}{4}(\bar{D}\bar{D})D_\alpha\Phi \\ &= W_\alpha - \frac{1}{4}\bar{D}_{\dot{\beta}}\{\bar{D}^{\dot{\beta}},\, D_\alpha\}\Phi \\ &= W_\alpha - \frac{i}{2}(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\bar{D}_{\dot{\beta}}\Phi \\ &= W_\alpha. \end{aligned} \tag{15.136}$$

Thus without any loss of generality we can compute the components of W_α in the Wess–Zumino gauge, viz

$$W_\alpha = -\frac{1}{4}(\bar{D}\bar{D})D_\alpha V_{WZ} \tag{15.137}$$

$$\bar{W}_{\dot{\alpha}} = -\frac{1}{4}(DD)\bar{D}_{\dot{\alpha}}V_{WZ}. \tag{15.138}$$

The chiral superfield $W_{\dot{\alpha}}$ can only be a function of y (not x) and θ, $\bar{\theta}$. Thus the covariant derivatives can be given simply by

$$D_\alpha = \partial_\alpha + 2i(\sigma^\mu\bar{\theta})_\alpha\partial^y_\mu,\;\; \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}}. \tag{15.139}$$

Furthermore

$$V_{WZ}(x^\mu,\, \theta,\, \bar{\theta}) = V_{WZ}(y^\mu - i\theta\sigma^\mu\bar{\theta},\, \theta,\, \bar{\theta}). \tag{15.140}$$

Recall that

$$\begin{aligned} V_{WZ} = {} & (\theta\sigma^\mu\bar{\theta})v_\mu(x) + (\theta\theta)(\bar{\theta}\bar{\lambda}(x)) \\ & + (\bar{\theta}\bar{\theta})(\theta\lambda(x)) + (\theta\theta)(\bar{\theta}\bar{\theta})d(x). \end{aligned} \tag{15.141}$$

Thus in terms of y we obtain

$$\begin{aligned} V_{WZ} = {} & (\theta\sigma^\mu\bar{\theta})v_\mu(y) + (\theta\theta)(\bar{\theta}\bar{\lambda}(y)) + (\bar{\theta}\bar{\theta})(\theta\lambda(y)) \\ & + (\theta\theta)(\bar{\theta}\bar{\theta})\left(d(y) - \frac{i}{2}\partial_\mu v^\mu(y)\right). \end{aligned} \tag{15.142}$$

We compute

$$\begin{aligned} D_\alpha((\theta\sigma^\mu\bar{\theta})v_\mu(y)) = {} & (\sigma^\mu\bar{\theta})_\alpha v_\mu + i\theta_\alpha(\bar{\theta}\bar{\theta})\partial^\mu v_\mu \\ & + (\sigma^{\mu\nu}\theta)_\alpha(\bar{\theta}\bar{\theta})(\partial_\mu v_\nu - \partial_\nu v_\mu) \end{aligned} \tag{15.143}$$

$$D_\alpha\left((\theta\theta)(\bar\theta\bar\theta)\left(d(y)-\frac{i}{2}\partial_\mu v^\mu(y)\right)\right)=2\theta_\alpha(\bar\theta\bar\theta)\left(d-\frac{i}{2}\partial_\mu v^\mu\right) \tag{15.144}$$

$$D_\alpha((\bar\theta\bar\theta)(\theta\lambda(y)))=\lambda_\alpha(\bar\theta\bar\theta) \tag{15.145}$$

$$D_\alpha((\theta\theta)(\bar\theta\bar\lambda(y)))=2\theta_\alpha(\bar\theta\bar\lambda)-i(\bar\theta\bar\theta)(\theta\theta)\left(\sigma^\mu\partial_\mu\bar\lambda\right)_\alpha. \tag{15.146}$$

Thus

$$-\frac{1}{4}(DD)D_\alpha((\theta\sigma^\mu\bar\theta)v_\mu(y))=i\theta_\alpha\partial^\mu v_\mu+(\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu v_\nu-\partial_\nu v_\mu) \tag{15.147}$$

$$-\frac{1}{4}(DD)D_\alpha\left((\theta\theta)(\bar\theta\bar\theta)\left(d(y)-\frac{i}{2}\partial_\mu v^\mu(y)\right)\right)=2\theta_\alpha\left(d-\frac{i}{2}\partial_\mu v^\mu\right) \tag{15.148}$$

$$-\frac{1}{4}(DD)D_\alpha((\bar\theta\bar\theta)(\theta\lambda(y)))=\lambda_\alpha \tag{15.149}$$

$$-\frac{1}{4}(DD)D_\alpha((\theta\theta)(\bar\theta\bar\lambda(y)))=-i(\theta\theta)\left(\sigma^\mu\partial_\mu\bar\lambda\right)_\alpha. \tag{15.150}$$

The final formula is

$$W_\alpha=\lambda_\alpha+2\theta_\alpha d+(\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu v_\nu-\partial_\nu v_\mu)-i(\theta\theta)\left(\sigma^\mu\partial_\mu\bar\lambda\right)_\alpha. \tag{15.151}$$

15.3.6 Non-abelian chiral, vector and spinor superfields

The infinitesimal abelian gauge transformation (15.126) corresponds to the finite transformation

$$e^V\longrightarrow e^{V'}=e^{\Phi^+}e^Ve^\Phi. \tag{15.152}$$

The non-abelian finite transformation will be given by this form with the understanding that $V=V_aT^a$, $\Phi=\Phi_aT^a$ where $[T^a,\,T^b]=if^{abc}T^c$ and $\mathrm{Tr}\;T^aT^b=\delta^{ab}$.

Next, we compute the infinitesimal non-abelian gauge transformation. We employ the Baker–Campbell–Hausdorff formula for e^Ae^B which includes all linear terms in B given by

$$e^Ae^B=e^{A+\mathcal{L}_{A/2}\cdot[B+\coth(\mathcal{L}_{A/2})\cdot B]+\cdots} \tag{15.153}$$

The Lie derivative is given by

$$\mathcal{L}_{A/2}\cdot B=[A/2,\,B]. \tag{15.154}$$

The hyperbolic cotangent is defined by its Taylor series expansion. For example, the order n is given by

$$(\mathcal{L}_{A/2})^n \cdot B = [A/2, [A/2, [A/2, \ldots, [A/2, B]\ldots]]], \ n \text{ commutators}. \tag{15.155}$$

We can compute

$$e^V e^\Phi = e^{V + \mathcal{L}_{V/2}\cdot[\Phi + \coth(\mathcal{L}_{V/2})\cdot\Phi] + \cdots} \tag{15.156}$$

$$e^{\Phi^+} e^V = e^{V + \mathcal{L}_{V/2}\cdot[-\Phi^+ + \coth(\mathcal{L}_{V/2})\cdot\Phi^+] + \cdots} \tag{15.157}$$

Thus

$$e^{V'} = e^{V + \mathcal{L}_{V/2}\cdot[-\Phi^+ + \coth(\mathcal{L}_{V/2})\cdot\Phi^+] + \mathcal{L}_{V/2}\cdot[\Phi + \coth(\mathcal{L}_{V/2})\cdot\Phi] + \cdots} \tag{15.158}$$

In other words, we have

$$\delta V = \mathcal{L}_{V/2} \cdot [\Phi - \Phi^+] + \mathcal{L}_{V/2} \cdot \coth(\mathcal{L}_{V/2}) \cdot [\Phi + \Phi^+]. \tag{15.159}$$

Recall that

$$x \coth x = 1 + \frac{x^2}{3} - \frac{x^4}{45} + \cdots \tag{15.160}$$

As it turns out, we can use the infinitesimal non-abelian gauge transformation (15.159) to bring the vector superfield V into the Wess–Zumino form, viz

$$\begin{aligned} V_{WZ} = {} & (\theta\sigma^\mu\bar{\theta})v_\mu(x) + (\theta\theta)(\bar{\theta}\bar{\lambda}(x)) + (\bar{\theta}\bar{\theta})(\theta\lambda(x)) \\ & + (\theta\theta)(\bar{\theta}\bar{\theta})d(x). \end{aligned} \tag{15.161}$$

The only difference between the abelian and non-abelian cases is the fact that the relations (15.128) between the components of V and $\Phi + \Phi^+$ which define the Wess–Zumino gauge become very complicated due to the non-linearity of the infinitesimal non-abelian gauge transformation (15.159). Clearly, the Wess–Zumino gauge in the non-abelian case, as in the abelian case, is not preserved by supersymmetry. Also, the Wess–Zumino gauge does not fix the non-abelian gauge freedom generated, as in the abelian case, by the scalar component of $\Phi - \Phi^+$.

We will, therefore, only consider vector superfields in the Wess–Zumino gauge, viz $V = V_{WZ}$. Furthermore, we will only consider chiral superfields Φ such that $\Phi - \Phi^+$ generate the remaining non-abelian gauge freedom, i.e. those $\Phi - \Phi^+$ with only the scalar component non-zero. In other words

$$\Phi - \Phi^+ = 2A - \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu A. \tag{15.162}$$

The corresponding vector superfield $\Phi + \Phi^+$ is given by

$$\Phi + \Phi^+ = 2i(\theta\sigma^\mu\bar{\theta})\partial_\mu A. \tag{15.163}$$

It is then clear that the infinitesimal non-abelian gauge transformation (15.159) reduces to

$$\delta V_{WZ} = \frac{1}{2}[V_{WZ}, \Phi - \Phi^+] + \Phi + \Phi^+. \tag{15.164}$$

In terms of components this reads

$$\begin{aligned}\delta(v_\mu/2) &= [v_\mu/2,\, A] + i\partial_\mu A,\;\; \delta\lambda_\alpha = [\lambda_\alpha,\, A],\\ \delta\bar{\lambda}^{\dot{\alpha}} &= [\bar{\lambda}^{\dot{\alpha}},\, A],\;\; \delta d = [d,\, A].\end{aligned} \tag{15.165}$$

The spinor superfields W_α and $\bar{W}_{\dot{\alpha}}$ in the non-abelian case are given by

$$W_\alpha = -\frac{1}{4}(\bar{D}\bar{D})e^{-V}D_\alpha e^{V} \tag{15.166}$$

$$\bar{W}_{\dot{\alpha}} = -\frac{1}{4}(DD)e^{-V}\bar{D}_{\dot{\alpha}}e^{V}. \tag{15.167}$$

These obviously reduce to the previous formulas when the vector superfield V is abelian. Next we check (using $D_\alpha\Phi^+ = 0$) that

$$\begin{aligned}e^{-V'}D_\alpha e^{V'} &= e^{-\Phi}e^{-V}e^{-\Phi^+}D_\alpha e^{\Phi^+}e^{V}e^{\Phi}\\ &= e^{-\Phi}e^{-V}\Big(D_\alpha e^{V}\Big)e^{\Phi} + e^{-\Phi}\Big(D_\alpha e^{\Phi}\Big).\end{aligned} \tag{15.168}$$

Then (using $\bar{D}_{\dot{\alpha}}\Phi = 0$)

$$\begin{aligned}\bar{D}\bar{D}\Big(e^{-V'}D_\alpha e^{V'}\Big) &= e^{-\Phi}\bar{D}\bar{D}\Big(e^{-V}\Big(D_\alpha e^{V}\Big)e^{\Phi}\Big) + e^{-\Phi}\bar{D}\bar{D}\Big(D_\alpha e^{\Phi}\Big)\\ &= e^{-\Phi}\bar{D}\bar{D}\Big(e^{-V}\Big(D_\alpha e^{V}\Big)\Big)e^{\Phi} + e^{-\Phi}\bar{D}(\{\bar{D},\, D_\alpha\}e^{\Phi})\end{aligned} \tag{15.169}$$

The second term is obviously zero. The first term yields the result

$$W'_\alpha = e^{-\Phi}W_\alpha e^{\Phi}. \tag{15.170}$$

Similarly, we should find

$$\bar{W}'_{\dot{\alpha}} = e^{\Phi^+}\bar{W}_{\dot{\alpha}}e^{-\Phi^+}. \tag{15.171}$$

The next task is to compute W_α and $\bar{W}_{\dot{\alpha}}$ in terms of component fields in the Wess–Zumino gauge (for simplicity and without any loss of generality). First we note

$$e^{V_{WZ}} = 1 + V_{WZ} + \frac{1}{2}V_{WZ}^2,\;\; e^{-V_{WZ}} = 1 - V_{WZ} + \frac{1}{2}V_{WZ}^2. \tag{15.172}$$

Then

$$\begin{aligned}e^{-V_{WZ}}D_\alpha e^{V_{WZ}} &= D_\alpha V_{WZ} + \frac{1}{2}D_\alpha V_{WZ}^2 - V_{WZ}D_\alpha V_{WZ}\\ &= D_\alpha V_{WZ} + \frac{1}{2}[D_\alpha V_{WZ},\, V_{WZ}].\end{aligned} \tag{15.173}$$

The second term in this last equation will be absent in the abelian case. We have already computed $D_\alpha V_{WZ}$ in equations (15.143), (15.144), (15.145) and (15.146), viz

$$D_\alpha V_{WZ} = (\sigma^\mu\bar\theta)_\alpha v_\mu + 2\theta_\alpha(\bar\theta\bar\theta)d + (\sigma^{\mu\nu}\theta)_\alpha(\bar\theta\bar\theta)(\partial_\mu v_\nu - \partial_\nu v_\mu) + \lambda_\alpha(\bar\theta\bar\theta) + 2\theta_\alpha(\bar\theta\bar\lambda) - i(\bar\theta\bar\theta)(\theta\theta)\left(\sigma^\mu\partial_\mu\bar\lambda\right)_\alpha. \tag{15.174}$$

Next we compute

$$\begin{aligned}[(\sigma^\mu\bar\theta)_\alpha v_\mu,\ V_{WZ}] &= (\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta)[v_\mu,\ v_\nu] + \theta\theta[v_\mu,\ (\sigma^\mu\bar\theta)_\alpha(\bar\theta\bar\lambda)] \\ &= -i(\sigma^{\mu\nu}\theta)_\alpha(\bar\theta\bar\theta)[v_\mu,\ v_\nu] - \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)(\sigma^\mu)_{\alpha\dot\beta}[v_\mu,\ \bar\lambda^{\dot\beta}]\end{aligned} \tag{15.175}$$

$$\begin{aligned}[2\theta_\alpha(\bar\theta\bar\lambda),\ V_{WZ}] &= [2\theta_\alpha(\bar\theta\bar\lambda),\ \theta\sigma^\mu\bar\theta v_\mu] \\ &= -\bar\theta\bar\theta[\theta_\alpha(\theta\sigma^\mu\bar\lambda),\ v_\mu] \\ &= -\frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)(\sigma^\mu)_{\alpha\dot\beta}[v_\mu,\ \bar\lambda^{\dot\beta}]\end{aligned} \tag{15.176}$$

$$\begin{aligned}[\lambda_\alpha(\bar\theta\bar\theta),\ V_{WZ}] &= [2\theta_\alpha(\bar\theta\bar\theta)d,\ V_{WZ}] \\ &= [(\sigma^{\mu\nu}\theta)_\alpha(\bar\theta\bar\theta)(\partial_\mu v_\nu - \partial_\nu v_\mu),\ V_{WZ}] = 0.\end{aligned} \tag{15.177}$$

Thus

$$\begin{aligned}[D_\alpha V_{WZ},\ V_{WZ}] &= -i(\sigma^{\mu\nu}\theta)_\alpha(\bar\theta\bar\theta)[v_\mu,\ v_\nu] \\ &\quad -(\theta\theta)(\bar\theta\bar\theta)(\sigma^\mu)_{\alpha\dot\beta}[v_\mu,\ \bar\lambda^{\dot\beta}].\end{aligned} \tag{15.178}$$

Hence

$$\begin{aligned}e^{-V_{WZ}}D_\alpha e^{V_{WZ}} &= (\sigma^\mu\bar\theta)_\alpha v_\mu + 2\theta_\alpha(\bar\theta\bar\theta)d \\ &\quad + (\sigma^{\mu\nu}\theta)_\alpha(\bar\theta\bar\theta)\left(\partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu,\ v_\nu]\right) \\ &\quad + \lambda_\alpha(\bar\theta\bar\theta) + 2\theta_\alpha(\bar\theta\bar\lambda) \\ &\quad -i(\bar\theta\bar\theta)(\theta\theta)\left(\sigma^\mu\partial_\mu\bar\lambda - \frac{i}{2}\sigma^\mu[v_\mu,\ \bar\lambda]\right)_\alpha.\end{aligned} \tag{15.179}$$

The remaining calculation is identical to the abelian case. We get

$$\begin{aligned}W_\alpha &= \lambda_\alpha + 2\theta_\alpha d + (\sigma^{\mu\nu})_\alpha{}^\beta\theta_\beta\left(\partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu,\ v_\nu]\right) \\ &\quad -i(\theta\theta)(\sigma^\mu)_{\alpha\dot\beta}\left(\partial_\mu\bar\lambda^{\dot\beta} - \frac{i}{2}[v_\mu,\ \bar\lambda^{\dot\beta}]\right).\end{aligned} \tag{15.180}$$

15.3.7 The Wess–Zumino model

Recall that the d-term of a vector superfield transforms by a total derivative. Thus, given a vector superfield V a globally $N = 1$ supersymmetric action can be given by

$$\int d^4x \int d^4\theta \; V(x, \theta, \bar{\theta}). \tag{15.181}$$

Very simple vector superfields V can be constructed from a chiral superfield Φ as $V = \Phi^+\Phi$. We will be interested in actions of the form

$$\int d^4x \int d^4\theta \; \Phi^+\Phi. \tag{15.182}$$

This action will generate kinetic terms for the component fields of Φ. This term is called the Kahler potential.

Also, the F-term of a chiral superfield transforms by a total derivative. Thus, given a chiral superfield Φ, a globally $N = 1$ supersymmetric action can be given by

$$\int d^4x \int d^2\theta \; \Phi(y, \theta) + \int d^4x \int d^2\bar{\theta} \; \Phi^+(y^+, \bar{\theta}). \tag{15.183}$$

Any holomorphic function $W(\Phi)$ of a chiral superfield Φ is also a chiral superfield. Thus we will be interested in actions of the form

$$\int d^4x \left[\int d^2\theta \; W(\Phi) + \text{h.c.} \right]. \tag{15.184}$$

This will generate mass terms and interactions for the component fields of Φ. The holomorphic function $W(\Phi)$ is called the superpotential.

A general action of interest can be constructed from an elementary chiral superfield Φ as

$$S = \int d^4x \int d^4\theta \; \Phi^+\Phi - \int d^4x \left[\int d^2\theta \; W(\Phi) + \text{h.c.} \right]. \tag{15.185}$$

The Wess–Zumino model is the simplest globally $N = 1$ supersymmetric theory in four dimensions corresponding to $W(\Phi) = m\Phi^2/2 + g\Phi^3/3$. The total action is given by

$$S_{WZ} = \int d^4x \int d^4\theta \; \Phi^+\Phi - \int d^4x \left[\int d^2\theta \left[\frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3 \right] + \text{h.c.} \right]. \tag{15.186}$$

As it turns out, the component field F of Φ is auxiliary and, therefore, it can be set equal to its classical value. As a consequence, the bosonic part of the Wess–Zumino model will only depend on the component field A and it reads

$$(S_{WZ})_{\text{bosonic}} = \int d^4x \left[\partial^\mu A^+ \partial_\mu A - V_F(A^+, A) \right] \tag{15.187}$$

$$V_F(A^+, A) = F^+F = (mA + gA^2)^+(mA + gA^2). \tag{15.188}$$

This involves a quartic scalar interaction. Thus the Wess–Zumino model is the most general renormalizable globally $N = 1$ supersymmetric theory in four dimensions. Indeed the bosonic part of the more general model S reads

$$(S)_{\text{bosonic}} = \int d^4x \left[\partial^\mu A^+ \partial_\mu A - \left(\frac{\delta W}{\delta \Phi} \right)^+ \left(\frac{\delta W}{\delta \Phi} \right) \right]. \tag{15.189}$$

The potential in this case will involve higher powers in A and thus S is not renormalizable in four dimensions.

A generalization of the action S can be constructed from a set of elementary chiral superfields Φ^I as

$$S = \int d^4x \int d^4\theta\ K(\Phi^{I+}, \Phi^J) - \int d^4x \Big[\int d^2\theta\ W(\Phi^I) + \text{h.c.} \Big]. \tag{15.190}$$

The Kahler potential $K(\Phi^{I+}, \Phi^J)$ is a vector superfield. This will always generate kinetic terms for the components fields, i.e. terms with no more than two derivatives. This is the most general globally $N = 1$ supersymmetric theory in four dimensions (excluding higher derivative theories).

15.3.8 The $N = 1$ super Yang–Mills theory

Recall that the spinor superfield (also known as the gaugino superfield) W_α is given by (with $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu]/2$ and $D_\mu = \partial_\mu(\ldots) - i[v_\mu, \ldots]/2$)

$$W_\alpha = \lambda_\alpha + 2\theta_\alpha d + (\sigma^{\mu\nu})_\alpha{}^\beta \theta_\beta F_{\mu\nu} - i(\theta\theta)(\sigma^\mu)_{\alpha\dot\beta} D_\mu \bar\lambda^{\dot\beta}. \tag{15.191}$$

The component fields $\bar\lambda$, λ, d and $F_{\mu\nu}$ are functions of y^μ. Since W_α is a chiral superfield, the square $W^\alpha W_\alpha$ is also a chiral superfield and as a consequence their F-terms will transform by total derivatives under supersymmetry transformations. The F-term of the square $W^\alpha W_\alpha$ is given by

$$\begin{aligned}[\text{Tr}\ W^\alpha W_\alpha]_F \theta\theta = \epsilon^{\alpha\beta}\ \text{Tr}\Big[& -2i\lambda_\beta \cdot (\theta\theta)(\sigma^\mu)_{\alpha\dot\gamma} D_\mu \bar\lambda^{\dot\gamma} \\ & + 2\theta_\beta d \cdot 2\theta_\alpha d + 4\theta_\beta d \cdot (\sigma^{\mu\nu})_\alpha{}^\gamma \theta_\gamma F_{\mu\nu} \\ & + (\sigma^{\mu\nu})_\beta{}^\rho \theta_\rho F_{\mu\nu} \cdot (\sigma^{\mu'\nu'})_\alpha{}^\gamma \theta_\gamma F_{\mu'\nu'} \Big].\end{aligned} \tag{15.192}$$

The third term is zero because $\theta\sigma^{\mu\nu}\theta = 0$. By using $\theta_\alpha\theta_\beta = \epsilon_{\alpha\beta}\theta\theta/2$, $\hat\epsilon = -\tilde\epsilon$, $\tilde\epsilon(\sigma^{\mu\nu})^T\tilde\epsilon = \sigma^{\mu\nu}$ and

$$\text{tr}_2\, \sigma^{\mu\nu}\sigma^{\mu'\nu'} = \frac{1}{2}(\eta^{\mu\mu'}\eta^{\nu\nu'} - \eta^{\mu\nu'}\eta^{\nu\mu'}) + \frac{i}{2}\epsilon^{\mu\nu\mu'\nu'}, \tag{15.193}$$

we compute

$$\begin{aligned}\epsilon^{\alpha\beta}\Big[(\sigma^{\mu\nu})_\beta{}^\rho \theta_\rho F_{\mu\nu} \cdot (\sigma^{\mu'\nu'})_\alpha{}^\gamma \theta_\gamma F_{\mu'\nu'}\Big] &= \frac{1}{2}\theta\theta \cdot \epsilon^{\alpha\beta}(\sigma^{\mu\nu})_\beta{}^\rho \epsilon_{\rho\gamma}(\sigma^{\mu'\nu'})_\alpha{}^\gamma \cdot F_{\mu\nu}F_{\mu'\nu'} \\ &= -\frac{1}{2}\theta\theta \cdot \text{tr}_2\, \sigma^{\mu\nu}\tilde\epsilon(\sigma^{\mu'\nu'})^T\tilde\epsilon \cdot F_{\mu\nu}F_{\mu'\nu'} \\ &= -\frac{1}{2}\theta\theta \cdot \text{tr}_2\, \sigma^{\mu\nu}\sigma^{\mu'\nu'} \cdot F_{\mu\nu}F_{\mu'\nu'} \\ &= -\frac{1}{4}\theta\theta[2F_{\mu\nu}F^{\mu\nu} + i\epsilon^{\mu\nu\mu'\nu'}F_{\mu\nu}F_{\mu'\nu'}].\end{aligned} \tag{15.194}$$

We get then

$$[\mathrm{Tr}\ W^\alpha W_\alpha]_F = \mathrm{Tr}\left[-2i\lambda\sigma^\mu D_\mu\bar{\lambda} + 4d^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right]. \tag{15.195}$$

Let g and θ be the Yang–Mills gauge coupling and the Yang–Mills theta angle respectively. We define the complex coupling τ by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}. \tag{15.196}$$

The $N = 1$ super Yang–Mills action is then given by

$$\begin{aligned} S_{YM}^{N=1} &= \frac{1}{8\pi}\int d^4x\ \mathrm{Im}\left(\tau\int d^2\theta\ \mathrm{Tr}\ W^\alpha W_\alpha\right) \\ &= \frac{1}{8\pi}\int d^4x\ \mathrm{Im}\,[\tau\ \mathrm{Tr}\ W^\alpha W_\alpha]_F. \end{aligned} \tag{15.197}$$

This is obviously invariant under the gauge transformation (the chiral superfield which generates a finite gauge transformation is now denoted by Λ)

$$W_\alpha \longrightarrow e^{-\Lambda}W_\alpha e^{\Lambda}. \tag{15.198}$$

Indeed, explicitly we have (using the fact that the fermion term is real including the factor of i and $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2$)

$$\begin{aligned} S_{YM}^{N=1} = \frac{1}{g^2}\int d^4x\ \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + 2d^2\right] \\ -\frac{\theta}{32\pi^2}\int d^4x\ \mathrm{Tr}\ F_{\mu\nu}\tilde{F}^{\mu\nu}. \end{aligned} \tag{15.199}$$

Next, we add matter in the form of a chiral superfield Φ in some representation R of the gauge group. In other words, under a gauge transformation we must have

$$\Phi \longrightarrow e^{-\Lambda}\Phi,\ \ \Phi^+ \longrightarrow \Phi^+ e^{-\Lambda^+}, \tag{15.200}$$

with $\Lambda = \Lambda_a T_R^a$ where T_R^a are the generators of the gauge group in the representation R. The precise meaning of these gauge transformations is

$$\Phi_a \longrightarrow (e^{-\Lambda})_a{}^b\Phi_b,\ \ \Phi_a^+ \longrightarrow \Phi_b^+(e^{-\Lambda^+})_{ba}, \tag{15.201}$$

Recall that the vector superfield transforms under gauge transformations as

$$e^V \longrightarrow e^{\Lambda^+}e^V e^{\Lambda}. \tag{15.202}$$

Thus, V must be a matrix in the representation R, viz $V = V_a T_R^a$. Properly speaking, the vector superfield is actually V_a which carries an adjoint representation index and is independent of the representation R.

It is then obvious that the combination $\Phi^+ e^V\Phi$ is invariant under gauge transformations, viz

$$\Phi^+ e^V\Phi \longrightarrow \Phi^+ e^V\Phi. \tag{15.203}$$

The $N = 1$ supersymmetric matter action can then be given by the gauge-covariant Wess–Zumino model, viz

$$\begin{aligned} S_{\text{matter}} &= \int d^4x \int d^4\theta\ \Phi^+ e^V \Phi \\ &\quad - \int d^4x \left[\int d^2\theta \left[\frac{m_{ab}}{2} \Phi_a \Phi_b + \frac{g_{abc}}{3} \Phi_a \Phi_b \Phi_c \right] + \text{h.c.} \right] \\ &= \int d^4x [\Phi^+ e^V \Phi]_d \\ &\quad - \int d^4x \left[\frac{m_{ab}}{2} \Phi_a \Phi_b + \frac{g_{abc}}{3} \Phi_a \Phi_b \Phi_c + \text{h.c.} \right]_F . \end{aligned} \tag{15.204}$$

The mass and cubic terms are allowed by gauge invariance only if m_{ab} and g_{abc} are totally symmetric invariant tensors with respect to the gauge group. Equivalently m_{ab} and g_{abc} can only be (by gauge invariance) non-zero if the tensor products $R \times R$ and $R \times R \times R$, respectively, contain the singlet representation of the gauge group. Another way of stating this result is that one must contract the indices a, b and c correctly and properly in order to have gauge invariance.

The full $N = 1$ super Yang–Mills action with a chiral superfield in a representation R of the gauge group is, therefore, given by

$$\begin{aligned} S^{N=1} &= \frac{1}{8\pi} \int d^4x\ \text{Im}[\tau\ \text{Tr}\ W^\alpha W_\alpha]_F + \int d^4x [\Phi^+ e^V \Phi]_d \\ &\quad - \int d^4x [W(\Phi) + \text{h.c.}]_F . \end{aligned} \tag{15.205}$$

As before, the component field F of Φ is auxiliary and, therefore, it can be set equal to its classical value. In fact the F-term contribution to the scalar potential of the field components A and A^+ remains unchanged, viz

$$V_F(A^+, A) = F^+ F = (mA + gA^2)^+ (mA + gA^2). \tag{15.206}$$

As it turns out, the component field d of the gaugino spinor superfield W_α is also an auxiliary field. Thus d can be set to its classical value and one ends up with the extra contribution to the scalar potential

$$V_D(A^+, A) = \frac{2}{g^2} d^2 = \frac{g^2}{8} [A, A^+]^2 . \tag{15.207}$$

The total potential is, therefore,

$$V(A^+, A) = V_F(A^+, A) + V_D(A^+, A) = F^+ F + \frac{2}{g^2} d^2 . \tag{15.208}$$

For renormalizability in $d = 4$ the superpotential $W(\Phi)$ must be given by at most a cubic potential. If we forget about renormalizability we can take $W(\Phi)$ to be a generic superpotential. A further generalization can be obtained by considering several flavors of chiral superfields Φ^I transforming in different representations R^I of

the gauge group and also by replacing the kinetic term of the chiral superfields by a general Kahler potential. The most general $N = 1$ super Yang–Mills action with chiral superfields Φ^I in representations R^I is of the form

$$S^{N=1} = \frac{1}{8\pi}\int d^4x\ \mathrm{Im}\,[\tau\ \mathrm{Tr}\, f(\Phi^I)W^\alpha W_\alpha]_F + \int d^4x[K(\Phi^{I+}e^V,\ \Phi^J)]_d - \int d^4x[W(\Phi^I) + \mathrm{h.c.}]_F. \tag{15.209}$$

The holomorphic function $f(\Phi)$, which is called the gauge kinetic function, is assumed to transform under the symmetrized square of the adjoint representation. This simply means that $f(\phi)$ must transform like W_α, viz

$$f(\Phi^I) \longrightarrow e^{-\Lambda} f(\Phi^I) e^{\Lambda}. \tag{15.210}$$

The last remark concerns the so-called Fayet–Iliopoulos term. For an abelian gauge theory, we observe that the d-term of the vector superfield V, which by construction transforms under supersymmetry transformations by a total derivative, transforms also by a total derivative under gauge transformations. Thus, another gauge invariant $N = 1$ supersymmetric action can be given by

$$S_{FI} = \int d^4x\ \xi \int d^4\theta V = \int d^4x\ \xi[V]_d. \tag{15.211}$$

Generalization to non-abelian semi-simple gauge groups (groups which are products of simple groups and $U(1)$ factors) is straightforward.

15.4 $N = 2$ supersymmetry

We introduce eight anticommuting Grassmannian coordinates θ^α, $\bar\theta_{\dot\alpha}$ and $\tilde\theta^\alpha$, $\bar{\tilde\theta}_{\dot\alpha}$. By the global $SU(2)$ R-symmetry the two Weyl spinors θ^α and $\tilde\theta^\alpha$ form a doublet in the fundamental representation of $SU(2)$ while $\bar\theta_{\dot\alpha}$ and $\bar{\tilde\theta}_{\dot\alpha}$ form a doublet in the anti-fundamental representation of $SU(2)$. There will be, therefore, four supercharges Q_α, $\bar Q_{\dot\alpha}$ and $\tilde Q_\alpha$, $\bar{\tilde Q}_{\dot\alpha}$ and four covariant derivatives D_α, $\bar D_{\dot\alpha}$ and $\tilde D_\alpha$, $\bar{\tilde D}_{\dot\alpha}$. The Berezin measure for $N = 2$ superspace is defined in an obvious way by

$$\int d^2\theta d^2\tilde\theta d^2\bar\theta d^2\bar{\tilde\theta}. \tag{15.212}$$

We define a new bosonic coordinate $\tilde y^\mu$ by

$$\tilde y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta + i\bar{\tilde\theta}\sigma^\mu\bar{\tilde\theta}. \tag{15.213}$$

Clearly

$$\bar D_{\dot\alpha}\tilde y^\mu = \bar{\tilde D}_{\dot\alpha}\tilde y^\mu = 0. \tag{15.214}$$

A scalar superfield is the most general scalar function $\Psi(x, \theta, \bar{\theta}, \tilde{\theta}, \bar{\tilde{\theta}})$ on the $N = 2$ superspace. An $N = 2$ chiral superfield is an $N = 2$ scalar superfield $\Psi(x, \theta, \bar{\theta}, \tilde{\theta}, \bar{\tilde{\theta}})$ which is a singlet under the global $SU(2)$ R-symmetry and which satisfies the constraints

$$\bar{D}_{\dot{\alpha}}\Psi(x, \theta, \bar{\theta}, \tilde{\theta}, \bar{\tilde{\theta}}) = \bar{\tilde{D}}_{\dot{\alpha}}\Psi(x, \theta, \bar{\theta}, \tilde{\theta}, \bar{\tilde{\theta}}) = 0. \tag{15.215}$$

The most general solution to these constraints is $\Psi = \Psi(\tilde{y}, \theta, \tilde{\theta})$. Expanding in powers of $\tilde{\theta}$ we get

$$\begin{aligned}\Psi &= \Phi(\tilde{y}, \theta) + \sqrt{2}\tilde{\theta}^{\alpha}W_{\alpha}(\tilde{y}, \theta) + \tilde{\theta}\tilde{\theta}G(\tilde{y}, \theta) \\ &= \Phi(y + i\bar{\tilde{\theta}}\sigma^{\mu}\bar{\tilde{\theta}}, \theta) + \sqrt{2}\tilde{\theta}^{\alpha}W_{\alpha}(y + i\bar{\tilde{\theta}}\sigma^{\mu}\bar{\tilde{\theta}}, \theta) \\ &\quad + \tilde{\theta}\tilde{\theta}G(y + i\bar{\tilde{\theta}}\sigma^{\mu}\bar{\tilde{\theta}}, \theta).\end{aligned} \tag{15.216}$$

It is obvious that Φ and G are $N = 1$ chiral superfields while W_{α} is a chiral spinor superfield. We compute the F-term (with respect to $\tilde{\theta}$)

$$[\mathrm{Tr}\ \Psi^2]_{\tilde{\theta}\tilde{\theta}}\tilde{\theta}\tilde{\theta} = \mathrm{Tr}(W^{\alpha}W_{\alpha} + 2G\Phi)\tilde{\theta}\tilde{\theta}. \tag{15.217}$$

The $N = 2$ Yang–Mills action is, therefore,

$$\begin{aligned}S^{N=2} &= \frac{1}{8\pi}\int d^4x\ \mathrm{Im}\,[\tau\ \mathrm{Tr}\ \Psi^2]_F \\ &= \frac{1}{8\pi}\int d^4x\ \mathrm{Im}\left[\tau\int d^2\theta d^2\tilde{\theta}\ \mathrm{Tr}\ \Psi^2\right] \\ &= \frac{1}{8\pi}\int d^4x\ \mathrm{Im}\left[\tau\int d^2\theta\ \mathrm{Tr}(W^{\alpha}W_{\alpha} + 2G\Phi)\right].\end{aligned} \tag{15.218}$$

The $N = 2$ gauge multiplet is equivalent to the $N = 1$ gauge multiplet plus the $N = 1$ chiral multiplet all in the adjoint representations of the gauge group. The spinor component λ of W and the spinor component ψ of Φ form a doublet under $SU(2)_R$, whereas the gauge field v_{μ}, the auxiliary field d of W_{α} and the auxiliary field F of Φ transform as singlets under $SU(2)_R$.

The first term in equation (15.218) is precisely the $N = 1$ Yang–Mills action. The central point is that the above action must be invariant under $N = 1$ supersymmetry transformations considered in the previous subsections, and as a consequence we should think of the chiral superfield G as an auxiliary superfield. In other words, the superfield G must satisfy a particular constraint such that the second term in equation (15.218) becomes the action of a chiral superfield Φ in the adjoint representation of the gauge group (without a superpotential). By comparing with the first term of equation (15.204) we see that the chiral superfield G must be given by

$$G(y, \theta) = \frac{1}{2}\int d^2\bar{\theta}[\Phi(y - i\theta\sigma^{\mu}\bar{\theta}, \theta, \bar{\theta})]^{+}\, e^{V(y - i\theta\sigma^{\mu}\bar{\theta}, \theta, \bar{\theta})}. \tag{15.219}$$

This can be deduced from the constraints on the $N = 2$ superfield Ψ given by (with $D^a = (D, \tilde{D})$ and $\bar{D}^a = (\bar{D}, \bar{\tilde{D}})$)

$$D^{a\alpha}D^{b}_{\alpha}\Psi = \bar{D}^{a\dot{\alpha}}\bar{D}^{b}_{\dot{\alpha}}\Psi^{+}. \tag{15.220}$$

The $N = 2$ Yang–Mills action becomes

$$\begin{aligned} S^{N=2} = \frac{1}{8\pi}\int d^4x\ \mathrm{Im}\Big[\tau\int d^2\theta\ \mathrm{Tr}\ W^{\alpha}W_{\alpha} \\ + \tau\int d^2\theta\int d^2\bar{\theta}\ \mathrm{Tr}\ \Phi^{+}e^{V}\Phi\Big]. \end{aligned} \tag{15.221}$$

15.5 Simple supersymmetry in more detail

In this section we will give a detailed and explicit calculation of the F- and d-terms of simple $N = 1$ superysmmetry. We will employ the notation of [2].

15.5.1 The *F*- and *d*-terms

The most general real scalar superfield is given by

$$\begin{aligned} S(x,\theta) = C(x) - i(\bar{\theta}\gamma_5\omega(x)) - \frac{1}{2}(\bar{\theta}\theta)N(x) \\ -\frac{i}{2}(\bar{\theta}\gamma_5\theta)M(x) + \frac{i}{2}(\bar{\theta}\gamma_5\gamma_{\mu}\theta)V^{\mu}(x) \\ -i(\bar{\theta}\gamma_5\theta)\Big(\bar{\theta}\Big(\lambda(x) + \frac{1}{2}\gamma_{\mu}\partial^{\mu}\omega(x)\Big)\Big) \\ -\frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\Big(D(x) + \frac{1}{2}\Delta C(x)\Big). \end{aligned} \tag{15.222}$$

C, M, N and D are real scalar fields, V_{μ} is a real vector field, ω and λ are Majorana spinors satisfying $s^{*} = -\beta\epsilon\gamma_5 s$.

A 4-component Majorana spinor s is formed from a (1/2, 0) 2-component spinor u as $s = (u, -eu^{*})$. It can be formed from a (0, 1/2) 2-component spinor v as $s = (ev^{*}, v)$. The matrix e is defined by $e = i\sigma_2 = -e^{T}$. We note that under supersymmetry transformations only the d-component of S transforms in a trivial way, i.e. it changes by a total derivative. A supersymmetric action can thus be constructed as the integral over spacetime of the d-term of a general real scalar superfield.

A chiral superfield is a reduced superfield obtained by setting

$$\lambda = D = 0,\ \ V_{\mu} = \partial_{\mu}Z. \tag{15.223}$$

The corresponding supersymmetry transformations (with the infinitesimal Majorana 4-component constant spinor α as a parameter) are given by

$$\begin{aligned}
\omega &\longrightarrow \omega' = \omega + \left(-i\gamma_5\gamma_\mu\partial^\mu C + i\gamma_5 N - M + \gamma_\mu\partial^\mu Z\right)\alpha \\
C &\longrightarrow C' = C + i\bar{\alpha}\gamma_5\omega \\
N &\longrightarrow N' = N + i\bar{\alpha}\gamma_5\left(\gamma_\mu\partial^\mu\omega\right) \\
M &\longrightarrow M' = M - \bar{\alpha}\left(\gamma_\mu\partial^\mu\omega\right) \\
V_\mu &\longrightarrow Z' = Z + \bar{\alpha}\omega.
\end{aligned} \tag{15.224}$$

A chiral superfield X can also be given by $X = (\Phi + \tilde{\Phi})/\sqrt{2}$ where Φ and $\tilde{\Phi}$ are left-chiral and right-chiral superfields respectively. If the chiral superfield X is real then $\tilde{\Phi} = \Phi^+$. In general, the left-chiral superfield Φ and the right-chiral superfield $\tilde{\Phi}$ of a chiral superfield X are completely independent superfields. However, any right-chiral superfield is the complex conjugate of a left-chiral superfield and thus there is no loss of generality in considering only left-chiral superfields.

A left-chiral superfield can be written in two equivalent ways. The first is

$$\begin{aligned}
\Phi(x,\theta) = \phi(x) - \sqrt{2}(\bar{\theta}\psi_L(x)) + \mathcal{F}(x)\left(\bar{\theta}\frac{1+\gamma_5}{2}\theta\right) + \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\phi(x) \\
-\frac{1}{\sqrt{2}}(\bar{\theta}\gamma_5\theta)(\bar{\theta}\gamma_\mu\partial^\mu\psi_L(x)) - \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Delta\phi(x).
\end{aligned} \tag{15.225}$$

The second is

$$\Phi(x,\theta) = \phi(x_+) - \sqrt{2}\left(\theta_L^T\epsilon\,\psi_L(x_+)\right) + \mathcal{F}(x_+)\left(\theta_L^T\epsilon\theta_L\right). \tag{15.226}$$

We have introduced the coordinates $x_\pm^\mu = x^\mu \pm \frac{1}{2}\bar{\theta}\gamma_5\gamma^\mu\theta$ and for the Majorana spinor θ we have used the identity $\bar{\theta} = \theta^T\epsilon\gamma_5$ where ϵ is the matrix $\epsilon = -\gamma_2\beta\gamma_5$. Let us also recall the following simple supersymmetry transformations

$$\begin{aligned}
\psi_L &\longrightarrow \psi_L' = \psi_L + \sqrt{2}\left(\gamma_\mu\partial^\mu\phi\alpha_R + \mathcal{F}\alpha_L\right) \\
\mathcal{F} &\longrightarrow \mathcal{F}' = \mathcal{F} + \sqrt{2}\,\bar{\alpha}_L\gamma_\mu\partial^\mu\psi_L \\
\phi &\longrightarrow \phi' = \phi + \sqrt{2}\,\bar{\alpha}_R\psi_L.
\end{aligned} \tag{15.227}$$

The Kahler potential K is an arbitrary real scalar superfield. The kinetic part of a supersymmetric action is given by the d-term of K

$$\frac{1}{2}\int d^4x[K]_d. \tag{15.228}$$

K is a function of elementary left-chiral superfields Φ_i, their complex conjugates Φ_i^+ and their superderivatives and spacetime derivatives. For a renormalizable theory K can, at most, be quadratic in the superfields Φ_i and Φ_i^+ of the form $K = g_{ij}\Phi_i^+\Phi_j$. In this case K does not depend on the superderivatives and spacetime derivatives. For a single left-chiral superfield we have

$$K = \Phi^{+}\Phi \tag{15.229}$$

We note also that under supersymmetry transformations the F-component of the left-chiral superfield Φ changes by a total derivative. Another supersymmetric action can thus be constructed as the integral over spacetime of the real part of the F-term of a general left-chiral superfield f. In general f is a function of elementary left-chiral superfields Φ_i but not their superderivatives or spactime derivatives. It is known as the superpotential. Note that f does not depend on Φ_i^{+} nor on its superderivatives and spactime derivative. The corresponding supersymmetric action for a single left-chiral superfield Φ is

$$\frac{1}{2}\int d^4x[f(\Phi)]_F + \frac{1}{2}\int d^4x[f(\Phi)]_F^{*}. \tag{15.230}$$

15.5.2 Extended gauge symmetry

Gauge transformations leave the simple supersymmetry generator Q invariant. Indeed, since Q is a Majorana spinor it can only transform in the trivial representation of any semi-simple gauge group. Gauge transformations act on the left-chiral superfield Φ given by equation (15.226) in the usual way

$$\Phi(x,\theta) \longrightarrow \Phi'(x,\theta) = e^{it_A\Lambda^A(x_+)}\Phi(x,\theta). \tag{15.231}$$

In the above, t_A are the hermitian generators of the gauge group and Λ^A are left-chiral superfields with no $\psi_L(x_+)$- and $\mathcal{F}(x_+)$-components. Note that the functions Λ^A depend on x_+ and not directly on the coordinates x and hence we can conclude that the components $\phi(x_+)$, $\psi_L(x_+)$, $\mathcal{F}(x_+)$ of the left-chiral superfield Φ must transform precisely in the same way as in equation (15.231).

Let us note that the superpotential term is gauge invariant if and only if it is invariant under the (global) gauge transformation (15.231) with Λ independent of x_+. Gauge superfields are needed explicitly only in the construction of the gauge invariant Kahler potential. In these terms the fields Φ and Φ^{+} transform as $\Phi'(x,\theta) = e^{it_A\Lambda^A(x_+)}\Phi(x,\theta)$ and $\Phi'^{+}(x,\theta) = \Phi^{+}(x,\theta)e^{-it_A\Lambda^A(x_-)}$. We have used the fact that $x_+^{*} = x_-$ (since θ is a Majorana spinor) and hence $(\Lambda^A(x_+))^{+} = \Lambda^A(x_-)$. Because x_+^{μ} and x_-^{μ} are different the combination $\Phi^{+}\Phi$ is not gauge invariant. We introduce, therefore, a gauge connection $\Gamma(x,\theta)$ with the transformation rule

$$\Gamma(x,\theta) \longrightarrow \Gamma'(x,\theta) = e^{it_A\Lambda^A(x_-)}\Gamma(x,\theta)e^{-it_A\Lambda^A(x_+)}. \tag{15.232}$$

We immediately notice that the field $\Phi^{+}\Gamma$ transforms in exactly the desired way, viz

$$\Phi^{+}(x,\theta)\Gamma(x,\theta) \longrightarrow \Phi'^{+}(x,\theta)\Gamma'(x,\theta) = \Phi^{+}(x,\theta)\Gamma(x,\theta)e^{-it_A\Lambda^A(x_+)}. \tag{15.233}$$

Hence, the combination $\Phi^{+}\Gamma\Phi$ is precisely gauge invariant. This is the gauge invariant version of $\Phi^{+}\Phi$. In fact $\Phi^{+}\Gamma\Phi$ will also be invariant under the following extended gauge transformations

$$\begin{aligned}\Phi(x,\theta) &\longrightarrow \Phi'(x,\theta) = e^{it_A\Omega^A(x,\theta)}\Phi(x,\theta)\\ \Gamma(x,\theta) &\longrightarrow \Gamma'(x,\theta) = e^{it_A\Omega^A(x,\theta)^+}\Gamma(x,\theta)e^{-it_A\Omega^A(x,\theta)}.\end{aligned} \tag{15.234}$$

Ω^A are general left-chiral superfields. The choice of the connection Γ is not unique. Given a left-chiral superfield Σ with the transformation property $\Sigma'(x,\theta) = e^{it_A\Lambda^A(x_+)}\Sigma(x,\theta)e^{-it_A\Lambda^A(x_+)}$, we can easily show that $\Gamma\Sigma$ is also a connection. We use this freedom to choose Γ to be hermitian of the form

$$\Gamma(x,\theta) = e^{-2t_AV^A(x,\theta)}. \tag{15.235}$$

The gauge superfields $V^A(x,\theta)$ are real superfields which by construction do not depend on the representation t_A of the gauge group.

By using equation (15.235) in the second equation of (15.234) we obtain

$$e^{-2t_AV'^A(x,\theta)} = e^{it_A\Omega^A(x,\theta)^+}e^{-2t_AV^A(x,\theta)}e^{-it_A\Omega^A(x,\theta)}. \tag{15.236}$$

The extended gauge transformations of the gauge superfields $V^A(x,\theta)$ are given by

$$V'^A(x,\theta) = V^A(x,\theta) + \frac{i}{2}(\Omega^A(x,\theta) - \Omega^A(x,\theta)^+) + \cdots \tag{15.237}$$

The gauge superfields $V^A(x,\theta)$ are real superfields of the form (15.222), namely

$$\begin{aligned}V^A(x,\theta) = C^A(x) &- i\Big(\bar\theta\gamma_5\omega^A(x)\Big) - \frac{1}{2}(\bar\theta\theta)N^A(x) - \frac{i}{2}(\bar\theta\gamma_5\theta)M^A(x)\\ &+ \frac{i}{2}\Big(\bar\theta\gamma_5\gamma^\mu\theta\Big)V_\mu^A(x) - i(\bar\theta\gamma_5\theta)\left(\bar\theta\left(\lambda^A(x) + \frac{1}{2}\gamma_\mu\partial^\mu\omega^A(x)\right)\right)\\ &-\frac{1}{4}(\bar\theta\gamma_5\theta)^2\left(D^A(x) + \frac{1}{2}\Delta C^A(x)\right).\end{aligned} \tag{15.238}$$

We can use the extended gauge transformations (15.237) to put the gauge superfields $V^A(x,\theta)$ into a simple form known as the Wess–Zumino gauge obtained by setting $C^A = \omega^A = N^A = M^A = 0$, i.e.

$$\begin{aligned}V^A(x,\theta) = &\frac{i}{2}\Big(\bar\theta\gamma_5\gamma^\mu\theta\Big)V_\mu^A(x) - i(\bar\theta\gamma_5\theta)(\bar\theta\lambda^A(x))\\ &-\frac{1}{4}(\bar\theta\gamma_5\theta)^2D^A(x).\end{aligned} \tag{15.239}$$

The supersymmetry transformations for the components of the gauge superfields $V^A(x,\theta)$ become

$$\begin{aligned}V_\mu^A &\longrightarrow V'^A_\mu = V_\mu^A + \bar\alpha\gamma_\mu\lambda^A\\ \lambda^A &\longrightarrow \lambda'^A = \lambda^A + \left(i\gamma_5D^A + \frac{1}{2}\Big[\partial_\mu\gamma^\nu V_\nu^A,\,\gamma^\mu\Big]\right)\alpha\\ D^A &\longrightarrow D'^A = D^A + i\bar\alpha\gamma_5\gamma_\mu\partial^\mu\lambda^A\end{aligned} \tag{15.240}$$

and

$$\begin{aligned} C^A = 0 &\longrightarrow C'^A = 0 \\ \omega^A = 0 &\longrightarrow \omega'^A = \gamma^\mu V_\mu^A \alpha \\ N^A = 0 &\longrightarrow N'^A = i\bar{\alpha}\gamma_5 \lambda^A \\ M^A = 0 &\longrightarrow M'^A = -\bar{\alpha}\lambda^A. \end{aligned} \tag{15.241}$$

We can immediately notice that under supersymmetry the three conditions $\omega^A = N^A = M^A = 0$ of the Wess–Zumino gauge are not stable. We conclude that the Wess–Zumino gauge is not supersymmetric. Thus, in the Wess–Zumino gauge the action is not invariant under both extended gauge transformations and supersymmetry transformations. However, the action is still invariant under supersymmetry transformations which take us out of the Wess–Zumino gauge followed by particular extended gauge transformations which take us back to the Wess–Zumino gauge.

We get ordinary gauge transformations from extended gauge transformations when Ω^A are taken to be left-chiral superfields without the $\psi_L(x)$- and $\mathcal{F}(x)$-components. In particular, infinitesimal ordinary gauge transformations are left-chiral superfields Ω^A with $\phi(x_+)$-components given by infinitesimal real functions $\Lambda^A(x_+)$. Thus

$$\begin{aligned} \Omega^A(x_+) = \Lambda^A(x) &+ \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\Lambda^A(x) \\ &- \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Delta\Lambda^A(x) \equiv \Lambda^A(x_+). \end{aligned} \tag{15.242}$$

The transformation law (15.236) becomes

$$e^{-2t_A V'^A(x,\theta)} = e^{it_A\Lambda^A(x_+)^*} e^{-2t_A V^A(x,\theta)} e^{-it_A\Lambda^A(x_+)}. \tag{15.243}$$

In the Wess–Zumino gauge, the gauge superfield $V^A(x, \theta)$ is of the form (15.239). We will use the identity

$$e^a e^X e^b = e^{(X + L_X(b-a) + L_X \coth L_X(b+a) + \cdots)} \tag{15.244}$$

where quadratic and higher order terms in a and/or b are neglected and $L_X f = \frac{1}{2}[X, f]$. We have

$$\begin{aligned} X &= -2t_A V^A(x, \theta) \\ &= -2t_A\left(\frac{i}{2}\left(\bar{\theta}\gamma_5\gamma^\mu\theta\right)V_\mu^A(x) - i(\bar{\theta}\gamma_5\theta)(\bar{\theta}\lambda^A(x)) - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2 D^A(x)\right) \end{aligned} \tag{15.245}$$

$$\begin{aligned} a &= it_A\Lambda^A(x_+)^* \\ &= it_A\left(\Lambda^A(x) - \frac{1}{2}\left(\bar{\theta}\gamma_5\gamma^\mu\theta\right)\partial_\mu\Lambda^A - \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Delta\Lambda^A(x)\right) \end{aligned} \tag{15.246}$$

$$
\begin{aligned}
b &= -it_A\Lambda^A(x_+) \\
&= -it_A\Big(\Lambda^A(x) + \frac{1}{2}\big(\bar\theta\gamma_5\gamma^\mu\theta\big)\partial_\mu\Lambda^A - \frac{1}{8}(\bar\theta\gamma_5\theta)^2\Delta\Lambda^A(x)\Big).
\end{aligned}
\tag{15.247}
$$

We can easily compute

$$
\begin{aligned}
a + b &= -it_A\big(\bar\theta\gamma_5\gamma^\mu\theta\big)\partial_\mu\Lambda^A \\
a - b &= 2it_A\Big(\Lambda^A(x) - \frac{1}{8}(\bar\theta\gamma_5\theta)^2\Delta\Lambda^A(x)\Big).
\end{aligned}
\tag{15.248}
$$

Let us recall that $\theta = \theta_L + \theta_R$ where θ_L and θ_R are two-dimensional Majorana spinors. Hence, we compute $\bar\theta\gamma_5\gamma^\mu\theta = \bar\theta_L\gamma_5\gamma^\mu\theta_R + \bar\theta_R\gamma_5\gamma^\mu\theta_L$, $(\bar\theta\gamma_5\theta)\bar\theta = (\bar\theta_L\gamma_5\theta_L)\bar\theta_R + (\bar\theta_R\gamma_5\theta_R)\bar\theta_L$ and $(\bar\theta\gamma_5\theta)^2 = 2(\bar\theta_L\gamma_5\theta_L)(\bar\theta_R\gamma_5\theta_R)$. We remark that (1) every term in X has at least one factor of θ_L and at least one factor of θ_R, (2) the sum $a + b$ is proportional to the product of one θ_L and one θ_R. It is, therefore, evident that $L_X^2(a + b) = 0$ and hence $L_X \coth L_X(a + b) = a + b = -it_A(\bar\theta\gamma_5\gamma^\mu\theta)\partial_\mu\Lambda^A$. Next, we compute $L_X(b - a) = \frac{1}{2}[X, b - a]$. Again, every term in X has at least one factor of θ_L and at least one factor of θ_R, whereas the second term in $b - a$ is proportional to the product of two θ_Ls and two θ_Rs. Hence, the second term in $b - a$ will not contribute and only the first term will contribute. Explicitly we compute $L_X(b - a) = -2C^C_{AB}t_C V^A\Lambda^B$. The structure constants are defined by $[t_A, t_B] = iC^C_{AB}t_C$. From the transformation law (15.243) we get the final result

$$
V'^C = V^C + C^C_{AB}V^A\Lambda^B + \frac{i}{2}\big(\bar\theta\gamma_5\gamma^\mu\theta\big)\partial_\mu\Lambda^C. \tag{15.249}
$$

Note that under ordinary gauge transformations the Wess–Zumino gauge is stable. Indeed, the gauge transformed superfield V'^C given by equation (15.249) is of the same form as the superfield V^C, in other words it is of the form (15.239). In terms of the components we have

$$
\begin{aligned}
V'^C_\mu &= V^C_\mu + C^C_{AB}V^A_\mu\Lambda^B + \partial_\mu\Lambda^C \\
\lambda'^C &= \lambda^C + C^C_{AB}\lambda^A\Lambda^B \\
D'^C &= D^C + C^C_{AB}D^A\Lambda^B.
\end{aligned}
\tag{15.250}
$$

The first equation is precisely the usual gauge transformation rule $V'_\mu = gV_\mu g^+ + ig\partial_\mu g^+$ where the gauge group element g is related to the gauge parameter Λ by $g = 1 + i\Lambda$. The second and third equation indicate that the fields λ^C and D^C are matter fields which transform in the adjoint representation of the gauge group. The Majorana field λ^C is known as the gaugino field, whereas D^C will turn out to be an auxiliary field.

15.5.3 Calculation of the *F*-term in the abelian theory

To construct $N = 1$ supersymmetric $U(1)$ pure gauge action the starting point is the spinor superfield

$$W_\alpha(x, \theta) = \frac{i}{4}(\mathcal{D}^T \epsilon \mathcal{D})\mathcal{D}_\alpha V(x, \theta). \tag{15.251}$$

The real superfield $V(x, \theta)$ represents the vector gauge superfield. The superderivatives are defined by

$$\begin{aligned} \mathcal{D}_\alpha &= (\gamma_5\epsilon)_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} - (\gamma^\mu\theta)_\alpha\partial_\mu = -\frac{\partial}{\partial\bar{\theta}_\alpha} - (\gamma^\mu\theta)_\alpha\partial_\mu \\ \bar{\mathcal{D}}_\beta &= \mathcal{D}_\gamma(\gamma_5\epsilon)_{\gamma\beta} = \frac{\partial}{\partial\theta_\beta} + \left(\gamma_5\epsilon\gamma^\mu\theta\right)_\beta\partial_\mu. \end{aligned} \tag{15.252}$$

They satisfy

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = -2\gamma^\mu_{\alpha\beta}\partial_\mu. \tag{15.253}$$

The chirality matrix γ_5 is defined by $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \mathrm{diag}(1, 1, -1, -1)$. The matrix ϵ is defined by $\epsilon = \mathrm{diag}(e, e)$, $e = i\sigma_2$. We have $\epsilon^2 = -1$, $\gamma_5^2 = 1$, $\gamma_5\epsilon = \epsilon\gamma_5$, $\epsilon^T = -\epsilon$, $\gamma_5^T = \gamma_5$. In the above we have also used the fact that θ is a Majorana spinor, viz $\bar{\theta}_\alpha = (\gamma_5\epsilon)_{\beta\alpha}\theta_\beta$. We can compute

$$\begin{aligned} \mathcal{D}_\alpha &= \mathcal{D}_{L\alpha} + \mathcal{D}_{R\alpha} \\ \mathcal{D}^T\epsilon\mathcal{D} &= \mathcal{D}_L^T\epsilon\mathcal{D}_L + \mathcal{D}_R^T\epsilon\mathcal{D}_R \end{aligned} \tag{15.254}$$

$$\begin{aligned} \mathcal{D}_{L\alpha} &= \left(\frac{1+\gamma_5}{2}\mathcal{D}\right)_\alpha = \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_{L\beta}} - \left(\gamma^\mu\theta_R\right)_\alpha\partial_\mu \\ \mathcal{D}_{R\alpha} &= \left(\frac{1-\gamma_5}{2}\mathcal{D}\right)_\alpha = -\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_{R\beta}} - \left(\gamma^\mu\theta_L\right)_\alpha\partial_\mu. \end{aligned} \tag{15.255}$$

A left-chiral superfield Φ must satisfy $\mathcal{D}_{R\alpha}\Phi = 0$, whereas a right-chiral superfield $\tilde{\Phi}$ must satisfy $\mathcal{D}_{L\alpha}\tilde{\Phi} = 0$. We note that $\mathcal{D}_{R\alpha}x_+^\mu = 0$ and $\mathcal{D}_{L\alpha}x_-^\mu = 0$. By using $\gamma_\mu^T = -C\gamma_\mu C^{-1}$ where $C = -\gamma_5\epsilon$ and $C^{-1} = \epsilon\gamma_5$ we can show, after a long calculation, that

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2\left(\gamma^\mu\epsilon\gamma_5\right)_{\alpha\beta}\partial_\mu. \tag{15.256}$$

We can also verify after some algebra that

$$\begin{aligned} &\{\mathcal{D}_{L\alpha}, \mathcal{D}_{L\beta}\} = \{\mathcal{D}_{R\alpha}, \mathcal{D}_{R\beta}\} = 0 \\ &\{\mathcal{D}_{L\alpha}, \mathcal{D}_{R\beta}\} = -2(\gamma^\mu\epsilon)_{\alpha\beta}\partial_\mu \\ &\{\mathcal{D}_{R\alpha}, \mathcal{D}_{L\beta}\} = 2(\gamma^\mu\epsilon)_{\alpha\beta}\partial_\mu. \end{aligned} \tag{15.257}$$

Therefore,

$$\left(\mathcal{D}^T\epsilon\mathcal{D}\right)\mathcal{D} = \left(\mathcal{D}_L^T\epsilon\mathcal{D}_L\right)\mathcal{D}_R + \left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)\mathcal{D}_L. \tag{15.258}$$

This follows partially from the fact that there are only two independent left superderivatives which anticommute and only two independent right superderivatives which also anticommute. Hence, the spinor superfield W can be decomposed as follows

$$W_\alpha = W_{L\alpha} + W_{R\alpha} \tag{15.259}$$

$$W_{L\alpha} = \frac{i}{4}\left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)\mathcal{D}_{L\alpha}V, \;\; W_{R\alpha} = \frac{i}{4}\left(\mathcal{D}_L^T\epsilon\mathcal{D}_L\right)\mathcal{D}_{R\alpha}V. \tag{15.260}$$

In other words, the spinor superfield W is chiral. Extended abelian gauge transformations are given by

$$V(x,\theta) \longrightarrow V(x,\theta) + \delta V(x,\theta), \;\; \delta V(x,\theta) = \frac{i}{2}(\Omega(x,\theta) - \Omega^+(x,\theta)). \tag{15.261}$$

$\Omega(x,\theta)$ is an arbitrary left-chiral superfield so it is a function of x_+ and θ_L. Because Ω is a left-chiral superfield (viz $\mathcal{D}_R\Omega = \mathcal{D}_L\Omega^+ = 0$) we have $W_{L\alpha} \longrightarrow W_{L\alpha} + \delta W_{L\alpha}$ where

$$\begin{aligned}\delta W_{L\alpha} &= \frac{i}{4}\left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)\mathcal{D}_{L\alpha}\left(\frac{i}{2}\Omega\right)\\ &= -\frac{1}{8}[\epsilon_{\mu\nu}\mathcal{D}_{R\mu}\mathcal{D}_{R\nu},\, \mathcal{D}_{L\alpha}]\Omega.\end{aligned} \tag{15.262}$$

But we compute

$$[\epsilon_{\mu\nu}\mathcal{D}_{R\mu}\mathcal{D}_{R\nu},\, \mathcal{D}_{L\alpha}] = -4\left(\gamma^\mu\partial_\mu\mathcal{D}_R\right)_\alpha. \tag{15.263}$$

Hence $\delta W_{L\alpha} = 0$. Similarly, we can show that $\delta W_{R\alpha} = 0$. Therefore, the spinor superfield W_α is gauge invariant. In the Wess–Zumino gauge the gauge superfield V takes the form

$$\begin{aligned}V = &\frac{i}{2}\left(\bar\theta\gamma_5\gamma^\mu\theta\right)V_\mu(x) - i(\bar\theta\gamma_5\theta)(\bar\theta\lambda(x))\\ &-\frac{1}{4}(\bar\theta\gamma_5\theta)^2 D(x).\end{aligned} \tag{15.264}$$

We need to compute

$$W_\alpha = \frac{i}{4}\epsilon_{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{D}_\alpha V. \tag{15.265}$$

This is a very long calculation. We first compute

$$\mathcal{D}_\alpha V = (\gamma_5\epsilon)_{\alpha\beta}\frac{\partial V}{\partial\theta_\beta} - (\gamma^\mu\theta)_\alpha\partial_\mu V \tag{15.266}$$

$$\begin{aligned}\mathcal{D}_\nu\mathcal{D}_\alpha V = {}& (\gamma_5\epsilon)_{\nu\nu'}(\gamma_5\epsilon)_{\alpha\alpha'}\frac{\partial^2 V}{\partial\theta_{\nu'}\partial\theta_{\alpha'}} + (\gamma^\sigma\epsilon\gamma_5)_{\alpha\nu}\partial_\sigma V \\ & + (\gamma_5\epsilon)_{\nu\rho}(\gamma^\sigma\theta)_\alpha\frac{\partial}{\partial\theta_\rho}\partial_\sigma V \\ & -(\gamma^{\nu'}\theta)_\nu(\gamma_5\epsilon)_{\alpha\alpha'}\partial_{\nu'}\frac{\partial V}{\partial\theta_{\alpha'}} + (\gamma^{\nu'}\theta)_\nu(\gamma^{\alpha'}\theta)_\alpha\partial_{\nu'}\partial_{\alpha'}V.\end{aligned} \tag{15.267}$$

We act with one more superderivative $\mathcal{D}_\mu$ and contract with $\epsilon_{\mu\nu}$ to get

$$\begin{aligned}\epsilon_{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{D}_\alpha V = {}& 2\epsilon_{\mu\nu}\left(\gamma^{\alpha'}\epsilon\gamma_5\right)_{\alpha\mu}(\gamma^{\nu'}\theta)_\nu\partial_{\nu'}\partial_{\alpha'}V \\ & + 2\epsilon_{\mu\nu}(\gamma_5\epsilon)_{\mu\mu'}(\gamma^{\nu'}\theta)_\nu(\gamma^{\alpha'}\theta)_\alpha\partial_{\nu'}\partial_{\alpha'}\frac{\partial V}{\partial\theta_{\mu'}} \\ & + 2\epsilon_{\mu\nu}(\gamma_5\epsilon)_{\mu\mu'}(\gamma_5\epsilon)_{\alpha\alpha'}(\gamma^{\nu'}\theta)_\nu\partial_{\nu'}\frac{\partial^2 V}{\partial\theta_{\mu'}\partial\theta_{\alpha'}} \\ & + 2\epsilon_{\mu\nu}\left(\gamma^{\nu'}\epsilon\gamma_5\right)_{\alpha\nu}(\gamma_5\epsilon)_{\mu\mu'}\partial_{\nu'}\frac{\partial V}{\partial\theta_{\mu'}} \\ & + \epsilon_{\mu\nu}(\gamma_5\epsilon)_{\mu\mu'}(\gamma_5\epsilon)_{\nu\nu'}(\gamma_5\epsilon)_{\alpha\alpha'}\frac{\partial^3 V}{\partial\theta_{\mu'}\partial\theta_{\nu'}\partial\theta_{\alpha'}} \\ & + \epsilon_{\alpha'\nu'}(\gamma^{\mu'}\theta)_\alpha\partial_{\mu'}\frac{\partial^2 V}{\partial\theta_{\nu'}\partial\theta_{\alpha'}} \\ & + \epsilon_{\mu\nu}(\gamma_5\epsilon)_{\alpha\alpha'}(\gamma^{\mu'}\theta)_\mu(\gamma^{\nu'}\theta)_\nu\partial_{\mu'}\partial_{\nu'}\frac{\partial V}{\partial\theta_{\alpha'}}.\end{aligned} \tag{15.268}$$

The different terms are computed separately as follows

$$\begin{aligned}2\epsilon_{\mu\nu}\left(\gamma^{\alpha'}\epsilon\gamma_5\right)_{\alpha\mu}(\gamma^{\nu'}\theta)_\nu\partial_{\nu'}\partial_{\alpha'}V &= 2(\gamma_5\theta)_\alpha\partial^2 V \\ &= \frac{i}{2}(\theta^T\epsilon\theta)^2\partial^2\lambda_\alpha \\ &\quad -i(\bar\theta\gamma_5\theta)\left(\gamma_5\gamma^\mu\theta\right)_\alpha\partial^2 V_\mu\end{aligned} \tag{15.269}$$

$$
\begin{aligned}
2\epsilon_{\mu\nu}(\gamma_5\epsilon)_{\mu\mu'}(\gamma^{\nu'}\theta)_\nu(\gamma^{\alpha'}\theta)_\alpha\partial_{\nu'}\partial_{\alpha'}\frac{\partial V}{\partial\theta_{\mu'}} &= 2(\gamma_5\gamma^{\nu'}\theta)_{\mu'}(\gamma^{\alpha'}\theta)_\alpha\partial_{\nu'}\partial_{\alpha'}\frac{\partial V}{\partial\theta_{\mu'}} \\
&= -2i(\bar\theta\gamma_5\theta)\big(\gamma^\nu\gamma_5\theta\big)_\alpha\partial^\mu\partial_\nu V_\mu \\
&\quad -2i\big(\bar\theta\gamma_5\gamma^\nu\theta\big)\theta_\alpha\big(\bar\theta\gamma_5\gamma^\mu\partial_\mu\partial_\iota\lambda\big) \\
&= i(\theta^T\epsilon\theta)\big(\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\theta\big)_\alpha\partial_\sigma F_{\mu\nu} \\
&\quad + 2i(\theta^T\epsilon\theta)\big(\gamma_5\gamma^\mu\theta\big)_\alpha\partial^2 V_\mu \\
&\quad -2i\big(\bar\theta\gamma_5\gamma^\nu\theta\big)\theta_\alpha\big(\bar\theta\gamma_5\gamma^\mu\partial_\mu\partial_\iota\lambda\big) \\
&= i(\theta^T\epsilon\theta)\big(\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\theta\big)_\alpha\partial_\sigma F_{\mu\nu} \\
&\quad + 2i(\theta^T\epsilon\theta)\big(\gamma_5\gamma^\mu\theta\big)_\alpha\partial^2 V_\mu \\
&\quad -\frac{i}{2}(\bar\theta\gamma_5\theta)^2(\partial^2\lambda)_\alpha .
\end{aligned}
\tag{15.270}
$$

In the third equality we have used the Bianchi identity, whereas in the last equality we have used, among other things, the identity $(\gamma^\nu\theta)_\alpha(\bar\theta\gamma_5\theta) = -\theta_\alpha(\bar\theta\gamma_5\gamma^\nu\theta)$.

We also compute

$$
\begin{aligned}
2\epsilon_{\mu\nu}(\gamma^{\nu'}\epsilon\gamma_5)_{\alpha\nu}(\gamma_5\epsilon)_{\mu\mu'}\partial_{\nu'}\frac{\partial V}{\partial\theta_{\mu'}} &= 2(\gamma^{\nu'}\epsilon)_{\alpha\mu'}\partial_{\nu'}\frac{\partial V}{\partial\theta_{\mu'}} \\
&= -i(\gamma^\mu\gamma^\nu\theta)_\alpha F_{\mu\nu} - 2i\theta_\alpha\partial^\mu V_\mu \\
&\quad + 4i(\gamma^\mu\theta)_\alpha\big(\theta^T\epsilon\gamma_5\partial_\mu\lambda\big) \\
&\quad + 2i(\theta^T\epsilon\theta)\big(\gamma^\mu\gamma_5\partial_\mu\lambda\big)_\alpha \\
&\quad + 2(\gamma^\mu\theta)_\alpha(\theta^T\epsilon\theta)\partial_\mu D
\end{aligned}
\tag{15.271}
$$

$$
\begin{aligned}
2\epsilon_{\mu\nu}(\gamma_5\epsilon)_{\mu\mu'}(\gamma_5\epsilon)_{\alpha\alpha'}(\gamma^{\nu'}\theta)_\nu\partial_{\nu'}\frac{\partial^2 V}{\partial\theta_{\mu'}\partial\theta_{\alpha'}} &= 2(\gamma_5\epsilon)_{\alpha\alpha'}(\gamma_5\gamma^{\mu'}\theta)_{\nu'}\partial_{\mu'}\frac{\partial^2 V}{\partial\theta_{\nu'}\partial\theta_{\alpha'}} \\
&= -i(\gamma^\mu\gamma^\nu\theta)_\alpha F_{\mu\nu} + 2i\theta_\alpha\partial^\mu V_\mu \\
&\quad + 4i(\gamma^\mu\theta)_\alpha\big(\theta^T\epsilon\gamma_5\partial_\mu\lambda\big) \\
&\quad + 4i(\gamma_5\theta)_\alpha\big(\theta^T\epsilon\gamma^\mu\partial_\mu\lambda\big) \\
&\quad + 2(\gamma^\mu\theta)_\alpha(\theta^T\epsilon\theta)\partial_\mu D \\
&= -i(\gamma^\mu\gamma^\nu\theta)_\alpha F_{\mu\nu} + 2i\theta_\alpha\partial^\mu V_\mu \\
&\quad -2i\big(\gamma^\mu\partial_\mu\lambda\big)_\alpha\big(\theta^T\epsilon\gamma_5\theta\big) \\
&\quad -2i(\gamma_5\partial_\mu\lambda)_\alpha\big(\bar\theta\gamma_5\gamma^\mu\theta\big) \\
&\quad + 2(\gamma^\mu\theta)_\alpha(\theta^T\epsilon\theta)\partial_\mu D
\end{aligned}
\tag{15.272}
$$

$$\epsilon_{\mu\nu}(\gamma_5\epsilon)_{\mu\mu'}(\gamma_5\epsilon)_{\nu\nu'}(\gamma_5\epsilon)_{\alpha\alpha'}\frac{\partial^3 V}{\partial\theta_{\mu'}\partial\theta_{\nu'}\partial\theta_{\alpha'}} = \epsilon_{\mu'\nu'}(\gamma_5\epsilon)_{\alpha\alpha'}\frac{\partial^3 V}{\partial\theta_{\mu'}\partial\theta_{\nu'}\partial\theta_{\alpha'}} = -4i\lambda_\alpha - 4(\gamma_5\theta)_\alpha D \tag{15.273}$$

$$\epsilon_{\alpha'\nu'}(\gamma^{\mu'}\theta)_\alpha\partial_{\mu'}\frac{\partial^2 V}{\partial\theta_{\nu'}\partial\theta_{\alpha'}} = -4i(\gamma^{\mu'}\theta)_\alpha\left(\theta^T\epsilon\gamma_5\partial_{\mu'}\lambda\right) - 2(\gamma^{\mu'}\theta)_\alpha(\theta^T\epsilon\theta)\partial_{\mu'}D \tag{15.274}$$

$$\epsilon_{\mu\nu}(\gamma_5\epsilon)_{\alpha\alpha'}(\gamma^{\mu'}\theta)_\mu(\gamma^{\nu'}\theta)_\nu\partial_{\mu'}\partial_{\nu'}\frac{\partial V}{\partial\theta_{\alpha'}} = (\gamma_5\epsilon)_{\alpha\alpha'}(\theta^T\epsilon\theta)\frac{\partial}{\partial\theta_{\alpha'}}\partial^2 V = \frac{i}{2}(\bar\theta\gamma_5\theta)^2(\partial^2\lambda)_\alpha - i\left(\gamma_5\gamma^\mu\theta\right)_\alpha(\theta^T\epsilon\theta)\partial^2 V_\mu. \tag{15.275}$$

The spinor superfield W_α takes the form

$$\begin{aligned} W_\alpha = {} & \lambda_\alpha + \frac{1}{2}(\gamma^\mu\gamma^\nu\theta)_\alpha F_{\mu\nu} - i(\gamma_5\theta)_\alpha D \\ & -\frac{1}{2}(\theta^T\epsilon\theta)\left(\gamma^\mu\gamma_5\partial_\mu\lambda\right)_\alpha + \frac{1}{2}\left(\theta^T\epsilon\gamma_5\theta\right)\left(\gamma^\mu\partial_\mu\lambda\right)_\alpha \\ & + \frac{1}{2}\left(\theta^T\epsilon\gamma^\mu\theta\right)\left(\gamma_5\partial_\mu\lambda\right)_\alpha + \frac{i}{2}(\gamma^\mu\theta)_\alpha(\theta^T\epsilon\theta)\partial_\mu D \\ & -\frac{1}{4}(\theta^T\epsilon\theta)\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\theta\right)_\alpha\partial_\sigma F_{\mu\nu} - \frac{1}{8}(\theta^T\epsilon\theta)^2\partial^2\lambda_\alpha. \end{aligned} \tag{15.276}$$

We need also to compute the left-chiral and right-chiral spinor superfields $W_{L\alpha}$ and $W_{R\alpha}$. We have

$$\frac{1}{2}(\gamma^\mu\gamma^\nu\theta_L)_\alpha F_{\mu\nu}(x_+) = \left(\frac{1+\gamma_5}{2}\right)_{\alpha\beta} \times\left[\frac{1}{2}(\gamma^\mu\gamma^\nu\theta)_\beta F_{\mu\nu} - \frac{1}{4}(\theta^T\epsilon\theta)\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\theta\right)_\beta\partial_\sigma F_{\mu\nu}\right] \tag{15.277}$$

$$\lambda_{L\alpha}(x_+) = \left(\frac{1+\gamma_5}{2}\right)_{\alpha\beta}\left[\lambda_\beta + \frac{1}{2}(\theta^T\epsilon\gamma^\mu\theta)(\gamma_5\partial_\mu\lambda)_\beta - \frac{1}{8}(\theta^T\epsilon\theta)^2\partial^2\lambda_\beta\right] \tag{15.278}$$

$$\left(\theta_L^T\epsilon\theta_L\right)\left(\gamma^\mu\partial_\mu\lambda_R(x_+)\right)_\alpha = \left(\frac{1+\gamma_5}{2}\right)_{\alpha\beta} \times\left[-\frac{1}{2}(\theta^T\epsilon\theta)\left(\gamma^\mu\gamma_5\partial_\mu\lambda\right)_\beta + \frac{1}{2}\left(\theta^T\epsilon\gamma_5\theta\right)\left(\gamma^\mu\partial_\mu\lambda\right)_\beta\right] \tag{15.279}$$

$$-i\theta_{L\alpha}D(x_+) = \left(\frac{1+\gamma_5}{2}\right)_{\alpha\beta}\left[-(i\gamma_5\theta)_\beta D + \frac{i}{2}(\theta^T\epsilon\theta)(\gamma^\mu\theta)_\beta\partial_\mu D\right]. \tag{15.280}$$

Thus

$$\begin{aligned} W_{L\alpha} \equiv \left(\frac{1+\gamma_5}{2}\right)_{\alpha\beta} W_\beta = \lambda_{L\alpha}(x_+) + \frac{1}{2}\left(\gamma^\mu\gamma^\nu\theta_L\right)_\alpha F_{\mu\nu}(x_+) \\ + \left(\theta_L^T\epsilon\theta_L\right)\left(\gamma^\mu\partial_\mu\lambda_R\right)_\alpha(x_+) - i\theta_{L\alpha}D(x_+). \end{aligned} \tag{15.281}$$

Similarly, we obtain

$$\begin{aligned} W_{R\alpha} = \lambda_{R\alpha}(x_-) + \frac{1}{2}\left(\gamma^\mu\gamma^\nu\theta_R\right)_\alpha F_{\mu\nu}(x_-) \\ -\left(\theta_R^T\epsilon\theta_R\right)\left(\gamma^\mu\partial_\mu\lambda_L\right)_\alpha(x_-) - i\theta_{R\alpha}D(x_-). \end{aligned} \tag{15.282}$$

Let us now compute the following F-term:

$$[\epsilon_{\alpha\beta}W_{L\alpha}W_{L\beta}]_F. \tag{15.283}$$

Another long calculation yields

$$\begin{aligned} -\frac{1}{2}\left(\theta_L^T\epsilon\theta_L\right)[\epsilon_{\alpha\beta}W_{L\alpha}W_{L\beta}]_F = \left(\theta_L^T\epsilon\theta_L\right) \\ \times\left[-\frac{1}{2}\lambda^T\epsilon\gamma^\mu\partial_\mu\lambda - \frac{1}{2}\lambda^T\gamma_5\epsilon\gamma^\mu\partial_\mu\lambda\right. \\ \left. + \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{8}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right]. \end{aligned} \tag{15.284}$$

The supersymmetric $U(1)$ action is taken to be the real part of this F-term, viz

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\,\text{Re}[\epsilon_{\alpha\beta}W_{L\alpha}W_{L\beta}]_F = \left[-\frac{1}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda + \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{15.285}$$

The supersymmetry transformations for the components of the gauge superfields V are

$$\begin{aligned} V_\mu &\longrightarrow V'_\mu = V_\mu + \bar{\alpha}\gamma_\mu\lambda \\ \lambda &\longrightarrow \lambda' = \lambda + \left(i\gamma_5 D + \frac{1}{4}F_{\mu\nu}[\gamma^\nu, \gamma^\mu]\right)\alpha \\ D &\longrightarrow D' = D + i\bar{\alpha}\gamma_5\gamma_\mu\partial^\mu\lambda. \end{aligned} \tag{15.286}$$

We can verify the invariance of the action by using the identities

$$\begin{aligned}\delta F_{\mu\nu} &= \bar{\alpha}((\partial_\mu\gamma_\nu - \partial_\nu\gamma_\mu)\lambda)\\ \delta(\bar{\lambda}\gamma^\mu\partial_\mu\lambda) &= 2iD(\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\lambda)\\ &\quad + F_{\mu\nu}\bar{\alpha}((\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\lambda).\end{aligned} \tag{15.287}$$

15.5.4 Calculation of the *F*-term in the non-abelian theory

Under generalized gauge transformations the non-abelian gauge superfield $V(x, \theta) = t_A V_A(x, \theta)$ transforms as

$$\begin{aligned}e^{-2t_A V_A} &\longrightarrow e^{-it_A\Omega_A}e^{-2t_A V_A}e^{it_A\Omega_A^*}\\ e^{+2t_A V_A} &\longrightarrow e^{-it_A\Omega_A^*}e^{+2t_A V_A}e^{it_A\Omega_A}.\end{aligned} \tag{15.288}$$

Ω_A is a left-chiral superfield ($\mathcal{D}_{R\alpha}\Omega_A = 0$), whereas Ω_A^* is a right-chiral superfield ($\mathcal{D}_{L\alpha}\Omega_A^* = 0$). Thus

$$e^{-2t_A V_A}\mathcal{D}_{L\alpha}e^{2t_A V_A} \longrightarrow e^{-it_A\Omega_A}e^{-2t_A V_A}\mathcal{D}_{L\alpha}e^{2t_A V_A}e^{it_A\Omega_A}. \tag{15.289}$$

In the above $\mathcal{D}_{L\alpha}$ still acts on $e^{it_A\Omega_A}$. By analogy with the abelian case we define the left-chiral spinor superfields $W_{AL\alpha}$ by

$$2t_A W_{AL\alpha} = \frac{i}{4}\epsilon_{\beta\gamma}\mathcal{D}_{R\beta}\mathcal{D}_{R\gamma}e^{-2t_A V_A}\mathcal{D}_{L\alpha}e^{2t_A V_A}. \tag{15.290}$$

Obviously these are left-chiral superfields ($\mathcal{D}_{R\beta}W_{AL\alpha} = 0$) since the product of 3 $\mathcal{D}_R$s vanishes. The above definition of the spinor superfields is motivated by the requirement of covariance under gauge transformations. Indeed, since Ω_A is left-chiral we have

$$\begin{aligned}2t_A W_{AL\alpha} &= \frac{i}{4}\epsilon_{\beta\gamma}\mathcal{D}_{R\beta}\mathcal{D}_{R\gamma}e^{-2t_A V_A}\mathcal{D}_{L\alpha}e^{2t_A V_A}\\ &\longrightarrow \frac{i}{4}e^{-it_A\Omega_A}\epsilon_{\beta\gamma}\mathcal{D}_{R\beta}\mathcal{D}_{R\gamma}e^{-2t_A V_A}\mathcal{D}_{L\alpha}e^{2t_A V_A}e^{it_A\Omega_A}\\ &\longrightarrow e^{-it_A\Omega_A}2t_A W_{AL\alpha}e^{it_A\Omega_A}\\ &\quad + \frac{i}{4}e^{-it_A\Omega_A}[\epsilon_{\beta\gamma}\mathcal{D}_{R\beta}\mathcal{D}_{R\gamma},\ \mathcal{D}_{L\alpha}]e^{it_A\Omega_A}\\ &\longrightarrow e^{-it_A\Omega_A}2t_A W_{AL\alpha}e^{it_A\Omega_A}.\end{aligned} \tag{15.291}$$

In the last line we have also used the identity (15.263). Thus, the full spinor superfield $W_{L\alpha} = t_A W_{AL\alpha}$ transforms as

$$W_{L\alpha} \longrightarrow e^{-it_A\Omega_A}W_{L\alpha}e^{it_A\Omega_A}. \tag{15.292}$$

In the Wess–Zumino gauge the gauge superfields are given by

$$V_A(x,\theta) = \frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)V_A^\mu(x) - i(\bar{\theta}\gamma_5\theta)(\bar{\theta}\lambda_A(x)) - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2 D_A(x). \tag{15.293}$$

As in the abelian case it is crucial to compute the spinor superfields $W_{AL\alpha}$ in terms of the components of the gauge superfields V_A. We will only calculate the spinor field at some point $x^\mu = X^\mu$. Towards this end we will adopt a version of the Wess–Zumino gauge in which

$$V_A^\mu(X) = 0. \tag{15.294}$$

We have (with $V = t_A V_A$)

$$\left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)e^{-2t_AV_A}\mathcal{D}_\alpha e^{2t_AV_A} = 2\left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)(1 - 2V + 2V^2 + \cdots) \times \mathcal{D}_\alpha(V + V^2 + \cdots) \tag{15.295}$$

The terms $2(\mathcal{D}_R^T\epsilon\mathcal{D}_R)\mathcal{D}_\alpha(V^2)$ and $2(\mathcal{D}_R^T\epsilon\mathcal{D}_R)(2V^2)\mathcal{D}_\alpha(V)$ will lead to zero in the gauge (15.293) together with equation (15.294) and hence higher order terms must also be zero in this gauge. The calculation reduces to

$$\left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)e^{-2t_AV_A}\mathcal{D}_\alpha e^{2t_AV_A} = 2\left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)(1 - 2V)\mathcal{D}_\alpha(V) \tag{15.296}$$

The term $2(\mathcal{D}_R^T\epsilon\mathcal{D}_R)(-2V)\mathcal{D}_\alpha(V)$ is similar to $2(\mathcal{D}_R^T\epsilon\mathcal{D}_R)\mathcal{D}_\alpha(V^2)$ and hence it will also lead to zero. We get the result

$$\left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)e^{-2t_AV_A}\mathcal{D}_\alpha e^{2t_AV_A} = 2\left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)\mathcal{D}_\alpha(V) \tag{15.297}$$

Hence, in the gauge (15.293) together with equation (15.294) the spinor superfields are given by

$$W_{AL\alpha} = \frac{i}{4}\left(\mathcal{D}_R^T\epsilon\mathcal{D}_R\right)\mathcal{D}_{L\alpha}V_A. \tag{15.298}$$

In other words, in this gauge we have a number of copies of the abelian theory labeled by A. The calculation of this superfield proceeds, therefore, as in the abelian case with the result

$$W_{AL\alpha} = \lambda_{AL\alpha}(X_+) + \frac{1}{2}(\gamma_\mu\gamma_\nu\theta_L)_\alpha(\partial^\mu V_A^\nu(X_+) - \partial^\nu V_A^\mu(X_+)) + \left(\theta_L^T\epsilon\theta_L\right)\left(\gamma^\mu\partial_\mu\lambda_{AR}(X_+)\right)_\alpha - i\theta_{L\alpha}D_A(X_+). \tag{15.299}$$

Since $W_{L\alpha}$ is gauge-covariant, at a general point we must have

$$W_{AL\alpha} = \lambda_{AL\alpha}(x_+) + \frac{1}{2}(\gamma_\mu\gamma_\nu\theta_L)_\alpha F_A^{\mu\nu}(x_+) + \left(\theta_L^T\epsilon\theta_L\right)\left(\gamma^\mu(D_\mu\lambda_R)_A\right)_\alpha(x_+) - i\theta_{L\alpha}D_A(x_+). \tag{15.300}$$

The curvature $F_A^{\mu\nu}$ and the covariant derivatives of the Majorana spinors $(D^\mu\lambda_R)_A$ are defined by

$$F_A^{\mu\nu} = \partial^\mu V_A^\nu - \partial^\nu V_A^\mu + C_{ABC} V_B^\mu V_C^\nu \tag{15.301}$$

$$\left(D^\mu\lambda_R\right)_A = \partial^\mu\lambda_{RA} + C_{ABC} V_B^\mu \lambda_{RC}. \tag{15.302}$$

Let us now compute the following F-term:

$$\left(\theta_L^T \epsilon \theta_L\right)[\epsilon_{\alpha\beta} W_{AL\alpha} W_{AL\beta}]_{\mathcal{F}}. \tag{15.303}$$

We have

$$[\epsilon_{\alpha\beta} W_{AL\alpha}\lambda_{AL\beta}]_{\theta_L^2} = \left(\theta_L^T \epsilon\theta_L\right)\left(\lambda_{AL}^T \epsilon\gamma^\mu (D_\mu\lambda_R)_A\right) \tag{15.304}$$

$$\begin{aligned}\left[\epsilon_{\alpha\beta} W_{AL\alpha}\frac{1}{2}\left(\gamma^\mu\gamma^\nu\theta_L\right)_\beta F_{A\mu\nu}\right]_{\theta_L^2} &= \frac{1}{4}\left[\left(\gamma^\mu\gamma^\nu\theta_L\right)^T \epsilon\left(\gamma^\alpha\gamma^\beta\theta_L\right)\right]F_{A\mu\nu}F_{A\alpha\beta} \\ &\quad - iD_A\left[\theta_L^T\epsilon\gamma^\mu\gamma^\nu\theta_L\right]F_{A\mu\nu}.\end{aligned} \tag{15.305}$$

We use, among other things, the identities $\bar\theta\gamma_5 = \theta^T\epsilon$, $\gamma_\mu^T = -C\gamma^\mu C^{-1}$ with $C = -C^{-1} = \epsilon\gamma^5$ and $\epsilon^2 = -1$ to show that $\bar\theta[\gamma^\mu, \gamma^\nu]\theta = \bar\theta\gamma_5[\gamma^\mu, \gamma^\nu]\theta = 0$. Hence, the second contribution in equation (15.305) vanishes. The first contribution can be put in the form

$$\begin{aligned}\left[\epsilon_{\alpha\beta} W_{AL\alpha}\frac{1}{2}\left(\gamma^\mu\gamma^\nu\theta_L\right)_\beta F_{A\mu\nu}\right]_{\theta_L^2} &= -\frac{1}{16}\left[\left(\bar\theta_L[\gamma^\mu, \gamma^\nu][\gamma^\alpha, \gamma^\beta]\theta_L\right)\right]F_{A\mu\nu}F_{A\alpha\beta} \\ &= -\frac{1}{4}\left(\theta_L^T\epsilon\theta_L\right) \\ &\quad \times(-\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} + i\epsilon^{\mu\nu\alpha\beta})F_{A\mu\nu}F_{A\alpha\beta} \\ &= -\frac{1}{4}\left(\theta_L^T\epsilon\theta_L\right) \\ &\quad \times\left(-2F_{A\mu\nu}F^{A\mu\nu} + i\epsilon^{\mu\nu\alpha\beta}F_{A\mu\nu}F_{A\alpha\beta}\right).\end{aligned} \tag{15.306}$$

Finally, we compute

$$\left[\epsilon_{\alpha\beta} W_{AL\alpha}\left(\theta_L^T\epsilon\theta_L\right)(\gamma^\mu(D_\mu\lambda_R)_A)_\beta\right]_{\theta_L^2} = \left(\theta_L^T\epsilon\theta_L\right)\left(\lambda_{AL}^T\epsilon\gamma^\mu(D_\mu\lambda_R)_A\right) \tag{15.307}$$

$$[\epsilon_{\alpha\beta} W_{AL\alpha}(-i\theta_{L\beta}D_A)]_{\theta_L^2} = \left(\theta_L^T\epsilon\theta_L\right)D_A^2 \tag{15.308}$$

Hence, we obtain the F-term

$$-\frac{1}{2}\left(\theta_L^T\epsilon\theta\right)[\epsilon_{\alpha\beta}W_{AL\alpha}W_{AL\beta}]_F = -\frac{1}{2}\left(\theta_L^T\epsilon\theta\right)\Big[\bar{\lambda}_A\gamma^\mu(D_\mu(1-\gamma_5)\lambda)_A + \frac{1}{2}F_{A\mu\nu}F^{A\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\alpha\beta}F_{A\mu\nu}F_{A\alpha\beta} - D_A^2\Big]. \tag{15.309}$$

As in the abelian case, the gauge action is given by the real part of this term, viz

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\,\text{Re}[\epsilon_{\alpha\beta}W_{AL\alpha}W_{AL\beta}]_F = -\frac{1}{2}\bar{\lambda}_A\left(\gamma^\mu D_\mu\lambda\right)_A - \frac{1}{4}F_{A\mu\nu}F^{A\mu\nu} + \frac{1}{2}D_A^2. \tag{15.310}$$

Let us say that $\bar{\lambda}_A\gamma^\mu(D_\mu\gamma_5\lambda)_A = -(\epsilon\gamma^\mu)_{\alpha\beta}\lambda_{A\alpha}(D_\mu\lambda)_{A\beta}$. But since $(\epsilon\gamma^\mu)^T = -\epsilon\gamma^\mu$ we get $\bar{\lambda}_A\gamma^\mu(D_\mu\gamma_5\lambda)_A = -\partial_\mu(\bar{\lambda}_A\gamma_5\gamma^\mu\lambda_A)$. In other words, this term is a total derivative and it is pure imaginary since $(\bar{\lambda}_A\gamma_5\gamma^\mu\lambda_A)^+ = -\bar{\lambda}_A\gamma_5\gamma^\mu\lambda_A$. The imaginary part of the above F-term is, therefore, given by

$$-\frac{1}{2}\,\text{Im}[\epsilon_{\alpha\beta}W_{AL\alpha}W_{AL\beta}]_F = -\frac{i}{2}\bar{\lambda}_A\gamma^\mu(D_\mu\gamma_5\lambda)_A + \frac{1}{8}\epsilon^{\mu\nu\alpha\beta}F_{A\mu\nu}F_{A\alpha\beta} \tag{15.311}$$

this is essentially the famous theta term. Although it is a total derivative it does not vanish due to instantons. The theta term is defined by

$$\mathcal{L}_\theta = -\frac{g^2\theta}{16\pi^2}\,\text{Im}[\epsilon_{\alpha\beta}W_{AL\alpha}W_{AL\beta}]_F. \tag{15.312}$$

The structure constants are defined by $C_{ABC} = gf_{ABC}$. By absorbing a factor g into the gauge field, the Lagrangian density for the gauge field becomes multiplied by an overall factor $1/g^2$. The total action becomes

$$\mathcal{L}_{\text{gauge}} + \mathcal{L}_\theta = -\text{Re}\left[\frac{\tau}{8\pi i}\epsilon_{\alpha\beta}W_{AL\alpha}W_{AL\beta}\right]_F. \tag{15.313}$$

The complex coupling parameter τ is defined by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}. \tag{15.314}$$

The supersymmetry transformations of the action (15.310) are

$$\begin{aligned}\delta V_\mu^A &= \bar{\alpha}\gamma_\mu\lambda^A\\ \delta\lambda^A &= \left(i\gamma_5 D^A - \frac{1}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}^A\right)\alpha\\ \delta D^A &= i\bar{\alpha}\gamma_5\gamma_\mu\partial^\mu\lambda^A.\end{aligned} \tag{15.315}$$

15.5.5 Calculation of the *d*-term

The starting point is the left-chiral superfields (where n is a color index)

$$\begin{aligned}\Phi_n(x,\,\theta) = \phi_n(x) - \sqrt{2}(\bar{\theta}\psi_{nL}(x)) + \mathcal{F}_n(x)\left(\bar{\theta}\frac{1+\gamma_5}{2}\theta\right)\\ + \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\phi_n(x)\\ -\frac{1}{\sqrt{2}}(\bar{\theta}\gamma_5\theta)\left(\bar{\theta}\gamma_\mu\partial^\mu\psi_{nL}(x)\right) - \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\partial^2\phi_n(x).\end{aligned} \tag{15.316}$$

Their complex conjugates are given by

$$\begin{aligned}\Phi_n^*(x,\,\theta) = \phi_n^*(x) - \sqrt{2}(\bar{\psi}_{nL}(x)\theta) + \mathcal{F}_n^*(x)\left(\bar{\theta}\frac{1-\gamma_5}{2}\theta\right)\\ -\frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\phi_n^*(x)\\ -\frac{1}{\sqrt{2}}(\bar{\theta}\gamma_5\theta)\left(\partial^\mu\bar{\psi}_{nL}(x)\gamma^\mu\theta\right) - \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\partial^2\phi_n^*(x).\end{aligned} \tag{15.317}$$

We want to compute the following *d*-term:

$$[\Phi^+\Gamma\Phi]_d = [\Phi_n^*\Gamma_{nm}\Phi_m]_d. \tag{15.318}$$

The Kahler potential $\Phi^+\Gamma\Phi$ is by construction a real superfield. The gauge connection is given by

$$\begin{aligned}\Gamma(x,\,\theta) = e^{-2t_AV^A(x,\theta)} = e^{-2t_A\left(\frac{i}{2}(\bar{\theta}\gamma_5\gamma^\mu\theta)V_\mu^A - i(\bar{\theta}\gamma_5\theta)(\bar{\theta}\lambda^A) - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2D^A\right)}\\ = 1 - 2t_AV^A + \frac{1}{2}t_At_B(\bar{\theta}\gamma_5\theta)^2V_\mu^AV^{\mu B}.\end{aligned} \tag{15.319}$$

We have immediately

$$\Phi^+\Gamma\Phi = \Phi^+\Phi - 2\Phi^+t_A\Phi V^A + \frac{1}{2}\Phi^+t_At_B\Phi(\bar{\theta}\gamma_5\theta)^2V_\mu^AV^{\mu B}. \tag{15.320}$$

Thus

$$[\Phi^+\Gamma\Phi]_{\theta^0} = [\Phi^+\Phi]_{\theta^0} = \phi_n^*\phi_n \tag{15.321}$$

$$\begin{aligned}[\Phi^+\Gamma\Phi]_{\theta^4} = [\Phi^+\Phi]_{\theta^4} - 2(t_A)_{nm}\left[\Phi_n^*\Phi_mV^A\right]_{\theta^4}\\ + \frac{1}{2}\phi_n^*(t_At_B)_{nm}\phi_m(\bar{\theta}\gamma_5\theta)^2V_\mu^AV^{\mu B}.\end{aligned} \tag{15.322}$$

The first contribution is computed as follows

$$
\begin{aligned}
[\Phi^{+}\Phi]_{\theta^4} = & -\frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\left(\partial^2\phi_n^* \cdot \phi_n + \phi_n^* \cdot \partial^2\phi_n\right) \\
& + (\bar{\theta}\gamma_5\theta)\partial_\mu\left(\bar{\psi}_{nL}\gamma^\mu\theta\right)(\bar{\theta}\psi_{nL}) \\
& + (\bar{\psi}_{nL}\theta)(\bar{\theta}\gamma_5\theta)\left(\bar{\theta}\gamma^\mu\partial_\mu\psi_{nL}\right) \\
& + \left(\bar{\theta}\frac{1-\gamma_5}{2}\theta\right)\left(\bar{\theta}\frac{1+\gamma_5}{2}\theta\right)\mathcal{F}_n^*\mathcal{F}_n \\
& - \frac{1}{4}(\bar{\theta}\gamma_5\gamma_\mu\theta)(\bar{\theta}\gamma_5\gamma_\nu\theta)\partial^\mu\phi_n^*\partial^\nu\phi_n.
\end{aligned}
\tag{15.323}
$$

In the above we have used the fact

$$
\left(\bar{\theta}\frac{1-\gamma_5}{2}\theta\right)(\bar{\theta}\gamma_5\gamma_\mu\theta)\mathcal{F}_n^*\partial^\mu\phi_n = \left(\bar{\theta}\frac{1+\gamma_5}{2}\theta\right)(\bar{\theta}\gamma_5\gamma_\mu\theta)\mathcal{F}_n\partial^\mu\phi_n^* = 0. \tag{15.324}
$$

We need also to use the results

$$
(\bar{\theta}\gamma_5\theta)\partial_\mu\left(\bar{\psi}_{nL}\gamma^\mu\theta\right)(\bar{\theta}\psi_{nL}) = \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\partial_\mu\bar{\psi}_{nL}\gamma_5\gamma^\mu\psi_{nL} \tag{15.325}
$$

$$
(\bar{\psi}_{nL}\theta)(\bar{\theta}\gamma_5\theta)\left(\bar{\theta}\gamma^\mu\partial_\mu\psi_{nL}\right) = \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\bar{\psi}_{nL}\gamma^\mu\partial_\mu\psi_{nL} \tag{15.326}
$$

$$
\left(\bar{\theta}\frac{1-\gamma_5}{2}\theta\right)\left(\bar{\theta}\frac{1+\gamma_5}{2}\right)\mathcal{F}_n^*\mathcal{F}_n = -\frac{1}{2}(\bar{\theta}\gamma_5\theta)^2\mathcal{F}_n^*\mathcal{F}_n \tag{15.327}
$$

$$
-\frac{1}{4}(\bar{\theta}\gamma_5\gamma_\mu\theta)(\bar{\theta}\gamma_5\gamma_\nu\theta)\partial^\mu\phi_n^*\partial^\nu\phi_n = \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\partial^\mu\phi_n^*\partial_\mu\phi_n. \tag{15.328}
$$

By putting these things together we get

$$
\begin{aligned}
[\Phi^{+}\Phi]_{\theta^4} = & -\frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\left(\frac{1}{2}\partial^2\phi_n^* \cdot \phi_n + \frac{1}{2}\phi_n^* \cdot \partial^2\phi_n\right. \\
& \left. -\partial^\mu\phi_n^*\partial_\mu\phi_n - \bar{\psi}_{nL}\gamma^\mu\partial_\mu\psi_{nL} - \partial_\mu\bar{\psi}_{nL}\gamma_5\gamma^\mu\psi_{nL} + 2\mathcal{F}_n^*\mathcal{F}_n\right).
\end{aligned}
\tag{15.329}
$$

The second contribution (the second term in equation (15.322)) is computed as follows

$$\begin{aligned}\left[\Phi_n^*\Phi_m V^A\right]_{\theta^4} &= \frac{i}{2}V_\mu^A\left[\Phi_n^*\Phi_m\bar{\theta}\gamma_5\gamma^\mu\theta\right]_{\theta^4} - i\left[\Phi_n^*\Phi_m(\bar{\theta}\gamma_5\theta)(\bar{\theta}\lambda^A)\right]_{\theta^4}\\ &\quad -\frac{1}{4}\left[\Phi_n^*\Phi_m(\bar{\theta}\gamma_5\theta)^2 D^A\right]_{\theta^4}\\ &= \left(\frac{i}{2}V_\mu^A\right)\Phi_n^*\left(\frac{1}{2}\bar{\theta}\gamma_5\gamma_\nu\theta\cdot\partial^\nu\phi_m\right)\left(\bar{\theta}\gamma_5\gamma^\mu\theta\right)\\ &\quad +\left(\frac{i}{2}V_\mu^A\right)(-\sqrt{2}\,\bar{\psi}_{nL}\theta)(-\sqrt{2}\,\bar{\theta}\psi_{mL})\left(\bar{\theta}\gamma_5\gamma^\mu\theta\right)\\ &\quad +\left(\frac{i}{2}V_\mu^A\right)\left(-\frac{1}{2}\bar{\theta}\gamma_5\gamma_\nu\theta\cdot\partial^\nu\phi_n^*\right)\phi_m\left(\bar{\theta}\gamma_5\gamma^\mu\theta\right)\\ &\quad -i\phi_n^*(-\sqrt{2}\,\bar{\theta}\psi_{mL})(\bar{\theta}\gamma_5\theta)(\bar{\theta}\lambda^A)\\ &\quad -i(-\sqrt{2}\,\bar{\psi}_{nL}\theta)\phi_m(\bar{\theta}\gamma_5\theta)(\bar{\theta}\lambda^A) - \frac{1}{4}\phi_n^*\phi_m(\bar{\theta}\gamma_5\theta)^2 D^A.\end{aligned} \tag{15.330}$$

We use now

$$\left(\frac{i}{2}V_\mu^A\right)\Phi_n^*\left(\frac{1}{2}\bar{\theta}\gamma_5\gamma_\nu\theta\cdot\partial^\nu\phi_m\right)\left(\bar{\theta}\gamma_5\gamma^\mu\theta\right) = -\frac{i}{4}(\bar{\theta}\gamma_5\theta)^2 V_\mu^A\phi_n^*\partial^\mu\phi_m \tag{15.331}$$

$$\left(\frac{i}{2}V_\mu^A\right)(-\sqrt{2}\,\bar{\psi}_{nL}\theta)(-\sqrt{2}\,\bar{\theta}\psi_{mL})\left(\bar{\theta}\gamma_5\gamma^\mu\theta\right) = \frac{i}{2}(\bar{\theta}\gamma_5\theta)^2 V_\mu^A(\bar{\psi}_{nL}\gamma^\mu\psi_{mL}) \tag{15.332}$$

$$\left(\frac{i}{2}V_\mu^A\right)\left(-\frac{1}{2}\bar{\theta}\gamma_5\gamma_\nu\theta\cdot\partial^\nu\phi_n^*\right)\phi_m\left(\bar{\theta}\gamma_5\gamma^\mu\theta\right) = \frac{i}{4}(\bar{\theta}\gamma_5\theta)^2 V_\mu^A\partial^\mu\phi_n^*\cdot\phi_m \tag{15.333}$$

$$-i\phi_n^*(-\sqrt{2}\,\bar{\theta}\psi_{mL})(\bar{\theta}\gamma_5\theta)(\bar{\theta}\lambda^A) = -i\frac{\sqrt{2}}{4}(\bar{\theta}\gamma_5\theta)^2\phi_n^*\bar{\lambda}^A\psi_{mL} \tag{15.334}$$

$$-i(-\sqrt{2}\,\bar{\psi}_{nL}\theta)\phi_m(\bar{\theta}\gamma_5\theta)(\bar{\theta}\lambda^A) = i\frac{\sqrt{2}}{4}(\bar{\theta}\gamma_5\theta)^2\phi_m\bar{\psi}_{nL}\lambda^A. \tag{15.335}$$

Thus, the second contribution takes the final form

$$\begin{aligned}[\Phi_n^*\Phi_m V^A]_{\theta^4} = -\frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\Big(&iV_\mu^A\phi_n^*\partial^\mu\phi_m - iV_\mu^A\bar{\psi}_{nL}\gamma^\mu\psi_{mL}\\ &-iV_\mu^A\partial^\mu\phi_n^*\cdot\phi_m + i\sqrt{2}\,\phi_n^*\left(\bar{\lambda}^A\psi_{mL}\right)\\ &-i\sqrt{2}\,\phi_m\bar{\psi}_{nL}\lambda^A + \phi_n^*\phi_m D^A\Big).\end{aligned} \tag{15.336}$$

The final result from equations (15.322), (15.329) and (15.336) is given by

$$
\begin{aligned}
[\Phi^+\Gamma\Phi]_{\theta^4} = -\frac{1}{4}(\bar\theta\gamma_5\theta)^2\Big(&\frac{1}{2}\partial^2\phi_n^*\cdot\phi_n + \frac{1}{2}\phi_n^*\cdot\partial^2\phi_n - \partial^\mu\phi_n^*\partial_\mu\phi_n \\
&-\bar\psi_{nL}\gamma^\mu\partial_\mu\psi_{nL} + \partial_\mu\bar\psi_{nL}\gamma^\mu\psi_{nL} \\
&+ 2\mathcal{F}_n^*\mathcal{F}_n - 2i(t_A)_{nm}V_\mu^A\phi_n^*\partial^\mu\phi_m + 2i(t_A)_{nm}V_\mu^A\partial^\mu\phi_n^*\cdot\phi_m \\
&-2i\sqrt{2}\,(t_A)_{nm}\phi_n^*(\bar\lambda^A\psi_{mL}) \\
&+ 2i\sqrt{2}\,(t_A)_{nm}\phi_m\big(\bar\psi_{nL}\lambda^A\big) + 2i(t_A)_{nm}V_\mu^A\big(\bar\psi_{nL}\gamma^\mu\psi_{mL}\big) \\
&-2(t_A)_{nm}\phi_n^*\phi_m D^A - 2\phi_n^*(t_At_B)_{nm}\phi_m V_\mu^A V^{\mu B}\Big).
\end{aligned}
\tag{15.337}
$$

The d-term of $\Phi^+\Gamma\Phi$ is the coefficient of $-(\bar\theta\gamma_5\theta)^2/4$ in the expansion of $\Phi^+\Gamma\Phi$ minus $\partial^2/2$ acting on $[\Phi^+\Gamma\Phi]_{\theta^0} = \phi_n^*\phi_n$. We use the result

$$
-2\partial^\mu\phi_n^*\partial_\mu\phi_n = \partial^2\phi_n^*\cdot\phi_n + \phi_n^*\cdot\partial^2\phi_n - \partial^2\big(\phi_n^*\phi_n\big). \tag{15.338}
$$

We find the d-term

$$
\begin{aligned}
(\Phi^+\Gamma\Phi)_D = &-2\partial^\mu\phi_n^*\partial_\mu\phi_n - \bar\psi_{nL}\gamma^\mu\partial_\mu\psi_{nL} + \partial_\mu\bar\psi_{nL}\gamma^\mu\psi_{nL} \\
&+ 2\mathcal{F}_n^*\mathcal{F}_n - 2i(t_A)_{nm}V_\mu^A\phi_n^*\partial^\mu\phi_m \\
&+ 2i(t_A)_{nm}V_\mu^A\partial^\mu\phi_n^*\cdot\phi_m + 2i(t_A)_{nm}V_\mu^A\big(\bar\psi_{nL}\gamma^\mu\psi_{mL}\big) \\
&-2i\sqrt{2}\,(t_A)_{nm}\phi_n^*\big(\bar\lambda^A\psi_{mL}\big) \\
&+ 2i\sqrt{2}\,(t_A)_{nm}\phi_m\big(\bar\psi_{nL}\lambda^A\big) - 2(t_A)_{nm}\phi_n^*\phi_m D^A \\
&-2\phi_n^*(t_At_B)_{nm}\phi_m V_\mu^A V^{\mu B}.
\end{aligned}
\tag{15.339}
$$

This term can be put in the equivalent but more elegant form

$$
\begin{aligned}
(\Phi^+\Gamma\Phi)_D = &-2(D_\mu\phi_n)^*\big(D^\mu\phi_n\big) - \bar\psi_{nL}\gamma^\mu D_\mu\psi_{nL} \\
&+ \bar D_\mu\bar\psi_{nL}\gamma^\mu\psi_{nL} + 2\mathcal{F}_n^*\mathcal{F}_n \\
&-2i\sqrt{2}\,(t_A)_{nm}\phi_n^*\big(\bar\lambda^A\psi_{mL}\big) + 2i\sqrt{2}\,(t_A)_{nm}\phi_m\big(\bar\psi_{nL}\lambda^A\big) \\
&-2(t_A)_{nm}\phi_n^*\phi_m D^A.
\end{aligned}
\tag{15.340}
$$

The covariant derivatives are defined by

$$
\begin{aligned}
D_\mu\phi_n &= \partial_\mu\phi_n - i(t_A)_{nm}V_\mu^A\phi_m \\
(D_\mu\phi_n)^* &= \partial_\mu\phi_n^* + i(t_A)_{mn}V_\mu^A\phi_m^*.
\end{aligned}
\tag{15.341}
$$

Also,

$$
\begin{aligned}
D_\mu\psi_{nL} &= \partial_\mu\psi_{nL} - i(t_A)_{nm}V_\mu^A\psi_{mL} \\
(D_\mu\psi_{nL})^+ &= \partial_\mu\psi_{nL}^+ + i(t_A)_{mn}V_\mu^A\psi_{mL}^+ \\
&\Leftrightarrow \bar D_\mu\bar\psi_{nL} \equiv \partial_\mu\bar\psi_{nL} + i(t_A)_{mn}V_\mu^A\bar\psi_{mL}.
\end{aligned}
\tag{15.342}
$$

15.6 Exercises

Exercise 1: Show that P^2 and W^2 are Casimirs by computing their commutators with all generators. What are their eigenvalues for massive and massless particles?

Exercise 2: Verify that if a and b are two 4-vectors such that $a \cdot b = a \cdot a = b \cdot b = 0$ then $a^\mu = hb^\mu$.

Exercise 3:

- Check explicitly the summation convention

$$\begin{aligned}\psi\chi &= \psi^\alpha\chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta \\ &= -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi.\end{aligned} \tag{15.343}$$

- Check explicitly the Fierz identities (15.33)–(15.41).

Exercise 4:

- Verify that the supersymmetry generators commute with the generators of translations, viz

$$\left[P_\mu, Q_\alpha^{\ A}\right] = \left[P_\mu, \bar{Q}_A^{\dot{\alpha}}\right] = 0. \tag{15.344}$$

- Show that the central charges X^{AB} commute with $\bar{Q}_{\dot{\alpha}}^{\ A}$ and $\bar{Q}_{\dot{\beta}}^{\ B}$ by using the Jacobi identity.
- Verify explicitly that P^2 and C^2 are Casimirs of the supersymmetry algebra.

Exercise 5:

- Show that

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu. \tag{15.345}$$

- Show that the condition for a chiral superfield which is given by $\bar{D}_{\dot{\alpha}}\Phi = 0$ is equivalent to

$$\begin{aligned}&\bar{\chi}_{\dot{\alpha}} = 0,\ \ n = 0,\ \ v_\mu = i\partial_\mu f,\ \ \psi = 0, \\ &\bar{\lambda}_{\dot{\alpha}} = -\frac{i}{2}\partial_\mu\phi^\beta(\sigma^\mu)_{\beta\dot{\alpha}},\ \ d = -\frac{1}{4}\partial_\mu\partial^\mu f.\end{aligned} \tag{15.346}$$

- Show that the Wess–Zumino gauge given by equation (15.128) is not preserved by supersymmetry transformations.
- Show that chiral spinor superfield satisfies $\bar{D}_{\dot{\alpha}}W_\beta = 0$.
- Show that chiral and anti-chiral spinor superfields satisfy

$$\bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} = D^\alpha W_\alpha. \tag{15.347}$$

Exercise 6: Verify explicitly that by using the infinitesimal non-abelian gauge transformation (15.159) we can bring the vector superfield V into the Wess–Zumino form, viz

$$V_{WZ} = (\theta\sigma^{\mu}\bar{\theta})v_{\mu}(x) + (\theta\theta)(\bar{\theta}\bar{\lambda}(x)) + (\bar{\theta}\bar{\theta})(\theta\lambda(x)) + (\theta\theta)(\bar{\theta}\bar{\theta})d(x). \tag{15.348}$$

Exercise 7: Show that the Wess–Zumino gauge does not fix the non-abelian gauge freedom generated, as in the abelian case, by the scalar component of $\Phi - \Phi^{+}$.

Exercise 8: Show that the the component field F of the chiral superfield Φ is auxiliary and, therefore, it can be set equal to its classical value in the Wess–Zumino model. Verify then that the bosonic part of the Wess–Zumino model will only depend on the component field A and it reads

$$\begin{aligned}(S_{WZ})_{\text{bosonic}} &= \int d^4x\Big[\partial^{\mu}A^{+}\partial_{\mu}A - V_F(A^{+}, A)\Big],\\ V_F(A^{+}, A) &= F^{+}F = (mA + gA^2)^{+}(mA + gA^2).\end{aligned} \tag{15.349}$$

Exercise 9: Show that the total potential of the field components A and A^{+} in $N = 1$ super Yang–Mills action with matter given by a chiral superfield is given by

$$V(A^{+}, A) = V_F(A^{+}, A) + V_D(A^{+}, A) = F^{+}F + \frac{2}{g^2}d^2. \tag{15.350}$$

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Chapter 16

The AdS/CFT correspondence

The goal in this chapter is to provide a pedagogical presentation of the celebrated AdS/CFT correspondence adhering mostly to the language of quantum field theory (QFT). This is certainly possible, and perhaps even natural, if we recall that in this correspondence we are positing that quantum gravity in an anti-de Sitter spacetime AdS_{d+1} is nothing but a conformal field theory (CFT_d) at the boundary of AdS spacetime. Some of the reviews of the AdS/CFT correspondence which emphasize the QFT aspects and language include Kaplan [1], Zaffaroni [2] and Ramallo [3].

This chapter contains, therefore, a thorough introduction to conformal symmetries, anti-de Sitter spacetimes, conformal field theories and the AdS/CFT correspondence. The primary goal, however, in this chapter is the holographic entanglement entropy. In other words, how spacetime geometry as encoded in Einstein's equations in the bulk of AdS spacetime can emerge from the quantum entanglement entropy of the CFT living on the boundary of AdS.

A sample of the original literature for the holographic entanglement entropy is [4–7]. However, a very good concise and pedagogical review of the formalism relating spacetime geometry to quantum entanglement due to Van Raamsdonk and collaborators is found in [6] and [8].

16.1 Conformal symmetry

16.1.1 The conformal groups $SO(p+1, q+1)$

We assume a spacetime with signature $(-1, \ldots, +1, \ldots)$ with q minus signs and p plus signs where $q + p = d$ is the dimension of spacetime.

We have then $q = 0$ and $p = d$ for Euclidean, and $q = 1$ and $p = d - 1$ for Lorentzian.

We start with the group of diffeomorphisms, i.e. the group of general coordinate transformations. The conformal group is a subgroup of the diffeomorphism group which preserves the conformal flatness of the metric. The corresponding transformations preserve the angles but change the lengths.

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A manifold is called conformally flat if the metric takes the following form

$$ds^2 = \exp(\omega(x))dx^\mu dx_\mu. \tag{16.1}$$

Conformal transformations are given by

$$x^\mu \longrightarrow x'^\mu(x), \;\; dx'^\mu dx'_\mu = \Omega^2(x) dx_\mu dx^\mu. \tag{16.2}$$

Thus, if the manifold is conformally flat, it will remain so under conformal transformations. These conformal transformations are a generalization of the scale transformation

$$x^\mu \longrightarrow x'^\mu = \alpha x^\mu, \;\; dx'^\mu dx'_\mu = \alpha^2 dx_\mu dx^\mu. \tag{16.3}$$

We consider infinitesimal conformal transformations $x'_\mu = x_\mu + v_\mu$, $\Omega = 1 + \omega/2$. We get from equation (16.2) the conditions

$$\omega = \frac{2}{d}\partial_\mu v^\mu \tag{16.4}$$

$$\partial_\nu v^\mu + \partial^\mu v_\nu - \frac{2}{d}\partial_\rho v^\rho \eta_\nu^{\ \mu} = 0. \tag{16.5}$$

In $d = 2$ this equation admits an infinite number of solutions and thus the conformal group in two dimensions is infinite-dimensional.

In $d \neq 2$ there is a finite number of solutions given precisely by

$$v_\mu = \delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu bx. \tag{16.6}$$

The conformal group contains therefore:

1. Lorentz transformations $\Lambda \in SO(p, q)$, i.e. rotations and boosts, with parameters ω_ν^μ. The finite transformations are $x \longrightarrow x' = \Lambda x$. There are $d(d-1)/2$ generators denoted by $M_{\mu\nu}$. They satisfy

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}). \tag{16.7}$$

2. Translations with parameters a^μ. There are d generators denoted by P^μ.
3. The scale (dilatation) transformation $x^\mu \longrightarrow \lambda x^\mu$ where λ is a constant. The generator of dilatation is denoted by D. A field theory which is invariant under scale transformations will also, under mild conditions, be invariant under all conformal transformations.
4. The special conformal transformations with parameters b^μ given by

$$\delta x^\mu = b^\mu x^2 - 2x^\mu bx. \tag{16.8}$$

A special conformal transformation is obtained from the composition of an inversion, a translation by a vector b^μ and another inversion. The inversion is an element in the conformal group which is not connected to the identity given by

$$x^\mu \longrightarrow \frac{x^\mu}{x^2}. \tag{16.9}$$

There are d generators of special conformal transformations denoted by K^μ. The finite form of special conformal transformations is

$$x^\mu \longrightarrow x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2bx + b^2x^2} \Rightarrow x \longrightarrow x'^2 = \frac{x^2}{1 + 2bx + b^2x^2}. \tag{16.10}$$

Altogether we have $(d + 1)(d + 2)/2$ conformal transformations. This is exactly the number of generators of the rotation group $SO(d + 2)$. However, the conformal group must be a non-compact group. It is, therefore, given by $SO(p + 1, q + 1)$. In Lorentzian signature the conformal group is $SO(d, 2)$, whereas in Euclidean signature the conformal group is $SO(d + 1, 1)$. For $d > 2$ the conformal group is actually given by $O(p + 1, q + 1)$ and consists of two disconnected components since the inversion element is not infinitesimally generated.

16.1.2 Differential representation of the conformal algebra

The conformal group in Lorentzian signature $\eta = (-1, +1, \ldots, +1)$ is $O(d, 2)$ with generators P_μ, K_μ, $M_{\mu\nu}$ and D which satisfy the algebra:

1. $M_{\mu\nu}$ generate the algebra of the Lorentz group $SO(d - 1, 1)$, viz

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma}M_{\rho\nu} - \eta_{\mu\rho}M_{\sigma\nu} + \eta_{\sigma\nu}M_{\mu\rho} - \eta_{\rho\nu}M_{\mu\sigma}). \tag{16.11}$$

2. D is a scalar under the Lorentz group, viz

$$[M_{\mu\nu}, D] = 0. \tag{16.12}$$

3. P_μ, K_μ are vectors under the Lorentz group, viz

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \ [M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu). \tag{16.13}$$

4. D is the Hamiltonian and P_μ, K_μ are the raising and lowering operators since

$$[D, P_\mu] = P_\mu, \ [D, K_\mu] = -K_\mu. \tag{16.14}$$

5. P_μ, K_μ close on a dilatation and a Lorentz transformation, viz

$$[P_\mu, K_\nu] = 2(\eta_{\mu\nu}D + iM_{\mu\nu}). \tag{16.15}$$

We define the new generators $\mathcal{M}_{AB}$ by the relations

$$\mathcal{M}_{AB} = M_{\mu\nu},\ A = \mu = 0,\ \ldots,\ d-1,\ B = \nu = 0,\ \ldots,\ d-1 \tag{16.16}$$

$$\mathcal{M}_{\mu d} = -\mathcal{M}_{d\mu} = i\frac{K_\mu - P_\mu}{2},\ \ \mu = 0,\ \ldots,\ d-1 \tag{16.17}$$

$$\mathcal{M}_{\mu d+1} = -\mathcal{M}_{d+1\mu} = -i\frac{K_\mu + P_\mu}{2},\ \ \mu = 0,\ \ldots,\ d-1 \tag{16.18}$$

$$\mathcal{M}_{dd+1} = -\mathcal{M}_{d+1d} = iD. \tag{16.19}$$

The algebra becomes

$$[\mathcal{M}_{AB},\ \mathcal{M}_{CD}] = i(\eta_{AD}\mathcal{M}_{CB} - \eta_{AC}\mathcal{M}_{DB} + \eta_{DB}\mathcal{M}_{AC} - \eta_{CB}\mathcal{M}_{AD}). \tag{16.20}$$

The metric η_{AB} is a flat $(d+2)$-dimensional with signature $-1+1+1\cdots-1$. By going from Lorentzian to Euclidean spacetime the conformal group $SO(d, 2)$ becomes the group $SO(d+1, 1)$ while the Lorentz group $SO(d-1, 1)$ becomes $SO(d)$ and the metric η_{AB} becomes of signature $-1+1+1\cdots+1$. The Poincaré and dilatation generators together form a subgroup of the conformal group.

The mass operator $P_\mu P^\mu$ is a Casimir of the Poincaré group but it is not a Casimir of the conformal group. States in a conformal field theory are, therefore, not classified by their mass, as in the Poincaré group, but they are instead classified by their scaling dimension, i.e. by the eigenvalue of the dilatation operator

$$D|\Delta\rangle = i\Delta|\Delta\rangle. \tag{16.21}$$

The representation of the dilatation operator (which is a hermitian operator) on classical fields is not unitary and hence the factor i. Indeed, dilatations are not bounded transformations and as a consequence a finite-dimensional representation of a non-compact Lie algebra must be necessarily non-unitary. This happens also, for example, with boosts in the Lorentz group [2, 9, 10].

The scaling dimension Δ of a field Φ is defined by the action of dilatations on the field given by the equation

$$\Phi(\lambda x) = \lambda^{-\Delta}\Phi(x). \tag{16.22}$$

If we take, for example, a free scalar field theory in d dimensions given by the action

$$\begin{aligned} S &= \int d^d x \partial_\mu \Phi(x)\partial^\mu\Phi(x) \\ &= \lambda^{-d}\lambda^{2\Delta}\int d^d(\lambda x)\partial_\mu\Phi(\lambda x)\partial^\mu\Phi(\lambda x) \\ &= \lambda^{-d+2\Delta+2}\int d^d x'\partial'_\mu\Phi(x')\partial^{\mu'}\Phi(x'). \end{aligned} \tag{16.23}$$

Hence, this invariant is under dilatation if and only if

$$-d + 2\Delta + 2 = 0. \tag{16.24}$$

The differential representation of the dilatation operator on scalar fields with scaling dimension Δ is given by

$$D\Phi = -i\big(x^\mu \partial_\mu + \Delta\big)\Phi. \tag{16.25}$$

This is the analog of the differential representation of the momentum and Lorentz generators given by

$$P_\mu \Phi = -i\partial_\mu \Phi \tag{16.26}$$

$$M_{\mu\nu}\Phi = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi + S_{\mu\nu}\Phi. \tag{16.27}$$

The special conformal generator is represented differentially by

$$K_\mu \Phi = \big(-2i\Delta x_\mu - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu\big)\Phi. \tag{16.28}$$

By using equation (16.25) we can rewrite the quantum analog of the scaling transformation $\Phi(x) \longrightarrow \lambda^\Delta \phi(\lambda x)$ as

$$[D, \Phi] = -i\big(x^\mu \partial_\mu + \Delta\big)\Phi. \tag{16.29}$$

Under a coordinate transformation $x \longrightarrow x'$ the metric tensor transforms as $g_{\alpha\beta}(x) \longrightarrow g'_{\alpha\beta}(x') = (\partial x^\mu / \partial x'^\alpha)(\partial x^\nu / \partial x'^\beta) g_{\mu\nu}(x)$. The conformal group is the subgroup which satisfies $g_{\alpha\beta}(x) \longrightarrow g'_{\alpha\beta}(x') = \Omega^2(x) g_{\mu\nu}(x)$. From the invariance of the volume element we can see that the Jacobian of the conformal transformation $x \longrightarrow x'$, $dx'^\mu dx'_\mu = \Omega^2(x) dx_\mu dx^\mu$ is given by

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{\sqrt{\det g'_{\mu\nu}}} = \Omega^{-d}. \tag{16.30}$$

The action of the generators of the conformal group on the scalar field Φ with scaling dimension Δ given by the transformations (16.25), (16.26), (16.27), (16.28) translates into the transformation law

$$\Phi(x) \longrightarrow \Phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \Phi(x). \tag{16.31}$$

The scalar field Φ is called quasi-primary operator. The covariance of the theory on conformal transformations is then given by the behavior of the correlation functions under conformal transformations given by

$$\begin{aligned} &\langle \Phi_1(x_1) \dots \Phi_N(x_N) \rangle \longrightarrow \langle \Phi'(x'_1) \dots \Phi'(x'_N) \rangle \\ &= \left| \frac{\partial x'}{\partial x} \right|^{-\Delta_1/d}_{x=x_1} \dots \left| \frac{\partial x'}{\partial x} \right|^{-\Delta_N/d}_{x=x_N} \langle \Phi_1(x_1) \dots \Phi_N(x_N) \rangle. \end{aligned} \tag{16.32}$$

For a scalar field we have obviously $\Phi' = \Phi$.

16.1.3 Constraints of conformal symmetry

Conformal invariance fully constrains the two- and three-point functions of the conformal field theory. For the two-point function of two quasi-primary operators ϕ_1 and ϕ_2 we have

$$\begin{aligned}\langle\Phi_1(x_1)\Phi_2(x_2)\rangle \longrightarrow \langle\Phi(x_1')\Phi(x_2')\rangle = &\left|\frac{\partial x'}{\partial x}\right|^{-\Delta_1/d}_{x=x_1} \\ &\times \left|\frac{\partial x'}{\partial x}\right|^{-\Delta_2/d}_{x=x_2} \langle\Phi_1(x_1)\Phi_2(x_2)\rangle.\end{aligned} \tag{16.33}$$

In other words, we have

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_1/d}_{x=x_1} \left|\frac{\partial x'}{\partial x}\right|^{\Delta_2/d}_{x=x_2} \langle\Phi(x_1')\Phi(x_2')\rangle. \tag{16.34}$$

Invariance under translations and rotations (for which the Jacobian is equal to 1) yields a dependence only on the combination $r_{12} = |x_1 - x_2|$ (the difference is due to translations and the modulus is due to rotations). Invariance under the scaling transformations $x \longrightarrow x' = \lambda x$ leads to

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle\Phi(x_1')\Phi(x_2')\rangle. \tag{16.35}$$

This gives the behavior

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{C_{12}}{r_{12}^{\Delta_1+\Delta_2}}. \tag{16.36}$$

Under special conformal transformation we have

$$\begin{aligned}x^\mu \longrightarrow x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2bx + b^2x^2} &\Rightarrow r'^2_{12} \\ &= \frac{r_{12}^2}{\left(1 + 2bx_1 + b^2x_1^{\,2}\right)\left(1 + 2bx_2 + b^2x_2^2\right)}\end{aligned} \tag{16.37}$$

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 + 2bx + b^2x^2)^d}. \tag{16.38}$$

We can then easily verify that we must have $\Delta_1 = \Delta_2 = \Delta$. Hence, the two-point function of a conformal field theory must be constrained such that

$$\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{C_{12}}{r_{12}^{2\Delta}}. \tag{16.39}$$

However, if $\Delta_1 \neq \Delta_2$ then we must have $\langle\Phi_1(x_1)\Phi_2(x_2)\rangle = 0$.

Similarly, the three-point function of a conformal field theory must be constrained by the invariance under translations, rotations, scalings and special conformal transformations such that

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{13}^{\Delta_1+\Delta_3-\Delta_2} r_{23}^{\Delta_2+\Delta_3-\Delta_1}}. \tag{16.40}$$

16.1.4 Conformal algebra in two dimensions

We consider a free boson Φ in two dimensions with the Euclidean action (with $z = \sigma^1 + i\sigma^2$)

$$S = \frac{1}{4\pi}\int d^2\sigma \partial_\mu \Phi \partial^\mu \Phi = \frac{1}{2\pi}\int dz d\bar{z} \partial\Phi\bar{\partial}\Phi. \tag{16.41}$$

The symmetries of this action are given by the conformal mappings

$$z \longrightarrow f(z),\ \ \bar{z} \longrightarrow \bar{z} = \bar{f}(\bar{z}). \tag{16.42}$$

These are angle-preserving transformations when f and its inverse are both holomorphic, i.e. f is biholomorphic. For example, $z \longrightarrow z + a$ is a translation, $z \longrightarrow \zeta z$ where $|\zeta| = 1$ is a rotation, and $z \longrightarrow \zeta z$ where ζ is real and not equal to 1 is a scale transformation, also called a dilatation.

We work with the complex coordinates

$$z = \sigma^1 + i\sigma^2 \longrightarrow z = e^{2(\sigma^2 - i\sigma^1)},\ \ \bar{z} = \sigma^1 - i\sigma^2 \longrightarrow \bar{z} = e^{2(\sigma^2 + i\sigma^1)}. \tag{16.43}$$

The factor of 2 in the exponent is included for consistency with the periodicity of the closed string given by $\sigma^1 \longrightarrow \sigma^1 + \pi$. The worldsheet for a closed string is a cylinder which is topologically an $\mathbf{R}^2$. It can also be regarded as a Riemann surface, i.e. as a deformation of the complex plane. The Euclidean time σ^2 on the worldsheet corresponds to the radial distance $r = \exp(2\sigma^2)$ on the complex plane, with the infinite past $\sigma^2 = -\infty$ at $r = 0$, and the infinite future $\sigma^2 = +\infty$ is a circle at $r = \infty$.

The generators of conformal mappings are given by the infinitesimal transformations

$$z \longrightarrow z' = z - \epsilon_n z^{n+1},\ \ \bar{z} \longrightarrow \bar{z}' = \bar{z} - \bar{\epsilon}_n \bar{z}^{n+1},\ \ n \in Z. \tag{16.44}$$

The generators are given by

$$l_n = -z^{n+1}\partial,\ \ \bar{l}_n = -\bar{z}^{n+1}\bar{\partial},\ \ n \in Z. \tag{16.45}$$

The generators with $n < -1$ are defined on the punctured complex plane, whereas the generators with $n > 1$ are defined on the complex plane with the point at infinity removed. The generators l_{-1}, l_0, l_1 are defined on the whole Riemann sphere, i.e. the complex plane + the point at infinity. They satisfy the classical Virasoro algebra

$$[l_m, l_n] = (m-n)l_{m+n},\ \ [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n},\ \ [l_m, \bar{l}_n] = 0. \tag{16.46}$$

It is also easily seen that the Virasoro algebra is the same as the algebra of infinitesimal diffeomorphisms of the circle $\mathbf{S}^1$.

The group $SO(3, 1)$ is called the restricted conformal group. The full conformal group in two dimensions is infinite-dimensional. The finite-dimensional subgroup $SO(3, 1)$ is generated by $l_0, l_{\pm 1}, \bar{l}_0, \bar{l}_{\pm 1}$. These are given explicitly by

$$\begin{aligned} l_{-1} &: \; z \longrightarrow z - \epsilon \\ l_0 &: \; z \longrightarrow z - \epsilon z \\ l_1 &: \; z \longrightarrow z - \epsilon z^2. \end{aligned} \tag{16.47}$$

And similarly for $\bar{l}_{-1}, \bar{l}_0, \bar{l}_1$. We have the following interpretation:

$$\begin{aligned} &l_{-1}, \; \bar{l}_{-1} \text{ translations} \\ &l_0 - \bar{l}_0 \text{ rotations} \\ &l_0 + \bar{l}_0 \text{ scalings} \\ &l_1, \; \bar{l}_1 \text{ special conformal transformations.} \end{aligned} \tag{16.48}$$

The finite or global form of these transformations are:

$$\begin{aligned} l_{-1} &: \; z \longrightarrow z + \alpha \\ l_0 &: \; z \longrightarrow \lambda z \\ l_1 &: \; z \longrightarrow \frac{z}{1 - \beta z}. \end{aligned} \tag{16.49}$$

By combining these transformations we obtain

$$z \longrightarrow \frac{az + b}{cz + d}, \;\; ad - bc = 1. \tag{16.50}$$

For infinitesimal z we obtain

$$z \longrightarrow \frac{b}{d} + \frac{1}{d^2} z - \frac{c}{d^3} z^2 \tag{16.51}$$

Only three parameters are independent, which is as it should be. We obtain a linear combination of the transformations (16.47).

The group given by the relation (16.50) is $SO(3, 1) = SL(2, C)/\mathbf{Z}_2$. The division by $\mathbf{Z}_2$ is to take into account the property that the above transformations remain unchanged if $a, b, c, d \longrightarrow -a, -b, -c, -d$. The Lorentzian analog is the group $SO(2, 2) = SL(2, R) \times SL(2, R)$ where one factor of $SL(2, R)$ stands for left-movers and the other factor stands for right-movers.

16.2 The AdS spacetime

For an extensive discussion of anti-de Sitter spacetimes and their uses see [11, 12].

16.2.1 Maximally symmetric spaces

The maximally symmetric manifolds of dimension d are those spaces with the maximum number $d(d+1)/2$ of Killing vector fields generating isometries, i.e. symmetries, consisting of d translations and $d(d-1)/2$ rotations/boosts. A space is a maximally symmetric manifold if and only if its Riemann curvature tensor is given by

$$R_{\mu\nu\alpha\beta} = \frac{R}{d(d-1)}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}), \tag{16.52}$$

where g is of course the metric tensor. This means that in a maximally symmetric space the curvature tensor looks the same everywhere and in every direction and thus it is fully specified locally by the Ricci scalar curvature R. Hence, there are only three possible maximally symmetric spaces locally specified by the sign of the Ricci scalar R (since the scale of R specifies only the size of the space) which are given by:

1. **Positive curvature:** The spheres $\mathbf{S}^d$ in Euclidean signature and de Sitter spacetimes dS_d in Lorentzian signature. The de Sitter spacetime plays a crucial role in cosmology in the early Universe during inflation (when the cosmological constant was very large and the expansion was exponential) as well as in the final state of the Universe which is seen to be dominated by a very small cosmological constant and an accelerated expansion.
2. **Zero curvature:** The Euclidean spaces $\mathbf{R}^d$ in Euclidean signature and Minkowski spacetimes $\mathbf{M}^d$ in Lorentzian signature.
3. **Negative curvature:** The hyperboloids $\mathbf{H}^d$ in Euclidean signature and anti-de Sitter spacetimes AdS_d in Lorentzian signature. The anti-de Sitter spacetime is crucial for the holographic principle and the AdS/CFT correspondence. This is the primary point of interest in this chapter.

For more detail see chapters 3 and 8 in [13].

16.2.2 Global and Poincaré coordinates

Global coordinates

We are interested mostly in AdS_5. Thus, we start from a six-dimensional flat spacetime $\mathbf{R}^{2,4}$ with metric

$$ds_5^2 = -dX_0^2 - dX_5^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2. \tag{16.53}$$

The isometry group of this metric is obviously $SO(2,4)$, which is precisely the conformal group in four dimensions. For AdS_{d+1} we need to start from $\mathbf{R}^{2,d}$ with isometry group $SO(2,d)$. We Lobachevski-like embed in this Minkowski spacetime $\mathbf{R}^{2,4}$ the following hyperboloid

$$-X_0^2 - X_5^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -L^2. \tag{16.54}$$

This hyperboloid is obviously five-dimensional. Now we can induce global coordinates $(\tau, \rho, \hat{x}_i)$ on this hyperboloid where $\hat{x}_i$ define an $\mathbf{S}^3$, i.e. $\sum_{i=1}^4 \hat{x}_i^2 = 1$, by the relations

$$\begin{aligned} X_0 &= L \cos\tau \cosh\rho \\ X_5 &= L \sin\tau \cosh\rho \\ X_i &= L \sinh\rho \hat{x}_i. \end{aligned} \tag{16.55}$$

The metric becomes (with $d\Omega_3$ is the solid angle on $\mathbf{S}^3$)

$$ds_5^2 = L^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3). \tag{16.56}$$

The coordinate ρ plays the role of a radius in AdS_5 since $\rho \in [0, +\infty[$, whereas the coordinate τ is time-like with $\tau \in [0, 2\pi]$. These coordinates cover the whole of AdS_5 which is the reason why they are called global. On the other hand, τ is periodic which signals the existence of closed time-like curves. However, this property is not intrinsic to the AdS_5 space but it is an artifact of this system of coordinates. Instead of the above space we will then take its universal cover space, in which we allow τ to run over the unrestricted range $]-\infty, +\infty[$, as the definition of anti-de Sitter spacetime AdS_5. The isometry group becomes a cover of $SO(2, 4)$.

The anti-de Sitter spacetime, as opposed to de Sitter spacetime, is not a globally hyperbolic spacetime. This means that AdS does not admit a well-defined time evolution starting from suitable initial data on a space-like (Cauchy) hypersurface. Indeed, by specifying the initial data on a space-like hypersurface in AdS (together with the knowledge of the equations of motion) is not sufficient to determine the future evolution uniquely and deterministically. This is due to the existence of a boundary at time-like infinity (see below) and thus information can flow in from infinity. See [14] for a systematic discussion of this issue.

The other difference between dS and AdS is with regard to topology. AdS_5 spacetime is topologically equivalent to $\mathbf{R}^4 \times \mathbf{S}^1$, whereas dS_5 spacetime is topologically equivalent to $\mathbf{R} \times \mathbf{S}^4$. This can be fleshed out using Penrose diagrams. See [12, 13] for an explicit discussion.

Poincaré coordinates

The next set of coordinates is much more important for quantum field theory. We introduce the following so-called Poincaré coordinates given by a Minkowski 4-vector x_μ and a radial coordinate u defined by the relations

$$\begin{aligned} X_0 &= \frac{1}{2u}(1 + u^2(L^2 + \vec{x}^2 - t^2)) \\ X_i &= Lux_i \\ X_4 &= \frac{1}{2u}(1 - u^2(L^2 - \vec{x}^2 + t^2)) \\ X_5 &= Lut. \end{aligned} \tag{16.57}$$

We get the metric

$$ds_5^2 = L^2\left(\frac{du^2}{u^2} + u^2\left(dx_i^2 - dt^2\right)\right) = L^2\left(\frac{du^2}{u^2} + u^2\left(dx_\mu dx^\mu\right)\right). \tag{16.58}$$

Thus, for each fixed value of the radial coordinate u we have a four-dimensional ordinary Minkowski spacetime, i.e. the ordinary spacetime is foliated over the radial coordinate u which takes the value from 0 to infinity. However, because of the overall conformal factor R^2u^2 multiplying the Minkowski metric all distances in the four-dimensional theory on the Minkowski slices or branes are rescaled by a factor of Ru.

The point $u = \infty$ is a conformal boundary since it is the conformally equivalent metric ds_5^2/u^2 which is seen to have a boundary $\mathbf{M}^4$ at $u = \infty$. However, the point $u = 0$ is a horizon since the 00 component of the metric vanishes there and hence the Killing vector $\partial/\partial t$ becomes of zero norm at this point. Furthermore, the metric can be extended beyond the horizon $u = 0$ which is only a coordinate singularity and the Poincaré coordinates cover only half of the hyperboloid.

Another form of the metric can be obtained by the substitution $u = 1/z$ and thus the conformal boundary becomes located at $z = 0$, whereas the horizon becomes located at $z = \infty$. The metric takes the form

$$ds_5^2 = \frac{L^2}{z^2}\left(dz^2 + dx_\mu dx^\mu\right). \tag{16.59}$$

Another system of coordinates can be obtained by the substitution $\sinh\rho = 2r/(1 - r^2)$ in the system of coordinates, i.e. r runs from 0 to 1. We obtain the metric

$$ds_5^2 = \frac{L^2}{(1 - r^2)^2}(-(1 + r^2)^2 d\tau^2 + 4dr^2 + 4r^2 d\Omega_3). \tag{16.60}$$

The metric (16.59) should be thought of as an approximation of the metric (16.60) in the vicinity of a point on the boundary located now at $r = 1$. In this approximation the 3-sphere is replaced with the flat coordinates x_i, t is replaced with τ, and the radial coordinate z is replaced with $z = 1 - r$. In this system of coordinates $r = 0$ is the center of anti-de Sitter and there is no horizon since the metric (16.60) is geodesically complete. However, the metric (16.59) is geodesically incomplete because we can reach $z = \infty$ along a time-like geodesic in a finite proper time, i.e. $z = \infty$ is a horizon.

Furthermore, we can check that we can travel from the center $r = 0$ of anti-de Sitter to the boundary $r = 1$ (which is an infinite proper distance) and back along a null curve satisfying $(1 + r^2)d\tau = 2dr$ in a finite proper time given by $\tau = \pi$. This means that anti-de Sitter space is causally finite and it behaves as a finite box of size L. See [15] for more discussion on the relation between the metrics (16.59) and (16.60).

Generalization

We generalize to anti-de Sitter in $d+1$ dimensions, i.e. AdS_{d+1}, with isometry group $SO(2, d)$ which is precisely the conformal group in d dimensions. AdS_{d+1} spacetime is topologically equivalent to $\mathbf{R}^d \times \mathbf{S}^1$ (to be contrasted with the topology of dS_{d+1} spacetime given by $\mathbf{R} \times \mathbf{S}^d$). We embed AdS_{d+1} in the Minkowski spacetime $\mathbf{R}^{2,d}$ by

$$-X_0^2 - X_{d+1}^2 + X_1^2 + X_2^2 + X_3^2 + \cdots + X_d^2 = -L^2. \tag{16.61}$$

Now we can induce global coordinates $(\tau, \rho, \hat{x}_i)$, or the slightly different ones $(\tau, r, \hat{x}_i)$, on this hyperboloid by the relations

$$\begin{aligned} X_0 &= L\cos\tau\cosh\rho = L\frac{\cos\tau}{\cos r} \\ X_{d+1} &= L\sin\tau\cosh\rho = L\frac{\sin\tau}{\cos r} \\ X_i &= L\sinh\rho\hat{x}_i = L\tan r\hat{x}_i. \end{aligned} \tag{16.62}$$

The range is $\rho \in [0, +\infty[$ or correspondingly $r \in [0, \pi/2[$, and $\tau \in [0, 2\pi] \longrightarrow \tau \in [-\infty, \infty]$ as we unwrap to the universal cover, and $\hat{x}_i$ defines a $(d-1)$-dimensional sphere $\mathbf{S}^{d-1}$, i.e. $\sum_{i=1}^{4}\hat{x}_i^2 = 1$. The metric reads

$$\begin{aligned} ds_{d+1}^2 &= L^2\Big(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}\Big) \\ &= \frac{L^2}{\cos^2 r}(-d\tau^2 + dr^2 + \sin^2 r d\Omega_{d-1}). \end{aligned} \tag{16.63}$$

Thus, AdS_{d+1} can be viewed as a cylinder with bases at $\tau = -\infty$ and $\tau = +\infty$, a center at $\rho = r = 0$, whereas the spatial infinity $\rho = \infty$ is at $r = \pi/2$, while going around the cylinder is given by the angular variables Ω_{d-1}. The anti-de Sitter space is, therefore, not compact both in time and space yet it behaves as a box, as we will discuss.

The symmetry group of AdS_{d+1} is given by the conformal group $SO(2, d)$. We have $SO(d)$ rotations among the X_i, $SO(2)$ rotation in the plane X_0X_{d+1}, d boosts in the planes X_0X_i and d boosts in the planes $X_{d+1}X_i$. In total we have $d(d-1)/2 + 1 + 2d = (d+1)(d+2)/2$ generators. These generators can be represented in terms of the coordinates X_A as

$$L_B^{\ A} = X^A\frac{\partial}{\partial X^B} - X_B\frac{\partial}{\partial X_A}. \tag{16.64}$$

We will also need later

$$L^{AB} = X^A\frac{\partial}{\partial X_B} - X^B\frac{\partial}{\partial X_A}, \quad L_{AB} = X_A\frac{\partial}{\partial X^B} - X_B\frac{\partial}{\partial X^A}. \tag{16.65}$$

For example, the rotation in the time-like plane X_0X_{d+1} is given by (recall the metric signature $-++\cdots-$)

$$L^{0}_{d+1} = X_0 \frac{\partial}{\partial X_{d+1}} - X_{d+1} \frac{\partial}{\partial X_0}. \tag{16.66}$$

On the other hand, we have

$$\frac{\partial}{\partial t} = \frac{\partial X_0}{\partial t}\frac{\partial}{\partial X_0} + \frac{\partial X_{d+1}}{\partial t}\frac{\partial}{\partial X_{d+1}} = -X_{d+1}\frac{\partial}{\partial X_0} + X_0 \frac{\partial}{\partial X_{d+1}}. \tag{16.67}$$

Hence, the Hamiltonian in anti-de Sitter spacetime is given by

$$\text{Hamiltonian} \equiv -i\frac{\partial}{\partial t} = -iL^{0}_{d+1}. \tag{16.68}$$

16.2.3 Euclidean Poincaré patch and RG equation

Poincaré coordinates revisited

Let us start with the most general metric in $(d+1)$-dimension which enjoys Poincaré invariance in d dimensions given by

$$ds^2_{d+1} = \Omega^2(z)\big(dz^2 + dx_\mu dx^\mu\big) \;=\; \Omega^2(z)(dz^2 + d\vec{x}^2 - dt^2). \tag{16.69}$$

As we will see, the extra coordinate z corresponds to an energy scale of a conformally invariant theory. Hence, the above metric must be invariant under $z \longrightarrow z$, $\vec{x} \longrightarrow \lambda\vec{x}$ and $t \longrightarrow \lambda t$. This leads to the requirement

$$\begin{aligned} ds^2_{d+1} &= \Omega^2(z)(dz^2 + d\vec{x}^2 - dt^2) \longrightarrow \lambda^2\Omega^2(\lambda z)(dz^2 + d\vec{x}^2 - dt^2) \\ &= \Omega^2(z)(dz^2 + d\vec{x}^2 - dt^2). \end{aligned} \tag{16.70}$$

The function Ω must then satisfy

$$\Omega(z) = \frac{L}{z}. \tag{16.71}$$

We get then

$$ds^2_{d+1} = \frac{L^2}{z^2}\big(dz^2 + dx_\mu dx^\mu\big) = \frac{L^2}{z^2}(dz^2 + d\vec{x}^2 - dt^2). \tag{16.72}$$

This is precisely anti-de Sitter spacetime AdS_{d+1} in Poincaré coordinates with boundary at $z = 0$ and horizon at $z = \infty$. The constant L is the radius of anti-de Sitter. This metric solves Einstein equations with cosmological constant Λ, viz

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu}. \tag{16.73}$$

The Ricci tensor and Ricci scalar of the metric (16.165) are given by

$$R_{\mu\nu} = -\frac{d}{L^2}g_{\mu\nu} \Rightarrow R = -\frac{d(d+1)}{L^2}. \tag{16.74}$$

Thus, the metric (16.165) defines an Einstein space. However, by contracting both sides of the Einstein equations (16.73) we obtain the Ricci scalar

$$R = 2\frac{d+1}{d-1}\Lambda. \tag{16.75}$$

By comparing the previous two equations we can determine the cosmological constant in terms of the radius of anti-de Sitter by the relation

$$\Lambda = -\frac{d(d-1)}{2L^2}. \tag{16.76}$$

Euclidean Poincaré coordinates

The relationship between the global coordinates τ, ρ and $\hat{x}_i$ (with a manifest $SO(2) \times SO(d)$ symmetry subgroup) and the Poincaré coordinates t, z, $\vec{x}$ (with the d-dimensional Poincaré symmetry subgroup manifest) is given by

$$\begin{aligned} X_0 &= L\cos\tau\cosh\rho = \frac{z}{2}\left(1 + \frac{L^2 + \vec{x}^2 - t^2}{z^2}\right) \\ X_i &= L\sinh\rho\hat{x}_i = L\frac{x_i}{z}, \;\; i < d \\ X_d &= L\sinh\rho\hat{x}_d = \frac{z}{2}\left(1 - \frac{L^2 - \vec{x}^2 + t^2}{z^2}\right) \\ X_{d+1} &= L\sin\tau\cosh\rho = L\frac{t}{z}. \end{aligned} \tag{16.77}$$

Although these Poincaré coordinates cover only half of the AdS_{d+1} spacetime (with conformal group $SO(2, d)$) their Euclidean analogs cover all of the Euclidean AdS_{d+1} space (with conformal group $SO(1, d+1)$). The Euclidean AdS_{d+1} space is obtained by the Wick rotation $X_{d+1} \longrightarrow -iX_{d+1}$, i.e. it is given by the embedding

$$-X_0^2 + X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = L^2. \tag{16.78}$$

This corresponds to the Wick rotations $t \longrightarrow -it$ and $\tau \longrightarrow -i\tau$. The metric becomes

$$ds_{d+1}^2 = R^2\left(\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}\right) = \Omega^2(z)(dz^2 + d\vec{x}^2 + dt^2). \tag{16.79}$$

The map between these systems becomes then given by

$$\begin{aligned} X_0 &= L\cosh\tau\cosh\rho = \frac{z}{2}\left(1 + \frac{L^2 + \vec{x}^2 + t^2}{z^2}\right) \\ X_i &= L\sinh\rho\hat{x}_i = L\frac{x_i}{z}, \;\; i < d+1 \\ X_{d+1} &= L\sinh\tau\cosh\rho = \frac{z}{2}\left(1 - \frac{L^2 - \vec{x}^2 - t^2}{z^2}\right). \end{aligned} \tag{16.80}$$

The boundary $z = 0$ in the Minkowski metric becomes $\mathbf{R}^d$ in the Euclidean metric, whereas the horizon $z = \infty$ in the Minkowski metric shrinks to a point in the Euclidean metric. By adding the point $z = \infty$ to the boundary $\mathbf{R}^d$ we obtain a sphere $\mathbf{S}^d$. This compactified Euclidean AdS_{d+1} is thus the solid $(d + 1)$-dimensional ball.

Renormalization group equation

As we said, the extra coordinate z corresponds to an energy scale of a conformal field theory, i.e. it defines a lattice spacing a. The d-dimensional slices or branes, defined by the points $x = (x_0, x_1, \cdots, x_d)$ in the higher-dimensional AdS_{d+1} spacetime, should then be regarded as lattices of increasing size in a Kadanoff–Wilson renormalization group approach [16, 17].

To exhibit this crucial point in some detail we start with a field theory Hamiltonian on a lattice a, with coupling constants or sources $J_i(x, a)$ and field operators $\mathcal{O}_i$, given by [3]

$$H = \sum_{i,x} J_i(x, a)\mathcal{O}_i(x). \tag{16.81}$$

Then, under the Kadanoff–Wilson renormalization group approach, the lattice is coarse grained, i.e. we increase the lattice spacing successively as $\mu = a \longrightarrow 2a \longrightarrow 4a \cdots$ and replace at each step the spins (fields) by block spins (averages of fields) [16, 17]. As a result the Hamiltonian will change in such a way that its form remains invariant but only the coupling constants $J_i(x, \mu)$ change or flow, i.e. the weightings of the field operators $\mathcal{O}_i$ flow, according to the renormalization group equation

$$\mu \frac{\partial}{\partial \mu} J_i(x, \mu) = \beta_i(J_j(x, \mu), \mu). \tag{16.82}$$

In the AdS/CFT proposal we view the lattice scale μ as an extra dimension and as a consequence the collection of the lattices $\mu = a, 2a, 4a, \ldots$ should be viewed as slices of a higher-dimensional space. The coupling constants $J_i(x, \mu)$ should then be reinterpreted as fields $\Phi_i(x, \mu)$ in this new higher-dimensional space with the equation of motion in the extra dimension given by the above renormalization group equation. The dynamics of these bulk fields $\Phi_i(x, \mu)$ is determined by a gravitational action with their boundary values equated with the microscopic/continuum values of the coupling constants $J_i(x, \mu)$ of the field theory (corresponding to the operators $\mathcal{O}_i$) in the UV [3]. See figure 16.1.

16.3 Scalar field in AdS_{d+1}

16.3.1 The Klein–Gordon equation

Recall that AdS_{d+1} in this system can be viewed as a cylinder with bases at $\tau = -\infty$ and $\tau = +\infty$, a center at $\rho = r = 0$, whereas the spatial infinity is at $r = \pi/2$, while going around the cylinder is given by the angular variables Ω_{d-1}. The embedding is given explicitly by (the radius is denoted here by R)

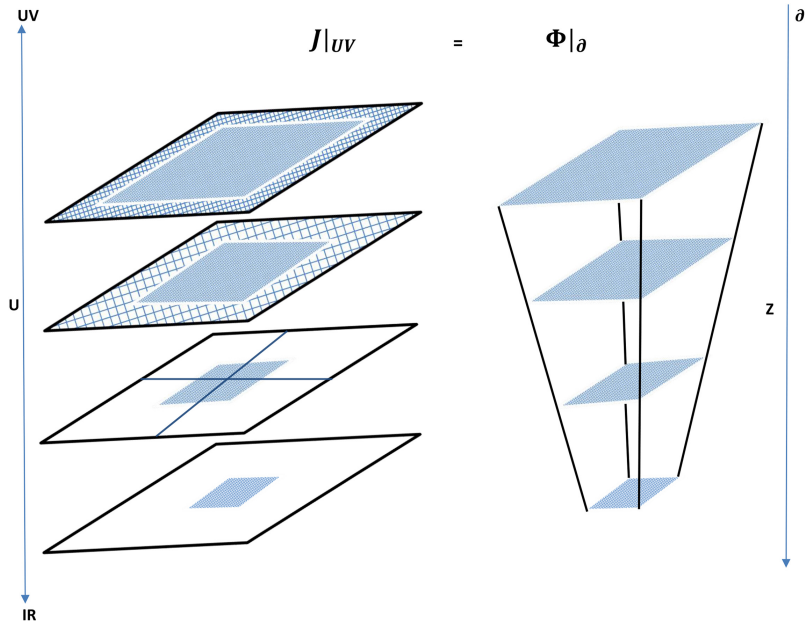

Figure 16.1. AdS_{d+1} as a foliation of M_d over the energy scale.

$$\begin{aligned} X_0 &= R\frac{\cos\tau}{\cos r} \\ X_{d+1} &= R\frac{\sin\tau}{\cos r} \\ X_\mu &= R\tan r\hat{x}_\mu. \end{aligned} \tag{16.83}$$

Recall also the ranges $r \in [0, \pi/2[$, and $\tau \in [-\infty, \infty]$ as we unwrap to the universal cover and $\hat{x}_\mu$ defines a $(d-1)$-dimensional sphere $\mathbf{S}^{d-1}$. We will consider global coordinates on AdS_{d+1} given by

$$ds^2_{d+1} = \frac{R^2}{\cos^2 r}\Big(-d\tau^2 + dr^2 + \sin^2 r d\Omega_{d-1}\Big). \tag{16.84}$$

We start by writing the action of a scalar field ϕ in AdS_{d+1} with $SO(2, d)$ invariance given by

$$S = \int d^{d+1}x\sqrt{g}\left[-\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - \frac{1}{2}m^2\phi^2\right]. \tag{16.85}$$

The Klein–Gordon equation is the Euler–Lagrange equation derived from this action given by

$$\frac{1}{\sqrt{g}}\partial_M\Big(\sqrt{g}\,g^{MN}\partial_N\phi\Big) - m^2\phi = 0. \tag{16.86}$$

16.3.2 Example of AdS_2

For simplicity we consider AdS_2 with isometry given by $SO(2, 1)$ and metric $ds_2^2 = R^2(-d\tau^2 + dr^2)/\cos^2 r$. We obtain the equation

$$-\partial_\tau^2\phi = \left(-\partial_r^2 + \frac{R^2m^2}{\cos^2 r}\right)\phi. \tag{16.87}$$

We pull the time dependence as $\phi = \exp(i\Delta\tau)\chi(r)$ where E is the energy. We get

$$\Delta^2\chi = \left(-\partial_r^2 + \frac{R^2m^2}{\cos^2 r}\right)\chi. \tag{16.88}$$

A solution is given by the ansatz

$$\chi(r) = \cos^\alpha r. \tag{16.89}$$

The equation of motion becomes algebraic given by

$$(\Delta^2 - \alpha^2)\chi = (-\alpha(\alpha - 1) + R^2m^2)\chi \Rightarrow \Delta^2 - \alpha^2 = -\alpha(\alpha - 1) + R^2m^2 = 0. \tag{16.90}$$

In other words, $\chi(r) = \cos^\alpha r$ is a solution iff the mass m of the scalar field is related to its energy Δ (in the ground state) by the relation

$$R^2m^2 = \Delta(\Delta - 1). \tag{16.91}$$

This crucial result can also be found using group theoretic method as follows.

Recall the signature $-++\cdots-$ and that the number of generators of $SO(2, d)$ is $(d + 1)(d + 2)/2$. Thus, the number of generators of $SO(2, 1)$ is three and they are given by L_2^0, L_1^0 and L_2^1. Explicitly we have

$$L_2^0 = X^0\frac{\partial}{\partial X^2} - X_2\frac{\partial}{\partial X_0} = \partial_\tau \tag{16.92}$$

$$L_1^0 = X^0\frac{\partial}{\partial X^1} - X_1\frac{\partial}{\partial X_0} = \sin\tau\sin r\partial_\tau - \cos\tau\cos r\partial_r \tag{16.93}$$

$$L_2^1 = X^1\frac{\partial}{\partial X^2} - X_2\frac{\partial}{\partial X_1} = -\cos\tau\sin r\partial_\tau - \sin\tau\cos r\partial_r. \tag{16.94}$$

They satisfy the algebra

$$\left[L_2^0, L_1^0\right] = -L_2^1,\ \left[L_2^0, L_2^1\right] = L_1^0,\ \left[L_1^0, L_2^1\right] = L_2^0. \tag{16.95}$$

Or equivalently

$$[D, P] = P,\ [D, K] = -K,\ [K, P] = -2D. \tag{16.96}$$

The operators D, P and K are given in terms of the operators L_B^A by the relations:

- The dilatation generator or Hamiltonian operator:

$$D = -iL_2^0. \tag{16.97}$$

The representations of the $SO(2, 1)$ algebra are then labeled by the eigenvalues Δ of D, viz

$$D|\psi\rangle = \Delta|\psi\rangle. \tag{16.98}$$

- The special conformal generator or lowering operator

$$K = L_1^0 - iL_2^1. \tag{16.99}$$

Thus, by acting on the ground state $|\psi_0\rangle$ with the lowering operator K, we must have

$$K|\psi_0\rangle = 0. \tag{16.100}$$

- The momentum generator or raising operator

$$P = L_1^0 + iL_2^1. \tag{16.101}$$

All other states above the ground state $|\psi_0\rangle$ are obtained by acting successively with the raising operator P on $|\psi_0\rangle$.

The conditions $D|\psi_0\rangle = \Delta|\psi_0\rangle$ and $K|\psi_0\rangle = 0$ read in the global coordinates τ, r as follows

$$-i\partial_\tau\psi_0(t, r) = \Delta\psi_0(t, r) \Rightarrow \psi_0(t, r) = \exp(i\Delta t)\chi(r) \tag{16.102}$$

$$(i \sin r\partial_\tau - \cos r\partial_r)\psi_0(t, r) = 0 \Rightarrow \chi(r) = \cos^\Delta r. \tag{16.103}$$

All other states $|\psi_n\rangle$ can now be obtained by the action of the raising operator P on this ground state $|\psi_0\rangle$, i.e. $|\psi_n\rangle = P^n|\psi_0\rangle$. The energy is found to be quantized as

$$E_n = \Delta + n. \tag{16.104}$$

In other words, the energy levels are integer spaced analogously to the harmonic oscillator motion and hence all AdS_2 orbits have the same period with respect to the AdS_2 time τ.

16.3.3 Generalization

The generators of the conformal algebra $SO(d, 2)$ are:

- The dilatation generator $D = -iL_{d+1}^0$ which plays the role of the Hamiltonian.
- The momentum generators $P_\mu = L_\mu^0 + iL_\mu^{d+1}$ which play the role of raising operators. Note that $L_\mu^{d+1} = L_{d+1}^\mu$.
- The special conformal generators $K_\mu = L_\mu^0 - iL_\mu^{d+1}$ which play the role of lowering operators.
- The rotation generators $M_{\mu\nu} = iL_\mu^\nu$ which generate the Lie algebra $SO(d)$.

The generators $L_A^{\ B}$ satisfy the algebra

$$\left[L_A^{\ B}, L_C^{\ D}\right] = \eta_A^{\ D} L_C^{\ B} - \eta_{AC} L^{BD} - \eta^{DB} L_{AC} - \eta_C^{\ B} L_A^{\ D}. \tag{16.105}$$

The rotation generators were absent in $SO(1, 2)$. Also, we have d momentum generators and d special conformal generators in the case of $SO(d, 2)$, i.e. P_μ and K_μ transform as vectors under $SO(d)$, viz

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), \ [M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu). \tag{16.106}$$

The generators P_μ and K_μ are indeed the raising and lowering operators respectively, viz

$$[D, P_\mu] = P_\mu, \ [D, K_\mu] = -K_\mu. \tag{16.107}$$

Also, we have

$$[P_\mu, K_\nu] = 2(\eta_{\mu\nu} D + i M_{\mu\nu}). \tag{16.108}$$

The $SO(d)$ rotation generators satisfy

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma} M_{\rho\nu} - \eta_{\mu\rho} M_{\sigma\nu} + \eta_{\sigma\nu} M_{\mu\rho} - \eta_{\rho\nu} M_{\mu\sigma}). \tag{16.109}$$

The dilatation generator D is a scalar under $SO(d)$ rotations, viz

$$[M_{\mu\nu}, D] = 0. \tag{16.110}$$

This means that the Hamiltonian and the angular momentum operators can be diagonalized simultaneously.

The ground state $|\psi_0\rangle$ will correspond to the smallest possible eigenvalue Δ of the Hamiltonian D and it must be annihilated by all the lowering operators K_μ, viz

$$D|\psi_0\rangle = \Delta|\psi_0\rangle, \ K_\mu|\psi_0\rangle = 0. \tag{16.111}$$

The ground state or highest weight state $|\psi_0\rangle$ is called a primary state. All other states (descendant states) can be obtained by acting successively with the raising operators P_μ (which commute among themselves), viz

$$|\psi_{nlm}\rangle \sim \left(P_\mu^2\right)^n P_{\mu_1} \ldots P_{\mu_l}|\psi_0\rangle. \tag{16.112}$$

The energy of this state is clearly (since each P_μ raises the energy by a single unit)

$$E_{nl} = \Delta + 2n + l. \tag{16.113}$$

These states carry $SO(d)$ indices and hence they are characterized by an angular momentum quantum number equal exactly to the integer l, i.e. to the number of uncontracted vector indices. If $l = 0$ then the state does not carry a free $SO(d)$ index, i.e. it is a scalar, and as consequence its spin is given directly by $l = 0$, whereas if

$l = 1$, the state carries a single free vector index, i.e. it transforms as a vector under $SO(d)$, and as a consequence its spin must be $l = 1$. The quantum number l is, therefore, the angular momentum or spin quantum number.

The quantum numbers m[1] appearing in (16.112) relate to the spherical harmonics $Y_{lm}(\Omega)$ on AdS_{d+1} which, by rotational symmetry, carry the angular dependence of the wave functions $\psi_{nlm}(\tau, r, \Omega) = \langle \tau, r, \Omega | \psi_{nlm} \rangle$. For example, for $d = 3$ we have a single (integer) number m which is the usual magnetic quantum number and $Y_{lm}(\Omega) = Y_{lm}(\theta, \phi)$ are the usual spherical harmonics. Thus, the wave functions $\psi_{nlm}(\tau, r, \Omega)$ are of the general form [1]

$$\psi_{nlm}(\tau, r, \Omega) = \exp(iE_{nl}\tau)\, Y_{lm}(\Omega)\psi_{nl}(r). \tag{16.114}$$

The radial part $\psi_{nl}(r)$ is proportional to a hypergeometric function which for the ground state reduces to

$$\psi_{00}(r) \sim \cos^{\Delta} r. \tag{16.115}$$

This can be shown as follows. We start from the identity

$$\begin{aligned} K_\mu &= -(X_0 - iX_{d+1})\frac{\partial}{\partial X_\mu} + X_\mu\left(\frac{\partial}{\partial X_0} - i\frac{\partial}{\partial X_{d+1}}\right) \\ &= -\frac{\exp(-i\tau)}{r}\frac{\cos^2 r\hat{x}_\mu}{\cos r}\frac{\partial}{\partial \hat{x}_\mu} + X_\mu\left(\frac{\partial}{\partial X_0} - i\frac{\partial}{\partial X_{d+1}}\right). \end{aligned} \tag{16.116}$$

The ground state $|\psi_0\rangle$ is characterized by zero angular momentum, i.e. $l = 0$, and hence it does not depend on the angles $\hat{x}_\mu$ or equivalently Ω. The condition $K_\mu|\psi_0\rangle = 0$ reduces then to

$$\left(\frac{\partial}{\partial X_0} - i\frac{\partial}{\partial X_{d+1}}\right)|\psi_0\rangle = 0. \tag{16.117}$$

We compute (by dropping the derivative with respect to $\hat{x}_\mu$ in $\partial/\partial r$)

$$\frac{\partial}{\partial X_0} = \frac{\cos r}{R}\left(-\sin\tau\frac{\partial}{\partial\tau} + \tan^{-1} r\cos\tau\frac{\partial}{\partial r}\right) \tag{16.118}$$

$$\frac{\partial}{\partial X_{d+1}} = \frac{\cos r}{R}\left(\cos\tau\frac{\partial}{\partial\tau} + \tan^{-1} r\sin\tau\frac{\partial}{\partial r}\right). \tag{16.119}$$

The above condition becomes then

$$\left(-i\sin r\frac{\partial}{\partial\tau} + \cos r\frac{\partial}{\partial r}\right)|\psi_0\rangle = 0. \tag{16.120}$$

[1] It should not be confused with the mass m. But it should also be clear from the context which one is which.

This is immediately solved by

$$\psi_0 \sim \exp(i\Delta t)\cos^{\Delta} r. \tag{16.121}$$

This fundamental result can be found from another route. The generalization of the Klein–Gordon equation (16.87) to AdS_{d+1} is given by

$$-\partial_\tau^2\phi = \left(-\partial_r^2 + \frac{1-d}{\cos r \sin r}\partial_r + \frac{l(l+d-2)}{\sin^2\rho} + \frac{R^2m^2}{\cos^2 r}\right)\phi. \tag{16.122}$$

In this equation $l(l+d-2)$ are the eigenvalues of the Laplacian on the sphere $\mathbf{S}^{d-1}$ with corresponding eigenfunctions given by the d-dimensional spherical harmonics Y_{lm} where m is the corresponding set of magnetic quantum numbers on AdS_{d+1}. In other words, the complete scalar field on AdS_{d+1} is actually $Y_{lm}(\Omega)\phi(\tau, r)$. We separate the remaining variables as $\phi(\tau, r) = \exp(iE\tau)\chi(\rho)$. For the ground state we must have $E = \Delta$ and $l = 0$ and the Klein–Gordon equation reduces to

$$-\Delta^2\chi = \left(\partial_r^2 + \frac{d-1}{\cos r \sin r}\partial_r - \frac{R^2m^2}{\cos^2 r}\right)\chi. \tag{16.123}$$

We make the change of variables

$$\chi = \sin^{\alpha} r \cos^{\beta} r\hat{\chi}\,. \tag{16.124}$$

The exponents α and β are determined from the requirement that the linear derivative vanishes. We have

$$\sin^{\alpha-1} r \cos^{\beta-1} r\partial_r\hat{\chi}(-2\beta + d - 1 + (2\beta + 2\alpha)\cos^2 r) = 0 \Rightarrow \beta = -\alpha = \frac{d-1}{2}. \tag{16.125}$$

Hence

$$\begin{aligned}\partial_r^2 + \frac{d-1}{\cos r \sin r}\partial_r = \sin^{\alpha} r \cos^{\beta} r\partial_r^2\hat{\chi} \; + \sin^{\alpha} r \cos^{\beta} r\hat{\chi} \\ \times\left[\frac{\beta(\beta-d)}{\cos^2 r} + \frac{\beta(\beta-d+2)}{\sin^2 r}\right].\end{aligned} \tag{16.126}$$

The equation of motion becomes

$$-\Delta^2\hat{\chi} = \partial_r^2\hat{\chi} - \frac{1}{4}\left[\frac{d^2 - 1 + 4R^2m^2}{\cos^2 r} + \frac{(d-1)(d-3)}{\sin^2 r}\right]\hat{\chi}\,. \tag{16.127}$$

We propose the solution

$$\hat{\chi} = \sin^{\alpha_1} r \cos^{\beta_1} r. \tag{16.128}$$

We compute immediately the second derivative

$$-(\alpha_1+\beta_1)^2\hat{\chi}=\partial_r^2\hat{\chi}-\left[\frac{\beta_1(\beta_1-1)}{\cos^2 r}+\frac{\alpha_1(\alpha_1-1)}{\sin^2 r}\right]\hat{\chi}. \tag{16.129}$$

By comparing the previous two equations we obtain

$$\alpha_1(\alpha_1-1)=\frac{1}{4}(d-1)(d-3)\Rightarrow\alpha_1=\frac{d-1}{2} \tag{16.130}$$

$$\beta_1(\beta_1-1)=\frac{1}{4}(d^2-1+4R^2m^2)\Rightarrow\beta_1=\frac{1}{2}+\frac{1}{2}\sqrt{d^2+4R^2m^2} \tag{16.131}$$

$$\begin{aligned}\alpha_1+\beta_1=\Delta\Rightarrow&\frac{d-1}{2}+\frac{1}{2}+\frac{1}{2}\sqrt{d^2+4R^2m^2}\\&=\Delta\Rightarrow R^2m^2=\Delta(\Delta-d).\end{aligned} \tag{16.132}$$

16.4 Representation theory of the conformal group

16.4.1 More on dilatation operator and primary/descendant operators

Hamiltonian quantization of ordinary quantum field theory involves foliation of the d-dimensional spacetime by equal time $(d-1)$-dimensional surfaces characterized by the same Hilbert space. The unitary evolution operator $U=\exp(iH(t_2-t_1))$ allows us to advance from the surface $t=t_1$ to the surface $t=t_2$. The theory in this case is covariant under the Poincaré group and the states of the Hilbert space are specified by two quantum numbers: mass m and spin s.

In conformal field theory the dilatation operator is what plays the role of the Hamiltonian and the scaling dimension Δ is what plays the role of the momentum. In this case Euclidean spacetime is foliated using spheres $\mathbf{S}^{d-1}$ characterized by the same Hilbert space. This is called radial quantization. By the action of the dilatation operator we move from one sphere to another. States of the Hilbert space are specified now by the scaling dimension Δ and the spin s, viz

$$D|\Delta\rangle=i\Delta|\Delta\rangle \tag{16.133}$$

$$M_{\mu\nu}|\Delta,s\rangle=\Sigma_{\mu\nu}|\Delta,s\rangle. \tag{16.134}$$

The metric reads (with Ω the solid angle on $\mathbf{S}^{d-1}$ and $\tau=\log r$)

$$\begin{aligned}ds^2&=dr^2+r^2d\Omega^2\\&=e^{2\tau}(d\tau^2+d\Omega^2).\end{aligned} \tag{16.135}$$

This metric is conformally equivalent to the metric on the cylinder. In other words, the transformation $r\longrightarrow\tau=\log r$ maps $\mathbf{R}\times\mathbf{S}^{d-1}$ to $\mathbf{R}^d$. The parameter τ plays then the role of the time parameter. The lower base of the cylinder is at the infinite past $\tau\longrightarrow-\infty$ $(r=0)$, whereas the upper base of the cylinder is at the infinite future

$\tau \longrightarrow +\infty$ ($r = +\infty$). Going around the cylinder is given by the solid angle Ω. The evolution operator is given by

$$U|\Delta\rangle = \exp(i\tau D)|\Delta\rangle = r^{-\Delta}|\Delta\rangle. \tag{16.136}$$

There is a unique vacuum state $|0\rangle$ which is invariant under the global conformal group. This corresponds to no operator insertion in the cylinder which would create a state at a given time τ (corresponding to a given radius r).

Let $\mathcal{O}_\Delta(x)$ be some operator with scaling dimension Δ. The insertion of this operator at the origin $r = 0$ (or infinite past $\tau = -\infty$) creates the state $|\Delta\rangle = \mathcal{O}_\Delta(0)|0\rangle$ with scaling dimension Δ. By inserting the operator $\mathcal{O}_\Delta(x)$ at an arbitrary point x will create the state

$$\begin{aligned} |\chi\rangle &= \mathcal{O}_\Delta(x)|0\rangle \\ &= \exp(iPx)\mathcal{O}_\Delta(0)\exp(-iPx)|0\rangle \\ &= \exp(iPx)|\Delta\rangle. \end{aligned} \tag{16.137}$$

The momentum operator P_μ is a raising operator with respect to the eigenvalues of the dilatation operator, i.e. it raises the scaling dimension Δ by unity. Thus, by expanding the exponential $\exp(iPx)$ and acting with the momentum operator we obtain a linear superposition of states with different eigenvalues Δ.

Similarly, the special conformal generator K_μ is a lowering operator with respect to the eigenvalues of the dilatation operator, i.e. it lowers the scaling dimension Δ by unity. An operator annihilated by K_μ is called a primary operator. By acting on this primary operator with P_μ we obtain the so-called descendant operators. The primary operator and its descendant operators form a conformal family.

Each state then corresponds to an operator and vice versa. This state-operator correspondence is one-to-one. For example, by inserting a primary operator with scaling dimension Δ at the origin we obtain a state with scaling dimension Δ annihilated by K_μ. Conversely, given a state with a scaling dimension Δ annihilated by K_μ we can construct a local primary operator at the origin by constructing its correlators with other operators, viz

$$\langle\phi(x_1)\phi(x_2)\ldots\mathcal{O}_\Delta(0)\rangle = \langle 0|\phi(x_1)\phi(x_2)\ldots|\Delta\rangle \tag{16.138}$$

16.4.2 Representation theory of $O(4, 2)$

A concise description of this topic can be found in [2, 10] and references therein. See also [18–22].

The conformal group in four dimensions in Lorentzian signature is given by $SO(4, 2)$ or more precisely $O(4, 2)$. There are 15 generators $\mathcal{M}_{AB} = -\mathcal{M}_{BA}$. The irreducible representations of the conformal group are infinite-dimensional. They are characterized by the eigenvalues of the three Casimir (quadratic, cubic and quartic) operators

$$C_I = \mathcal{M}_{AB}\mathcal{M}^{AB} \tag{16.139}$$

$$C_{II} = \epsilon_{ABCDEF}\mathcal{M}^{AB}\mathcal{M}^{CD}\mathcal{M}^{EF} \tag{16.140}$$

$$C_{III} = \mathcal{M}^{B}_{A}\mathcal{M}^{C}_{B}\mathcal{M}^{D}_{C}\mathcal{M}^{A}_{D}. \tag{16.141}$$

An infinite-dimensional irreducible representation of the conformal group is determined by an irreducible representation of the Lorentz group with definite conformal dimension and annihilated by the special conformal operators K_μ. The stability algebra at the origin consists of the generators D, K_μ and $M_{\mu\nu}$. A primary conformal operator O (the lowest weight state) in a given representation of the Lorentz group is defined by

$$[D,\, O(0)] = i\Delta O(0) \tag{16.142}$$

$$[K,\, O(0)] = 0. \tag{16.143}$$

The descendants $\partial \ldots \partial O(0)$ are obtained by the repeated action of the momentum operators P_μ. The eigenvalues of the Lorentz operators $M_{\mu\nu}$ on the primary operator O are spin quantum numbers denoted for example by j_L and j_R. This defines an irreducible representation of the conformal group characterized by Δ, j_L and j_R.

The three Casimirs of the stability algebra are

$$\mathcal{D} = \Delta \tag{16.144}$$

$$\frac{1}{2}M_{\mu\nu}M^{\mu\nu} = j_L(j_L + 1) + j_R(j_R + 1) \tag{16.145}$$

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}M_{\mu\nu}M^{\alpha\beta} = j_L(j_L + 1) - j_R(j_R + 1). \tag{16.146}$$

The eigenvalues of the conformal group $O(4, 2)$ are then given by

$$C_I = \Delta(\Delta - 4) + 2j_L(j_L + 1) + 2j_R(j_R + 1) \tag{16.147}$$

$$C_{II} = (\Delta - 2)(j_L(j_L + 1) - j_R(j_R + 1)) \tag{16.148}$$

$$\begin{aligned} C_{III} = {} & (\Delta - 2)^4 - 4(\Delta - 2)^2(j_L(j_L + 1) + j_R(j_R + 1) + 1) \\ & + 16j_Lj_R(j_L + 1)(j_R + 1). \end{aligned} \tag{16.149}$$

In particular for tensor representations of spin s associated with the quantum numbers $(\Delta, j_L = s/2, j_R = s/2)$ we get

$$C_I = \Delta(\Delta - 4) + s(s + 2) \tag{16.150}$$

$$C_{II} = 0 \tag{16.151}$$

$$C_{III} = (\Delta(\Delta - 2) - s(s + 2))((\Delta - 2)(\Delta - 4) - s(s + 2)). \tag{16.152}$$

The requirement of unitarity imposes the following constraints on the possible values of Δ, j_L and j_R. We have

$$\Delta \geqslant j + 1, \ j_L j_R = 0 \tag{16.153}$$

$$\Delta \geqslant j_L + j_R + 2, \ j_L j_R \neq 0. \tag{16.154}$$

The first constraint (16.153) is saturated by massless fields satisfying $\partial^2 \Phi_{(0,\, j)} = 0$. Indeed, this wave equation is conformally covariant only if $\Delta = j + 1$. Similarly, the second constraint (16.154) is saturated by conserved tensor fields satisfying $\partial^{\alpha_1 \dot{\alpha}_1} O_{\alpha_1 \ldots \alpha_{2j_L},\, \dot{\alpha}_1 \ldots \dot{\alpha}_{2j_R}} = 0$ which is a conformally covariant equation only if $\Delta = 2 + j_L + j_R$.

16.4.3 $O(4, 2)$ and isometries of AdS_5

The conformal group $O(d, 2)$ is the isometry group of AdS_{d+1}. By the AdS/CFT correspondence there is a d-dimensional conformal field theory (CFT_d) living on the boundary of AdS_{d+1} where gauge invariant composite operators in the CFT_d are associated with fields in AdS_{d+1}.

This means in particular that the scaling dimension Δ can be reinterpreted as the energy of a particle moving in anti-de Sitter space. Indeed, particles in AdS_5 are characterized by the quantum numbers (E, j_L, j_R) where the energy E is identified with the scaling dimension of the corresponding conformal primary operator living on the boundary.

Indeed, the covariant wave equation of a particle in AdS_5 can be re-expressed in terms of the Casimir of the conformal group and hence the mass of the particle can be determined in terms of the quantum numbers (E, j_L, j_R). For example, for a scalar field with $j_L = j_R = 0$ the Laplacian operator in AdS_5 is precisely the Casimir operator C_I and hence from equation (16.150) we obtain the mass squared $m^2 = \Delta(\Delta - 4)$ which will be obtained more directly in due course. By using also equation (16.150) we obtain for fermions of spin 1/2 with $j_L = 1/2$, $j_R = 0$ or $j_L = 0$, $j_R = 1/2$ the mass $m = \Delta - 2$, and for vector fields of spin 1 with $j_L = j_R = 1/2$ we get the mass squared $m^2 = \Delta(\Delta - 4) + 3 = (\Delta - 1)(\Delta - 3)$, whereas for symmetric tensor fields of spin 2 with $j_L = j_R = 1$ we obtain the mass squared $m^2 = \Delta(\Delta - 4)$. See [10] for more detail.

The generators D and $M_{\mu\nu}$, giving the quantum numbers (Δ, j_L, j_R), correspond to the non-compact subgroup $O(3, 1)$ of the conformal group $O(4, 2)$. However, we observe that the operators in the representations (Δ, j_L, j_R) yields, when applied to the vacuum, non-normalizable states. This is because these states cannot furnish a finite-dimensional unitary representation of a non-compact group.

Another set of good quantum numbers corresponds to the maximal compact subgroup $O(2) \times O(4)$ of the conformal group $O(4, 2)$. This compact group $O(2) \times O(4)$ allows us to obtain finite-dimensional unitary representations of the conformal group using states with finite norm. Indeed, the quantum numbers (E, j_L, j_R) can now be viewed as the eigenvalues of the Cartan generators of $O(2) \times O(4)$. In particular, the quantum number E is associated with the $O(2)$

generator $H = (K_0 + P_0)/2$ which is called the conformal energy. This can also be seen by going to the Euclidean spacetime $\mathbf{R}^4$ which can be mapped via a conformal transformation to $\mathbf{R} \times \mathbf{S}^3$ (radial quantization). In this case the group $O(2)$ is seen acting on $\mathbf{R}$ and hence H is the Hamiltonian corresponding to translations in this direction, whereas the factor $O(4)$ acts on $\mathbf{S}^3$.

Furthermore, we note that since P_0 is a raising operator and K_0 is a lowering operator the eigenvalue E seems to be integer-valued. But in the quantum theory it is the covering space of the conformal group that is being realized and is obtained by unwinding the factor $O(2)$ giving rise to a continuous spectrum of E. Hence, the identification of E with Δ.

16.4.4 The fields and operators in CFT_d

Although the scaling dimension Δ in the CFT_4 can be identified with the conformal energy E in the AdS_5, strictly speaking we do not have particle states in a conformal field theory. The first obvious reason is that the mass operator $P_\mu P^\mu$ is not a Casimir of the conformal group, i.e. it does not commute with the dilatation operator. Hence, if a state in a given representation of the conformal group has an energy E_0, then by the action of the dilatation operator we can obtain states in this representation with any other value of the energy between 0 and ∞.

This can be understood more precisely by means of the Kallen–Lehmann spectral representation of the two-point function of a general interacting QFT_d given in terms of the free propagator $\Delta(p, \mu^2)$ and the spectral density $\rho(\mu^2)$ by the relation [23, 24]

$$\Delta(p) = \int_0^\infty \rho(\mu^2)\Delta_0(p, \mu^2)d\mu^2, \;\; \Delta(p, \mu^2) = \frac{1}{p^2 - \mu^2 + i\epsilon}. \tag{16.155}$$

The spectral density $\rho(\mu^2)$ encodes the contributions of the states $|s\rangle$ with momenta p_s to the two-point function and it is given explicitly by

$$\rho(p^2) = \sum_s \delta^d(p - p_s)|\langle s|\phi\rangle|^2. \tag{16.156}$$

For a free scalar field ϕ we have

$$\rho(p^2) \;=\; \delta^d(p^2 - m^2). \tag{16.157}$$

This corresponds to a single massive excitation. If ϕ overlaps with heavier states then there will be other terms with $p^2 > m^2$.

On the other hand, for a conformal field theory in four dimensions we know that the two-point function should behave as

$$\Delta(x) \sim \frac{1}{x^2} \Rightarrow \rho(p^2) \;=\; \delta^4(p^2). \tag{16.158}$$

This also corresponds to a single massless excitation. But for a generic scalar operator in CFT_4 characterized by a non-trivial scaling dimension (anomalous dimension) the behavior of the two-point function is altered as

$$\Delta(x) \sim \frac{1}{x^{2+2\delta}} \Rightarrow \rho(p^2) = (p^2)^{\delta-1}, \ \ \delta > 0. \tag{16.159}$$

This is a continuous power-law spectrum characterized by δ. In other words, there is no mass scale nor a discrete set of particles but the operator just creates a scale-invariant continuous set of states.

Hence, fields in an ordinary QFT_d are local operators which furnish a representation of the Lorentz group, generate the Hilbert space, and create particle states. But in a CFT_d the basic objects are operators which are not necessarily fields since they do not create particle-like excitations. Thus, the formalism of scattering theory and the S-matrix does not apply for conformal field theory.

16.4.5 Unitary bounds revisited

The states which saturate the unitarity bound (16.153) are called singleton and they are topological configurations living at the boundary of AdS_5 associated with fundamental fields (and not gauge invariant operators) of the CFT_4.

On the other hand, the constraint equation (16.154) enjoys a profound physical meaning. Recall that this inequality is saturated by conserved tensor fields satisfying $\partial^{\alpha_1\dot{\alpha}_1} O_{\alpha_1 \dots \alpha_{2j_L},\, \dot{\alpha}_1 \dots \dot{\alpha}_{2j_R}} = 0$, which is a conformally covariant equation only if $\Delta = 2 + j_L + j_R$. These conserved tensor fields are precisely the conserved currents in the CFT_4 which are associated with massless fields in AdS_5 with local gauge invariance. In other words, global symmetries in CFT_4 correspond to local symmetries in AdS_5. For example, the energy–momentum tensor $T_{\mu\nu}$ is associated with the graviton field $g_{\mu\nu}$ and the global current J_μ is associated with the gauge field A_μ, etc. The equation $\partial^{\alpha_1\dot{\alpha}_1} O_{\alpha_1 \dots \alpha_{2j_L},\, \dot{\alpha}_1 \dots \dot{\alpha}_{2j_R}} = 0$ satisfied by these conserved currents means that the number of degrees of freedom contained in the conserved tensor in CFT_4 is precisely the number of degrees of freedom contained in the massless field in AdS_5. Obviously, the tensor contains $(2j_L + 1)(2j_R + 1) - (2j_L)(2j_R) = 2(j_L + j_R) + 1$ degrees of freedom, i.e. a massless particle of spin $j_L + j_R$. For $\Delta > 2 + j_L + j_R$ the fields in AdS_5 are massive and the corresponding tensor fields are not conserved. See [10] for more detail.

16.5 Holography

The number of degrees of freedom, or equivalently the amount of information, contained in a quantum system is measured as we know by thermodynamic entropy. In quantum mechanics the entropy is an extensive quantity and thus the entropy of a d-dimensional spatial region $\mathbf{R}^d$ is proportional to its volume V_d.

However, in quantum gravity the entropy is sub-extensive, i.e. the entropy of a d-dimensional spatial region $\mathbf{R}^d$ is actually proportional to the surface area $S_{d-1} = \partial V_d$ which bounds its volume V_d and not proportional to the volume V_d itself. In other words, the entropy of a d-dimensional spatial region, in a gravitational theory, is bounded by the entropy of the black hole which fits inside that spatial region. This is essentially what is called the holographic principle introduced first by 't Hooft [25] and then extended to string theory by Susskind [26] (see also

[27]). As we can see, this principle is largely inspired by the Bekenstein–Hawking formula which states that the entropy of a black hole $\mathcal{S}_{BH}$ is proportional to the surface area A_H of the black hole horizon with the constant of proportionality equal to $1/4G_N$ where G_N is Newton's constant, viz

$$\mathcal{S}_{BH} = \frac{A_H}{4G_N}. \tag{16.160}$$

The holographic principle provides, therefore, a partial answer to the question of how could a higher-dimensional gravity theory (AdS_5) contain the same number of degrees of freedom, the same amount information, and have the same entropy as a lower dimensional quantum field theory (CFT_4), i.e. it lies at the heart of the celebrated AdS/CFT correspondence [28].

This can be seen more explicitly as follows. By the AdS/CFT correspondence, the AdS space AdS_{d+1} is the gravity dual of a d-dimensional conformal field theory CFT_d living on the boundary $z = \epsilon$ of AdS space. The radial coordinate z should be thought of as a lattice spacing, i.e. as a UV cutoff. Thus, the boundary theory is a quantum field theory on a d-dimensional lattice with lattice spacing ϵ. At any given instant of time, the boundary theory is also regulated by placing it in a spatial box of size R (IR cutoff). Hence, the number of cells in the box is given by $(R/L)^{d-1}$.

The central charge c_{QFT} of the CFT is by definition equal to the number of degrees of freedom per lattice site. Thus, the total number of degrees of freedom contained in the box is given by

$$N_{\text{QFT}} = \left(\frac{R}{\epsilon}\right)^{d-1} c_{\text{QFT}}. \tag{16.161}$$

From the AdS space side the estimation of the degrees of freedom can be carried out as follows. The metric at $z = \epsilon$ is

$$ds_d^2 = \frac{L^2}{\epsilon^2} dx_\mu dx^\mu. \tag{16.162}$$

By using the holographic principle, i.e. the Bekenstein–Hawking formula, the number of degrees of freedom at a given instant of time contained in the spatial volume of AdS space is given by the maximum entropy given by

$$N_{\text{AdS}} = \frac{A_H}{4G_N}. \tag{16.163}$$

Here, A_H is the area of the spatial boundary of AdS_{d+1} delimiting the spatial volume of AdS space. We use the metric to compute this area as follows

$$A_H = \int_{z=\epsilon} d^{d-1}x\sqrt{g} = \left(\frac{L}{\epsilon}\right)^{d-1} \int d^{d-1}x = \left(\frac{L}{\epsilon}\right)^{d-1} R^{d-1}. \tag{16.164}$$

By using also the fact that for gravity in $d + 1$ dimensions the Newton constant is given by $G_N = l_P^{d-1} = 1/M_P^{d-1}$ we arrive at the result

$$N_{\text{AdS}} = \frac{1}{4}\left(\frac{R}{\epsilon}\right)^{d-1}\left(\frac{L}{l_P}\right)^{d-1}. \tag{16.165}$$

By comparing equations (16.161) and (16.165) we obtain the central charge

$$c_{\text{QFT}} = \frac{1}{4}\left(\frac{L}{l_P}\right)^{d-1}. \tag{16.166}$$

Hence, semi-classical gravity corresponding to $L \gg l_P$ is dual to a CFT with a large central charge. If the conformal field theory is an $SU(N)$ gauge theory then the central charge is proportional to N^2 and as a consequence semi-classical gravity is dual in this case to a large N gauge theory.

16.6 The AdS/CFT correspondence

In this section we follow the presentation of [3].

16.6.1 Approaching the AdS boundary

We go back to Euclidean AdS_{d+1} in the Poincaré patch:

$$ds^2_{d+1} = \frac{L^2}{z^2}(dz^2 + d\vec{x}^2 + dt^2). \tag{16.167}$$

The action of a scalar field in AdS_{d+1} is given by

$$S = \int d^{d+1}x\sqrt{g}\left[-\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - \frac{1}{2}m^2\phi^2\right]. \tag{16.168}$$

The Klein–Gordon equation in the AdS_{d+1} background reads explicitly

$$z^{d+1}\partial_z(z^{1-d}\partial_z\phi) + z^2\partial^2\phi - m^2L^2\phi = 0. \tag{16.169}$$

We perform Fourier transform in the x-space, viz

$$\phi(z, x) = \int \frac{d^dk}{(2\pi)^d}\exp(ikx)f_k(z). \tag{16.170}$$

The Klein–Gordon equation reduces to

$$z^{d+1}\partial_z(z^{1-d}\partial_z f_k) - k^2z^2f_k - m^2L^2f_k = 0. \tag{16.171}$$

Near the conformal boundary $z = 0$ we may expect $f_k \sim z^\beta$. This gives

$$\beta(\beta - d) - m^2L^2 = 0 \Rightarrow \beta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}. \tag{16.172}$$

The solution near the boundary is, therefore, of the general form

$$f_k(z) \longrightarrow A(k)z^{d-\Delta} + B(k)z^{\Delta},\ z \longrightarrow 0. \tag{16.173}$$

The exponent Δ is the so-called scaling dimension of the field and it is given by

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2L^2}. \tag{16.174}$$

The scaling dimension Δ is real iff the mass m satisfies the Breitenlohner–Freedman (BF) bound

$$m^2 > -\frac{d^2}{4L^2}. \tag{16.175}$$

In this case $d - \Delta \leqslant \Delta$ and hence $z^{d-\Delta}$ is the dominant term as $z \longrightarrow 0$. We place the boundary at $z = \epsilon \longrightarrow 0$. Then the behavior of the scalar field on the boundary is given by

$$\phi(z = \epsilon, x) = A(x)\epsilon^{d-\Delta}. \tag{16.176}$$

For $m^2 > 0$ the exponent $d - \Delta$ is negative and hence this field is divergent. The quantum field theory source $\varphi(x)$, i.e. the scalar field living on the boundary, is then identified with $A(x)$, viz

$$\varphi(x) = \lim_{\epsilon \longrightarrow 0} \epsilon^{\Delta-d}\phi(\epsilon, x). \tag{16.177}$$

This is the scalar field representing (or dual to) the anti-de Sitter scalar field $\phi(z, x)$ at the boundary $z = 0$. The field φ is the holographic dual of the AdS field ϕ. The scaling dimension of the source φ is given by $d - \Delta$.

Let $\mathcal{O}(z, x)$ and $\mathcal{O}(x)$ be the dual operators to the scalar fields $\phi(z, x)$ and $\varphi(x)$ respectively. Their coupling is a boundary term of the form

$$\begin{aligned} S_{\text{bound}} &= \int d^d x \sqrt{\gamma}\phi(\epsilon, x)\mathcal{O}(\epsilon, x) \\ &= \int d^d x \frac{L^d}{\epsilon^d}\epsilon^{d-\Delta}\varphi(x)\mathcal{O}(\epsilon, x) \\ &= L^d \int d^d x \varphi(x)\mathcal{O}(x), \end{aligned} \tag{16.178}$$

where

$$\mathcal{O}(\epsilon, x) = \epsilon^{\Delta}\mathcal{O}(x). \tag{16.179}$$

This is the wavefunction renormalization of the operator $\mathcal{O}$ as we move into the bulk. This also shows explicitly that Δ is the scaling dimension of the dual operator $\mathcal{O}$ since going from $z = 0$ to $z = \epsilon$ is a dilatation operation in the quantum field theory.

In summary, the scaling dimensions of the source φ and its dual operator $\mathcal{O}$ are given by $d - \Delta$ and Δ respectively. Any deformation of the conformal field theory with the operator $\mathcal{O}$ takes then the form

$$\Delta S = \int d^d x M^{d-\Delta} \mathcal{O}. \tag{16.180}$$

We distinguish the usual three cases:

- For $m^2 > 0$ we have $\Delta > d$ and thus corrections to various amplitudes are of the form $(E/M)^\alpha$ for some positive α. In other words, these corrections are negligible for low energies and the operator $\mathcal{O}$ does not change the IR behavior of the theory. It is an irrelevant operator.
- For $m^2 = 0$ we have $\Delta = d$ and the operator $\mathcal{O}$ is marginal.
- For $m^2 < 0$ we have $\Delta < d$ and the operator $\mathcal{O}$ is relevant since it changes the IR behavior of the theory.

16.6.2 Correlation functions

We have now an operator $\mathcal{O}(x)$ living on the AdS boundary. We are interested in computing the correlation functions

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle. \tag{16.181}$$

These will determine the CFT living on the boundary completely. On the conformal field theory side with Lagrangian $\mathcal{L}$ the calculation of these correlation functions proceeds as usual by introducing the generating functional

$$Z_{\text{CFT}}[J] = \int \exp\left(\mathcal{L} + \int J(x)\mathcal{O}(x)\right) = \left\langle \exp\left(\int J(x)\mathcal{O}(x)\right)\right\rangle. \tag{16.182}$$

Then we have

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle = \frac{\delta^n \log Z_{\text{CFT}}[J]}{\delta J(x_1) \ldots \delta J(x_n)}|_{J=0}. \tag{16.183}$$

The operator $\mathcal{O}(x)$ is sourced by a scalar field $\varphi(x)$ living on the boundary which is related to the boundary value of the bulk scalar field $\phi(z, x)$ by the relation

$$\varphi(x) = \lim_{z \longrightarrow 0} z^{\Delta - d} \phi(z, x). \tag{16.184}$$

The boundary scalar field $\phi_0(x)$ is actually divergent and it is simply defined by

$$\phi_0(x) = \phi(0, x). \tag{16.185}$$

The AdS/CFT correspondence states then that the CFT generating functional with source $J = \phi_0$ is equal to the path integral on the gravity side evaluated over a bulk field which has the value ϕ_0 at the boundary of AdS [29, 30]. We write

$$\begin{aligned} Z_{\text{CFT}}[\phi_0] &= \left\langle \exp\left(\int \phi_0(x)\mathcal{O}(x)\right)\right\rangle \\ &= \int_{\phi \longrightarrow \phi_0} \mathcal{D}\phi \exp(S_{\text{grav}}) = Z_{\text{grav}}[\phi \longrightarrow \phi_0]. \end{aligned} \tag{16.186}$$

In the limit in which classical gravity is a good approximation the gravity path integral can be replaced by the classical amplitude given by the classical on-shell gravity action, i.e.

$$Z_{\rm CFT}[\phi_0] = \exp\Big(S_{\rm grav}^{\rm on\text{-}shell}[\phi \longrightarrow \phi_0]\Big). \tag{16.187}$$

Typically, the on-shell gravity action is divergent requiring thus regularization and renormalization using the so-called holographic renormalization [31–33]. The on-shell action gets renormalized and the above prescription becomes

$$Z_{\rm CFT}[\phi_0] = \exp(S_{\rm grav}^{\rm renor}[\phi \longrightarrow \phi_0]). \tag{16.188}$$

The correlation functions are then renormalized as

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n)\rangle = \frac{\delta^n S_{\rm grav}^{\rm renor}[\phi \longrightarrow \phi_0]}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)}|_{\varphi=0}. \tag{16.189}$$

Let us compute the one-point correlation function, viz

$$\langle \mathcal{O}(x)\rangle_\varphi = \frac{\delta S_{\rm grav}^{\rm renor}[\phi \longrightarrow \phi_0]}{\delta\varphi(x)} = \lim_{z \longrightarrow 0} z^{d-\Delta}\frac{\delta S_{\rm grav}^{\rm renor}[\phi \longrightarrow \phi_0]}{\delta\phi(z,\, x)}. \tag{16.190}$$

We can rewrite this in terms of an action of the form

$$S_{\rm grav} = \int_{\mathcal{M}} dz d^d x \mathcal{L}(\phi,\, \partial\phi). \tag{16.191}$$

Under $\phi \longrightarrow \phi + \delta\phi$, and assuming the Euler–Lagrange equations of motion, and also by assuming that the variation $\delta\phi$ vanishes whenever $x \longrightarrow \pm\infty$ or $z \longrightarrow \infty$, we obtain the on-shell variation

$$\delta S_{\rm grav}^{\rm on\text{-}shell} = \int_{\partial\mathcal{M}} d^d x \Pi \cdot \delta\phi|_{z=\epsilon}. \tag{16.192}$$

The canonical momentum with respect to z is given by

$$\Pi = \frac{\delta S_{\rm grav}^{\rm on\text{-}shell}}{\delta\phi(z,\, x)} = -\frac{\delta\mathcal{L}}{\delta\partial_z\phi(z,\, x)}. \tag{16.193}$$

The renormalized action is obtained by adding counter terms, viz

$$S_{\rm grav}^{\rm renor} = S_{\rm grav}^{\rm on\text{-}shell} + S_{\rm ct}. \tag{16.194}$$

We define in this case the renormalized canonical momentum as

$$\Pi^{\rm renor} = \frac{\delta S_{\rm grav}^{\rm renor}}{\delta\phi(z,\, x)}. \tag{16.195}$$

Hence

$$\langle \mathcal{O}(x)\rangle_\varphi = \lim_{z \longrightarrow 0} z^{d-\Delta}\Pi^{\rm renor}(z,\, x). \tag{16.196}$$

From the one-point function we can obtain the two-point function as follows. In the quantum field theory with fields ψ the one-point function is given by the path integral

$$\langle \mathcal{O}(x)\rangle_\varphi = \int [d\psi]\mathcal{O}(x)\exp\left(S_E[\psi] + \int d^d y\varphi(y)\mathcal{O}(y)\right). \tag{16.197}$$

By expanding in power series of the source φ we obtain

$$\langle \mathcal{O}(x)\rangle_\varphi = \langle \mathcal{O}(x)\rangle_{\varphi=0} + \int d^d y G_E(x-y)\varphi(y) + \cdot s \tag{16.198}$$

$G_E(x-y)$ is obviously the two-point function defined by

$$G_E(x-y) = \langle \mathcal{O}(x)\mathcal{O}(y)\rangle_{\varphi=0}. \tag{16.199}$$

By assuming that the observable $\mathcal{O}(x)$ has been normal-ordered we have $\langle \mathcal{O}(x)\rangle_{\varphi=0} = 0$. In other words, $\langle \mathcal{O}(x)\rangle_\varphi$ measures fluctuations of the observable away from its expectation value without source. Hence

$$\langle \mathcal{O}(x)\rangle_\varphi = \int d^d y G_E(x-y)\varphi(y) + \cdot s \tag{16.200}$$

In momentum space this reads

$$\langle \mathcal{O}(k)\rangle_\varphi = G_E(k)\varphi(k). \tag{16.201}$$

Therefore, the two-point function in momentum space is given immediately by

$$G_E(k) = \frac{\langle \mathcal{O}(k)\rangle_\varphi}{\varphi(k)}. \tag{16.202}$$

16.6.3 The two-point function

We return to the action of a scalar field in Euclidean AdS_{d+1} given by (including an overall constant η)

$$S_{\text{grav}} = \eta \int d^{d+1}x\sqrt{g}\left[-\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - \frac{1}{2}m^2\phi^2\right]. \tag{16.203}$$

By using the Euler–Lagrange equations of motion we obtain the on-shell action and variation as

$$S_{\text{grav}}^{\text{on-shell}} = \frac{\eta}{2}\int d^d x\left(\sqrt{g}\phi g^{zz}\partial_z\phi\right)_{z=\epsilon} \tag{16.204}$$

$$\delta S_{\text{grav}}^{\text{on-shell}} = \eta \int d^d x\left(\sqrt{g}\phi g^{zz}\partial_z\phi\right)_{z=\epsilon}. \tag{16.205}$$

The canonical momentum with respect to z is then given by

$$\Pi = \frac{\delta S_{\text{grav}}^{\text{on-shell}}}{\delta\phi(z,\,x)} = -\frac{\delta\mathcal{L}}{\delta\partial_z\phi(z,\,x)} = \eta\sqrt{g}\,g^{zz}\partial_z\phi. \tag{16.206}$$

Thus, the on-shell action is of the form

$$\begin{aligned} S_{\text{grav}}^{\text{on-shell}} &= \frac{1}{2}\int_{z=\epsilon} d^dx\Pi(z,\,x)\phi(z,\,x) \\ &= \frac{1}{2}\int_{z=\epsilon}\frac{d^dk}{(2\pi)^d}\pi_{-k}(z)f_k(z). \end{aligned} \tag{16.207}$$

The field $f_k(z)$ satisfies the equation of motion (16.171). We introduce another function $g_k(z)$ by $f_k(z) = z^{d/2}g_k(z)$ and substitute in equation (16.171) to obtain the differential equation

$$z^2\partial_z^2 g_k + z\partial_z g_k - (\nu^2 + k^2z^2)g_k = 0,\;\; \nu^2 = \frac{d^2}{4} + m^2L^2 = \left(\Delta - \frac{d}{2}\right)^2. \tag{16.208}$$

This is the modified Bessel equation. The solutions are, therefore, the modified Bessel functions $g_k(z) = I_{\pm\nu}(kz)$, viz $f_k(z) = z^{d/2}I_{\pm\nu}(kz)$. Recall the small and large z limits of the modified Bessel functions

$$I_{\pm\nu}(kz) \sim \frac{1}{\Gamma(1\pm\nu)}\left(\frac{kz}{2}\right)^{\pm\nu},\;\; z \longrightarrow 0 \tag{16.209}$$

$$I_{\pm\nu}(kz) \sim \frac{e^{kz}}{\sqrt{2\pi kz}},\;\; z \longrightarrow \infty. \tag{16.210}$$

The two independent solutions of equation (16.171) are taken to be given by

$$\phi_1(z,\,k) = \Gamma(1-\nu)\left(\frac{k}{2}\right)^{\nu} z^{d/2}I_{-\nu}(kz) \longrightarrow z^{d-\Delta},\;\; z \longrightarrow 0 \tag{16.211}$$

$$\phi_2(z,\,k) = \Gamma(1+\nu)\left(\frac{k}{2}\right)^{-\nu} z^{d/2}I_{\nu}(kz) \longrightarrow z^{\Delta},\;\; z \longrightarrow 0 \tag{16.212}$$

Note that the small z behavior agrees with that previously found in equation (16.173). Equation (16.173) should then be generalized as

$$\begin{aligned} f_k(z) &= A(k)\phi_1(z,\,k) + B(k)\phi_2(z,\,k) \\ &= z^{d/2}\Bigg[\Gamma(1-\nu)\left(\frac{k}{2}\right)^{\nu}A(k)I_{-\nu}(kz) \\ &\quad + \Gamma(1+\nu)\left(\frac{k}{2}\right)^{-\nu}B(k)I_{\nu}(kz)\Bigg]. \end{aligned} \tag{16.213}$$

The large z limit of this solution is given by

$$f_k(z) = z^{d/2}\frac{e^{kz}}{\sqrt{2\pi kz}}\left[\Gamma(1-\nu)\left(\frac{k}{2}\right)^{\nu}A(k) + \Gamma(1+\nu)\left(\frac{k}{2}\right)^{-\nu}B(k)\right], \quad \times z \longrightarrow \infty. \tag{16.214}$$

This diverges in the limit $z \longrightarrow \infty$ unless the term between brackets vanishes. Thus we obtain a regular field in the IR limit $z \longrightarrow \infty$ iff

$$\frac{B(k)}{A(k)} = -\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}. \tag{16.215}$$

Now in the small z limit the field behaves as

$$f_k(z) = A(k)z^{d-\Delta} + B(k)z^{\Delta}, \ z \longrightarrow 0. \tag{16.216}$$

We compute the small z limit of the conjugate field as

$$\pi_{-k}(z) = \eta L^{d-1}[(d-\Delta)A(-k)z^{-\Delta} + \Delta B(-k)z^{\Delta-d}], \ z \longrightarrow 0. \tag{16.217}$$

The on-shell action becomes

$$S_{\text{grav}}^{\text{on-shell}} = \frac{\eta}{2}L^{d-1}\int \frac{d^dk}{(2\pi)^d}((d-\Delta)A(k)A(-k)\epsilon^{-2\nu} + dA(k)B(-k)). \tag{16.218}$$

The first term is divergent and thus this action requires a renormalization. The required counter term is a quadratic local term living on the boundary of AdS space given by [3]

$$S_{\text{ct}} = \frac{\eta}{2}\eta_1\int \sqrt{\gamma}d^dx\phi^2. \tag{16.219}$$

The metric γ is the induced metric on the boundary, viz $ds^2 = L^2dx_\mu dx_\mu/\epsilon^2$. Indeed, we compute

$$S_{\text{ct}} = \frac{\eta}{2}\eta_1 L^d\int \frac{d^dk}{(2\pi)^d}(A(k)A(-k)\epsilon^{-2\nu} + 2A(k)B(-k)). \tag{16.220}$$

The divergence is canceled iff $\eta_1 = -(d-\Delta)/L$. The renormalized action is therefore

$$\begin{aligned} S_{\text{grav}}^{\text{renor}} &= S_{\text{grav}}^{\text{on-shell}} + S_{\text{ct}} \\ &= \frac{\eta}{2}L^{d-1}(2\Delta - d)\int \frac{d^dk}{(2\pi)^d}A(k)B(-k) \\ &= -\frac{\eta}{2}L^{d-1}(2\Delta - d)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\int \frac{d^dk}{(2\pi)^d}A(k)\left(\frac{k}{2}\right)^{2\nu}A(-k) \\ &= -\frac{\eta}{2}L^{d-1}(2\Delta - d)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\int \frac{d^dk}{(2\pi)^d}\varphi(k)\left(\frac{k}{2}\right)^{2\nu}\varphi(-k), \end{aligned} \tag{16.221}$$

where we have used equation (16.215) and

$$\varphi(x) = \lim_{z \longrightarrow 0} z^{\Delta - d}\phi(z, x) = A(x). \tag{16.222}$$

The one-point correlator of the dual operator on the boundary is given by equation (16.190) or equivalently

$$\langle \mathcal{O}(k) \rangle_{\varphi} = (2\pi)^d \frac{\delta S^{\text{renor}}_{\text{grav}}[\phi \longrightarrow \phi_0]}{\delta \varphi(-k)}. \tag{16.223}$$

We get

$$\langle \mathcal{O}(k) \rangle_{\varphi} = -\eta L^{d-1}(2\Delta - d)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}\varphi(k). \tag{16.224}$$

The two-point function is then given in terms of the one-point function by the formula (16.202), viz

$$G_E(k) = \frac{\langle \mathcal{O}(k) \rangle_{\varphi}}{\varphi(k)} = -\eta L^{d-1}(2\Delta - d)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{2}\right)^{2\nu}. \tag{16.225}$$

In position space this two-point function reads

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle = \frac{2\nu L^{d-1}\eta}{\pi^{d/2}}\frac{\Gamma\left(\frac{d}{2}+\nu\right)}{\Gamma(-\nu)}\frac{1}{|x|^{2\Delta}}. \tag{16.226}$$

This is the correct behavior of a conformal field of scaling dimension Δ, i.e. the exponent Δ is indeed the scaling dimension of the boundary operator $\mathcal{O}(x)$.

16.7 Conformal field theory on the torus

16.7.1 Modular invariance

We will follow [34, 35].

We go from the complex plane z to the cylinder w via the conformal exponential mapping, viz $w \longrightarrow z = \exp(w)$ or equivalently $r\exp(i\theta) = \exp(\sigma^2)\exp(-i\sigma^1)$. Thus, the Euclidean time $\sigma^2 = -\infty$ on the worldsheet corresponds to $r = 0$ on the plane while $\sigma^2 = +\infty$ corresponds to $r = +\infty$. The space-like worldsheet coordinate σ^1 is identified with the angle $-\theta$ on the plane, i.e. it is periodic.

A primary field $\Phi(z, \bar{z})$ of conformal dimension $(h, \bar{h})$ transforms under the conformal mapping $z \longrightarrow w$ covariantly as

$$\Phi(z, \bar{z}) \longrightarrow \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(w, \bar{w}). \tag{16.227}$$

Thus, $\Phi(z\bar{z})dz^h d\bar{z}^{\bar{h}}$ is invariant under the conformal mapping. On the cylinder the field is given by

$$\Phi_{\text{cyl}}(w, \bar{w}) = \left(\frac{\partial z}{\partial w}\right)^h \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\bar{h}} \Phi(z, \bar{z}) = z^h \bar{z}^{\bar{h}} \Phi(z, \bar{z}). \tag{16.228}$$

Laurent expansion on the complex plane becomes Fourier expansion on the cylinder under the conformal mapping. Indeed, we have (for holomorphic fields $\Phi(z)$ and $\Phi_{\rm cyl}(w)$)

$$\Phi(z) = \sum_n \frac{\phi_n}{z^{n+h}} \Rightarrow \Phi_{\rm cyl}(w) \;=\; z^h \Phi(z) = \sum_n \phi_n \exp(-nw). \tag{16.229}$$

It is obvious that under a full rotation in the plane $z \longrightarrow \exp(2\pi i)z$, $\bar{z} \longrightarrow \exp(-2\pi i)\bar{z}$ the field on the cylinder $\Phi_{\rm cyl}$ rotates as $\Phi_{\rm cyl}(w, \bar{w}) \longrightarrow \exp(2\pi i(h - \bar{h}))\Phi_{\rm cycl}(w, \bar{w})$. The spin of the field Φ is precisely $s = h - \bar{h}$ if you recall that $l_0 - \bar{l}_0$ generates rotation. Thus, a bosonic field Φ will satisfy the same boundary condition on the plane and on the cylinder, whereas for a fermionic field on the plane Φ with periodic (anti-periodic) boundary condition the field on the cylinder $\Phi_{\rm cyl}$ satisfies anti-periodic (periodic) boundary condition.

Next we construct the torus via discrete identification on the cylinder. Let H and P be the energy and momentum operators which generate translation in the time $\mathrm{Re}\, w = \sigma^2$ and space $\mathrm{Im}\, w = -\sigma^1$ directions respectively. On the plane we know that $l_0 + \bar{l}_0$ and $l_0 - \bar{l}_0$ are the generators of dilatations and rotations, respectively, and thus on the cylinder we must have $H = (l_0)_{\rm cyl} + (\bar{l}_0)_{\rm cyl}$ and $P = (l_0)_{\rm cyl} - (\bar{l}_0)_{\rm cyl}$.

The torus is defined by two periods in w. First we redefine w as $w \longrightarrow iw$, i.e. $z = \exp(iw)$ and $\mathrm{Re}\, w = \sigma^1$, $\mathrm{Im}\, w = \sigma^2$. The first period is then 1 since $w \longrightarrow w + 2\pi$ leaves z invariant. This corresponds to an identification along the real axis and rolling up the complex plane to a cylinder. The second period is taken to be equal to $\tau = \tau_1 + i\tau_2$ where τ is called the modular parameter. Thus the identification $w \equiv w + 2\pi\tau$ along the vector τ in the complex plane rolls up the cylinder to a torus. See figure 16.2. This torus is then effectively a lattice described by the complex number τ. Any field on this torus must then satisfy

$$\Phi(w + 1) = \Phi(w + \tau) = \Phi(w). \tag{16.230}$$

The complex number w can be rewritten conveniently in terms of two real numbers $\tilde{\sigma}^i \in [0, 1[$ each of period 1 as

$$w = \tilde{\sigma}^1 - \tau\tilde{\sigma}^2. \tag{16.231}$$

The torus is then defined by the more transparent equivalence relation

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \sim (\tilde{\sigma}^1 + n, \tilde{\sigma}^2 + m), \;\; n, m \in \mathbf{Z}. \tag{16.232}$$

This torus admits diffeomorphisms which are not connected continuously to the identity given by [36]

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \longrightarrow (a\tilde{\sigma}^1 + b\tilde{\sigma}^2, c\tilde{\sigma}^1 + d\tilde{\sigma}^2). \tag{16.233}$$

The points which are equivalent under (16.232) are mapped under the transformations (16.233) to other points which are also equivalent under (16.232) if and only of a, b, c and d are integers. The transformation (16.233) is invertible and one-to-one if and only if

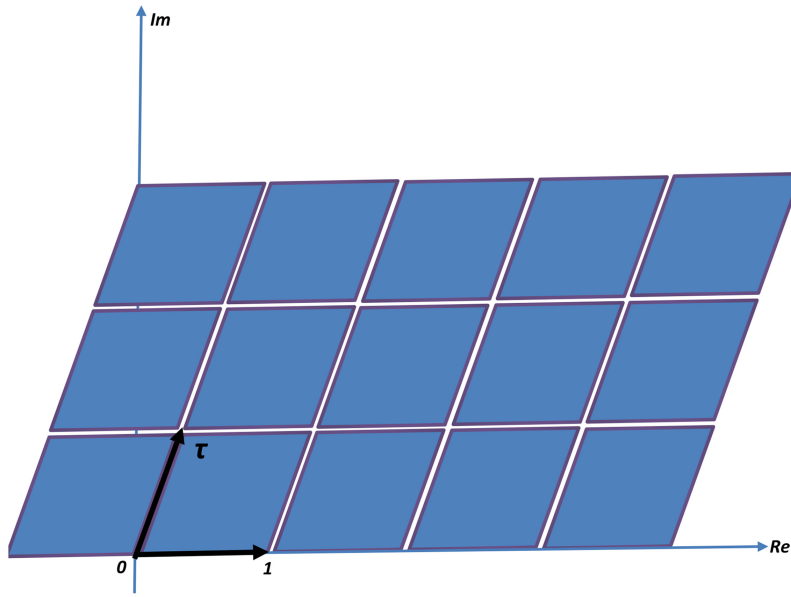

Figure 16.2. The torus.

$$ad - bc = 1. \tag{16.234}$$

This means that the transformation (16.233) defines an element of $SL(2, Z)$ which is the group of reparametrizations of the torus. This is what is called the modular group of the torus. These transformations cannot be reached continuously from the identity by exponentiating infinitesimal reparametrizations. This $SL(2, Z)$ is the part of the reparametrization invariance that is not taken into account in the Faddeev–Popov procedure [36].

Under the modular transformation (16.233) we also have, similar to equation (16.50), the transformation law

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d} \tag{16.235}$$

and

$$\begin{aligned} w \longrightarrow w' &= \tilde{\sigma}^{1'} - \tau' \tilde{\sigma}^{2'} \\ &= (a + c\tau')\tilde{\sigma}^1 + (b + d\tau')\tilde{\sigma}^2 \\ &= \frac{\tilde{\sigma}^1 - \tau\tilde{\sigma}^2}{c\tau + d} \\ &= \frac{w}{c\tau + d}. \end{aligned} \tag{16.236}$$

The modular group can be generated by two special modular transformations T and S. The modular transformation S consists of an inversion in the unit circle $r' = 1/r$

followed by a reflection with respect to the imaginary axis $\theta' = \pi - \theta$. Indeed, the action on the modular parameter τ is given by

$$S: \ \tau \longrightarrow \tau' = -\frac{1}{\tau}. \tag{16.237}$$

This corresponds explicitly to the $SL(2, Z)$ transformation

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \longrightarrow (\tilde{\sigma}^2, -\tilde{\sigma}^1). \tag{16.238}$$

This transformation interchanges then the two basis vectors with a change of sign.

The modular transformation T is a translation given explicitly by

$$T: \ \tau \longrightarrow \tau' = \tau + 1. \tag{16.239}$$

This corresponds explicitly to the $SL(2, Z)$ transformation

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \longrightarrow (\tilde{\sigma}^1 + \tilde{\sigma}^2, \tilde{\sigma}^2). \tag{16.240}$$

The transformations S and T generate the whole modular group. They satisfy $S^2 = (ST)^3 = 1$. The modular group is actually isomorphic to $SL(2, Z)/\mathbf{Z}_2$ since the two elements 1 and -1 are indistinguishable.

16.7.2 Free bosons on a torus

Let us then consider the action

$$S = \frac{1}{2\pi} \int \partial\Phi\bar{\partial}\Phi. \tag{16.241}$$

The measure is normalized such that (with $z = \sigma^1 + \tau\sigma^2$ and $\sigma^{1,2} \in [0, 1[$)

$$\int 1 = \int 2i dz \wedge d\bar{z} = \int 4\tau_2 d\sigma^1 \wedge d\sigma^2 = 4\tau_2. \tag{16.242}$$

We consider compactification on a circle of radius r, viz

$$\Phi = \Phi + 2\pi r. \tag{16.243}$$

The torus has actually two periods, τ and 1. Thus, the periodic boundary conditions are given explicitly by

$$\begin{aligned} \Phi_0(z + \tau, \bar{z} + \bar{\tau}) &= \Phi_0(z, \bar{z}) + 2\pi r n', \ \ \Phi_0(z + 1, \bar{z} + 1) \\ &= \Phi_0(z, \bar{z}) + 2\pi r n. \end{aligned} \tag{16.244}$$

The field space decomposes then into instanton sectors characterized by the pair (n', n). The solution of the equation of motion $\partial\bar{\partial}\Phi_0 = 0$ in the instanton sector (n', n) is given explicitly by

$$\Phi_0(z, \bar{z}) = \frac{2\pi r}{2i\tau_2}[n'(z - \bar{z}) + n(\tau\bar{z} - \bar{\tau}z)]. \tag{16.245}$$

We expand the field as $\Phi = \Phi_0 + Y$ where the fluctuation Y is assumed to vanish at infinity. The action becomes

$$S[\Phi] = S[\Phi_0] + \frac{1}{2\pi}\int Y \Box Y, \tag{16.246}$$

where $\Box = -\partial\bar{\partial}$. The relevant path integral is

$$\int \mathcal{D}\Phi \exp(-S[\Phi]) = \exp(-S[\Phi_0]) \int \mathcal{D}Y \exp\left(-\frac{1}{2\pi}\int Y \Box Y\right). \tag{16.247}$$

The fluctuation Y is split into a constant part $\tilde{Y}$ and a fluctuation Y' containing no zero mode. Thus, we have

$$\begin{aligned}\int \mathcal{D}\Phi \exp(-S[\Phi]) &= \exp(-S[\Phi_0]) \int d\tilde{Y}\mathcal{D}Y' \exp\left(-\frac{1}{2\pi}\int Y' \Box Y'\right) \\ &= 2\pi r \exp(-S[\Phi_0]) \int \mathcal{D}Y' \exp\left(-\frac{1}{2\pi}\int Y' \Box Y'\right).\end{aligned} \tag{16.248}$$

The path integral is normalized such that

$$\begin{aligned}1 &= \int \mathcal{D}X \exp\left(-\frac{1}{2\pi}\int X^2\right) \\ &= \int d\tilde{X} \exp\left(-\frac{1}{2\pi}(4\tau_2)\tilde{X}^2\right) \int \mathcal{D}X' \exp\left(-\frac{1}{2\pi}\int X'^2\right) \\ &= \frac{\pi}{\sqrt{2\tau_2}} \int \mathcal{D}X' \exp\left(-\frac{1}{2\pi}\int X'^2\right)\end{aligned} \tag{16.249}$$

Therefore, the partition function (16.248) on the torus becomes (by taking also the sum over all instanton sectors)

$$\begin{aligned}\int \mathcal{D}\Phi \exp(-S[\Phi]) &= 2\pi r \frac{\sqrt{2\tau_2}}{\pi} \frac{1}{\det{}'^{1/2}\,\Box} \sum_{n',n=-\infty}^{+\infty} \exp(-S[\Phi_0^{(n',n)}]) \\ &= 2\pi r \frac{\sqrt{2\tau_2}}{\pi} \frac{1}{\det{}'^{1/2}\,\Box} \sum_{n',n=-\infty}^{+\infty} \\ &\quad \times \exp\left(4\tau_2 \frac{1}{2\pi}\left(\frac{2\pi r}{2i\tau_2}\right)^2 (n' - \bar{\tau}n)(n' - \tau n)\right).\end{aligned} \tag{16.250}$$

The crucial piece is the determinant. We use the single-valued (under both $z \longrightarrow z + \tau$ and $z \longrightarrow z + 1$) eigenfunctions

$$\psi_{n,m}(z, \bar{z}) = \exp\left(\frac{2\pi r}{2i\tau_2}[n(z - \bar{z}) + m(\tau\bar{z} - \bar{\tau}z)]\right). \tag{16.251}$$

Indeed, we have

$$\Box\psi_{n,m}(z,\bar{z}) = \frac{\pi^2}{\tau_2^2}(n - m\bar{\tau})(n - m\tau)\psi_{n,m}(z,\bar{z}). \tag{16.252}$$

The determinant $\det'\Box$, which does not involve the zero mode, is then given by

$$\begin{aligned}\det'\Box &= \prod_{(m,n)\neq(0,0)} \frac{\pi^2}{\tau_2^2}(n - m\bar{\tau})(n - m\tau) \\ &= \prod_{(m,n)\neq(0,0)} \frac{\pi^2}{\tau_2^2} \prod_{n\neq 0} n^2 \prod_{m\neq 0, n\in Z} (n - m\bar{\tau})(n - m\tau).\end{aligned} \tag{16.253}$$

We use zeta-function regularization $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, $\zeta(-1) = -1/12$, $\zeta(0) = -1/2$, $\zeta'(0) = -\ln\sqrt{2\pi}$. Then

$$\prod_{n=1}^{\infty} a = a^{\zeta(0)} = a^{-1/2} \tag{16.254}$$

$$\prod_{n=1}^{\infty} n^{\alpha} = e^{\alpha \sum_{n=1}^{\infty} \ln n} = e^{-\alpha\zeta'(0)} = (2\pi)^{\alpha/2}. \tag{16.255}$$

The determinant becomes

$$\begin{aligned}\det'\Box &= \frac{\tau_2^2}{\pi^2}(2\pi)^2 \prod_{m>0, n\in Z} (n - m\bar{\tau})(n + m\bar{\tau})(n - m\tau)(n + m\tau) \\ &= \frac{\tau_2^2}{\pi^2}(2\pi)^2 \prod_{m>0} (e^{i\pi m\tau} - e^{-i\pi m\tau})^2 (e^{i\pi m\bar{\tau}} - e^{-i\pi m\bar{\tau}})^2,\end{aligned} \tag{16.256}$$

where we have used the identity

$$\prod_{n\in Z} (n + a) = a \prod_{n=1}^{\infty} (-n^2)(1 - a^2/n^2) = 2i \sin \pi a. \tag{16.257}$$

We get finally

$$\begin{aligned}\det'\Box &= \frac{\tau_2^2}{\pi^2}(2\pi)^2 \prod_{m>0} (q\bar{q})^{-m}(1 - q^m)^2(1 - \bar{q}^m)^2 \\ &= 4\tau_2^2 (q\bar{q})^{1/12} \prod_{m>0} (1 - q^m)^2(1 - \bar{q}^m)^2 \\ &= 4\tau_2^2 \eta^2 \bar{\eta}^2.\end{aligned} \tag{16.258}$$

We have defined

$$\eta = q^{1/24}\prod_{m>0}(1-q^m),\ \ \bar{\eta} = \bar{q}^{1/24}\prod_{m>0}(1-\bar{q}^m),$$
$$q = \exp(2i\pi\tau),\ \ \bar{q} = \exp(-2i\pi\bar{\tau}). \tag{16.259}$$

The partition function on the torus becomes

$$\int \mathcal{D}\Phi \exp(-S[\Phi]) = r\sqrt{\frac{2}{\tau_2}}\frac{1}{\eta\bar{\eta}}\sum_{n',n=-\infty}^{+\infty} \times \exp\left(-2\pi\left[\frac{1}{\tau_2}(n'r-\tau_1 nr)^2 + r^2n^2\tau_2\right]\right). \tag{16.260}$$

The summation over the winding n' in the time direction (corresponding to the period τ) will be converted into a summation over a conjugate momentum by means of Poisson resummation formula

$$\sum_{n'=-\infty}^{+\infty} f(n'r) = \frac{1}{r}\sum_{m=-\infty}^{+\infty}\tilde{f}\left(\frac{m}{r}\right),\ \ \tilde{f}(p) = \int_{-\infty}^{+\infty} dx e^{2\pi i p x} f(x). \tag{16.261}$$

We take the function f and its Fourier transform $\tilde{f}$ to be

$$f(n'r) = \exp\left(-2\pi\frac{1}{\tau_2}(n'r-\tau_1 nr)^2\right) \Rightarrow \tilde{f}(p) = \sqrt{\frac{\tau_2}{2}}\exp\left(2\pi i\tau_1 nrp - \frac{1}{2}\pi\tau_2 p^2\right). \tag{16.262}$$

The partition function becomes

$$\int \mathcal{D}\Phi \exp(-S[\Phi]) = \frac{1}{\eta\bar{\eta}}\sum_{m,n=-\infty}^{+\infty}\exp\left(-2\pi r^2n^2\tau_2 + 2\pi i\tau_1 nm - \frac{1}{2}\pi\tau_2\left(\frac{m}{r}\right)^2\right) = \frac{1}{\eta\bar{\eta}}\sum_{m,n=-\infty}^{+\infty} q^{\frac{1}{2}(\frac{m}{2r}+nr)^2}\bar{q}^{\frac{1}{2}(\frac{m}{2r}-nr)^2}. \tag{16.263}$$

As we know, compactification on a circle leads to a momentum and a winding defined by $p = m/r$ and $w = nr$. The zero modes are given by

$$\alpha_0 = p_L = \frac{p}{2} + w,\ \ \bar{\alpha}_0 = p_R = \frac{p}{2} - w. \tag{16.264}$$

Hence, we get the partition function

$$\int \mathcal{D}\Phi \exp(-S[\Phi]) = \frac{1}{\eta\bar{\eta}}\sum_{m,n=-\infty}^{+\infty} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}. \tag{16.265}$$

We recall that in the quantum theory the generators L_0 and $\bar{L}_0$ are given by

$$L_0 = \sum_{l=1} \alpha_{-l}\alpha_l + \frac{1}{2}\alpha_0^2, \;\; \bar{L}_0 = \sum_{l=1} \bar{\alpha}_{-l}\bar{\alpha}_l + \frac{1}{2}\bar{\alpha}_0{}^2. \tag{16.266}$$

The string ground states $|n, m\rangle$ are characterized by the fact that the oscillators α_l and $\bar{\alpha}_l$ are in their ground state, viz $\alpha_l|n, m\rangle = \bar{\alpha}_l|n, m\rangle = 0$, whereas the string center-of-mass has precisely a non-zero momentum equal m and non-zero winding equal n. In other words, we have

$$L_0|n, m\rangle = \frac{1}{2}\left(\frac{m}{2r} + nr\right)^2 |n, m\rangle, \;\; \bar{L}_0|n, m\rangle = \frac{1}{2}\left(\frac{m}{2r} - nr\right)^2 |n, m\rangle. \tag{16.267}$$

The Hamiltonian and momentum eigenvalues of the string ground states $|n, m\rangle$ are then given by

$$\begin{aligned} H|n, m\rangle &= (L_0 + \bar{L}_0)|n, m\rangle \\ &= \left(\frac{m^2}{4r^2} + n^2r^2\right)|n, m\rangle, \;\; P|n, m\rangle \\ &= (L_0 - \bar{L}_0)|n, m\rangle = nm|n, m\rangle. \end{aligned} \tag{16.268}$$

The partition function (16.265) can be rewritten in terms of the generators L_0 and $\bar{L}_0$ as

$$\int \mathcal{D}\Phi \exp(-S[\Phi]) = (q\bar{q})^{-c/24} \mathrm{tr} q^{L_0} \bar{q}^{\bar{L}_0}. \tag{16.269}$$

This is the most general form of the partition function of the conformal field theory of free bosons on the torus. We can verify modular invariance by studying the effect of the modular transformation $\tau \longrightarrow -1/\tau$ on η.

16.8 Holographic entanglement entropy

16.8.1 Entanglement entropy

In quantum mechanics it is shown by the EPR experiment for example that entanglement is at odds with locality. The action (due to a measurement) seems to propagate with an infinite velocity and although it cannot carry any energy we are left in an uncomfortable position. Entanglement as opposed to energy is not conserved and there are degrees of entanglement. Mathematically, entanglement means that the vector state is not separable, i.e. it cannot be written as a tensor product.

Quantum entanglement is measured by entropy or more precisely by entanglement entropy. However, entropy has actually two sources: statistical and quantum.

1. **Statistical/thermal entropy**: The thermal or Boltzmann entropy of a macroscopic state is the logarithm of the number n of microscopic states consistent with this state. Thus, this entropy measures the lack of resolution, i.e. the fact that a large number of microscopic configurations correspond to the same macroscopic thermodynamical state. The thermal entropy is defined in terms of the Boltzmann density matrix $\rho_{\text{ther}} = \exp(-\beta E)/Z$ by

$$\begin{aligned} S_{\text{ther}} &= -\,\text{tr}\rho_{\text{ther}}\,\log\rho_{\text{ther}} \\ &= \log n. \end{aligned} \tag{16.270}$$

The second equality holds if the microstates are equally probable.

2. **Entanglement entropy:**
   - **Measurement**: In quantum mechanics, there is another source of entropy associated with the restriction of observers, who are performing the experiments, to finite volume. Indeed, a typical observer performing an experiment on a closed system, which is supposed to be in a pure ground state $|\Psi\rangle$, will only be able to access a particular subsystem, i.e. a partial set of the relevant observables such as those with support in a restricted volume.

     We will denote the accessible subsystem by A (where the observers are restricted) and the inaccessible subsystem is B. The total system $\Sigma = A \cup B$ is in a pure ground state $|\Psi\rangle$. See figure 16.3.
   - **Reduced density matrix:** The state of the system will be given by a mixed density matrix ρ and the entropy will measure the correlation between the inaccessible subsystem B and the accessible part A of the closed system. The total Hilbert space is $\mathcal{H}_{\text{Tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$.

     The observer who cannot access the subsystem B will describe the total system by the reduced density matrix (obtained by tracing over the inaccessible degrees of freedom)

$$\rho_{\text{Red}} \equiv \rho_A = \text{tr}_B\rho_{\text{Tot}}\,. \tag{16.271}$$

     In other words, we trace (integrate) over the inaccessible subsystem B, i.e. we take average over the inaccessible degrees of freedom.
   - **Mixed versus pure:** The reduced density matrix is an incoherent (mixed) superposition (statistical ensemble, classical probabilities, no interference terms, random relative phases). It is not an idempotent and it satisfies $\text{Tr}\,\rho^2 < 1$.

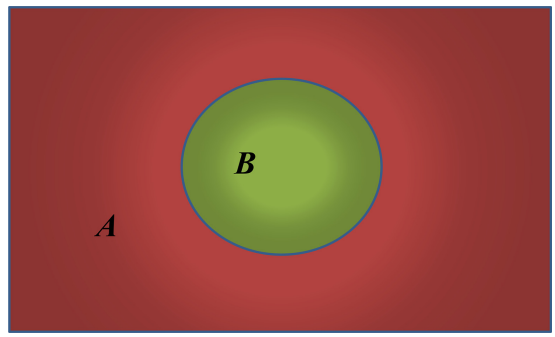

Figure 16.3. The accessible and inaccessible regions.

In contrast, a pure state is a vector in the Hilbert space which is a coherent superposition (interference terms, coherent relative phases) represented by a projector.

Mixed states are relevant if the exact initial state vector is unknown.

- **Entanglement entropy:** The entropy of the subsystem A which measures the correlation between the inaccessible subsystem and the accessible part of the closed system is defined by the von Neumann entropy of this reduced density matrix, viz

$$S_{\text{Red}} \equiv S_A = -\text{tr}_A \rho_A \log \rho_A = -\sum_i \rho_i \log \rho_i. \tag{16.272}$$

Thus, entanglement entropy is the logarithm of the number of microscopic states of the inaccessible subsystem B which are consistent with observations restricted to the accessible subsystem A, together with the assumption that the total system is in a pure state. It measures the degree of entanglement between A and B. This is different from the thermodynamic Boltzmann entropy.

- **Examples:** For a pure (separable) state, i.e. when all eigenvalues with the exception of one vanish, we get $S = 0$. For mixed states we have $S > 0$.

In the case of a totally incoherent mixed density matrix in which all the eigenvalues are equal to $1/N$ where N is the dimension of the Hilbert space we get the maximum value of the von Neumann entropy given by

$$S_{\text{Red}} = S_{\text{max}} = \log N. \tag{16.273}$$

In the case that ρ is proportional to a projection operator onto a subspace of dimension n we find

$$S_{\text{Red}} = \log n. \tag{16.274}$$

In other words, the von Neumann entropy measures the number of important states in the statistical ensemble, i.e. those states which have an appreciable probability. This entropy is also a measure of the degree of entanglement between subsystems A and B and hence, its other name, entanglement entropy.

- **Information:** The von Neumann entropy $S_V \equiv S_A$ is not additive as opposed to the thermal entropy S_B defined with respect to Boltzmann distribution. We have $S_B \geqslant S_V$, i.e. the Boltzmann thermal entropy (coarse grained, macroscopic) is always greater or equal to von Neumann entanglement (fine grained, microscopic) entropy.

The amount of information is the difference:

$$I = S_B - S_V. \tag{16.275}$$

If $\Sigma = A$ then there is no entanglement and the amount of information is maximal, i.e. $S_V = 0 \Rightarrow I = S_B$. If $A \ll B$ then in this case the amount

of information is zero, i.e. $S_V = S_B$. Equivalently, if $A \ll B$ then the entanglement entropy becomes maximal equal to the thermal entanglement.

Note that the von Neumann entropy of the total system is zero, viz $S_{\rm Tot} = {\rm tr}\rho_{\rm Tot} \log \rho_{\rm Tot} = 0$ since there is no inaccessible part here.

16.8.2 Entanglement entropy in quantum mechanics

For details of the formalism used here we refer to [37]. We will consider a Hamiltonian of the form

$$H = \frac{1}{2}\sum_{A,B}(\delta_{A,\,B}\pi^A\pi^B + V_{AB}\varphi^A\varphi^B). \tag{16.276}$$

In this equation V is a real symmetric matrix with positive definite eigenvalues. The normalized ground state of this model is given in the Schrödinger representation by

$$\langle\varphi^A|0\rangle = \left[\det\frac{W}{\pi}\right]^{1/4}\exp\left[-\frac{1}{2}W_{AB}\varphi^A\varphi^B\right]. \tag{16.277}$$

W is the square root of the matrix V. The corresponding density matrix is

$$\rho(\varphi,\,\varphi') = \left[\det\frac{W}{\pi}\right]^{1/2}\exp\left[-\frac{1}{2}W_{AB}(\varphi^A\varphi^B + \varphi'^A\varphi'^B)\right] \tag{16.278}$$

If we suppose that the field degrees of freedom φ^α, $\alpha = \overline{1,\,n}$, are inaccessible then the correct description of the state of the system will be given by the reduced density matrix in which we integrate out these inaccessible degrees of freedom, viz

$$\rho_{\rm red}(\varphi^{n+1},\,\varphi^{n+2},\,\ldots,\,\varphi'^{n+1},\,\varphi'^{n+2},\,\ldots) = \int\prod_{\alpha=1}^{n}d\varphi^\alpha\rho(\varphi,\,\varphi') \tag{16.279}$$

The entanglement entropy is the associated von Newman entropy of $\rho_{\rm red}$ defined by $S = -{\rm Tr}\,\rho_{\rm red}\log\rho_{\rm red}$. The entanglement entropy for any Hamiltonian of the form (16.276) can be shown to be given by [37]

$$S_{\rm ent} = \sum_i\left[\log\left(\frac{1}{2}\sqrt{\lambda_i}\right) + \sqrt{1+\lambda_i}\,\log\left(\frac{1}{\sqrt{\lambda_i}} + \sqrt{1+\frac{1}{\lambda_i}}\right)\right]. \tag{16.280}$$

The λ_i are the eigenvalues of the following matrix

$$\Lambda_{i,j} = -\sum_{\alpha=1}^{n}W^{-1}_{i\alpha}W_{\alpha j} \tag{16.281}$$

$W_{\alpha j}$ and $W^{-1}_{i\alpha}$ are elements of W and W^{-1}, respectively, with i, j running from $n+1$ to $\mathcal{N}$ and α from 1 to n, i.e. Λ is an $(\mathcal{N}-n)\times(\mathcal{N}-n)$ matrix and i, j run from $n+1$ to N.

16.8.3 Entanglement entropy in conformal field theory

In this section we follow mostly [38] and [39]. See also [40].

The entropy of a macroscopic state, in statistical mechanics, is defined by the logarithm of the number of microscopic states which are consistent with it. Thus, this entropy measures the lack of resolution, i.e. the fact that a large number of microscopic configurations correspond to the same macroscopic thermodynamical state.

However, in quantum mechanics, there is another source of entropy associated with the restriction of observers, who are performing the experiments, to finite volume. Indeed, a typical observer performing an experiment on a closed system, which is supposed to be in a pure ground state $|\Psi\rangle$, will only be able to access a particular subsystem, i.e. a partial set of the relevant observables such as those with support in a restricted volume.

We will denote the accessible subsystem by A and the inaccessible subsystem by B (see figure 16.3). In this case the state of the system will be given by a mixed density matrix ρ and the entropy will measure the correlation between the inaccessible subsystem and the accessible part of the closed system. The total Hilbert space is clearly given by $\mathcal{H}_{\text{Tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$. The observer who cannot access the subsystem B will describe the total system not by the ground state $|\Psi\rangle$ (or its corresponding density matrix $\rho_{\text{Tot}} = |\Psi\rangle\langle\Psi|$) but by the reduced density matrix

$$\rho_{\text{Red}} \equiv \rho_A = \text{tr}_B \rho_{\text{Tot}}. \tag{16.282}$$

In other words, we trace (integrate) over the inaccessible subsystem B, i.e. we take the average over the inaccessible degrees of freedom. The entanglement entropy of the subsystem A is defined by the von Neumann entropy of this reduced density matrix, viz

$$S_{\text{Red}} \equiv S_A = -\text{tr}_A \rho_A \log \rho_A. \tag{16.283}$$

The entanglement entropy is then essentially the logarithm of the number of microscopic states of the inaccessible part of the system which are consistent with the observations restricted to the accessible subsystem, together with the assumption that the total system is in a pure state. It measures, as we said, the degree of correlation (entanglement) between the accessible subsystem and the inaccessible part of the total system.

Note that the von Neumann entropy of the total system is zero, viz $S_{\text{Tot}} = \text{tr}\rho_{\text{Tot}} \log \rho_{\text{Tot}} = 0$ since there is no inaccessible part here.

We assume now a conformal field theory in two dimensions with complex coordinate $z = \sigma + i\tau$. The spatial dimension σ is given by $C = [0, L[$ where L is an infrared cutoff and we will assume periodic boundary condition, i.e. $L \equiv 0$. The subsystem playing the role of the accessible region (where measurements are performed) is $A = [0, \Sigma[$, whereas the unavailable region (to be traced over) is $B = [\Sigma, L[$. The ultraviolet cutoffs are introduced by considering instead the intervals $A = [\epsilon_1, \Sigma - \epsilon_2[$ and $B = [\Sigma + \epsilon_2, L - \epsilon_1[$. We perform the conformal mapping

$$z \longrightarrow w = -\frac{\sin\frac{\pi}{L}(z-\Sigma)}{\sin\frac{\pi}{L}z}. \tag{16.284}$$

The regularized intervals A and B become (with the assumption $\Sigma \ll L$) the positive half-axis and the negative half-axis, respectively, viz

$$A =]\epsilon_2/\Sigma,\ \Sigma/\epsilon_1],\ \ B =]-\Sigma/\epsilon_1,\ -\epsilon_2/\Sigma]. \tag{16.285}$$

This is spatial section at $\tau = 0$. Extrapolation into the past $\tau \longrightarrow -\infty$ corresponds to extrapolation to the lower half-plane with inner and outer radii ϵ_2/Σ and Σ/ϵ_1 respectively.

Lastly, we perform the conformal mapping

$$w \longrightarrow y = \frac{1}{\kappa}\ln w. \tag{16.286}$$

Our points from $\tau = -\infty$ to $\tau = 0$ are given by $w = R\exp(i\theta)$ where R ranges from ϵ_2/Σ to Σ/ϵ_1 and θ ranges from $-\pi$ to 0. Hence, $y = \ln R/\kappa + i\theta/\kappa$. We have then

$$\theta = \text{fixed},\ \ \Delta y = \frac{1}{\kappa}\ln\frac{\Sigma}{\epsilon_1} - \frac{1}{\kappa}\ln\frac{\epsilon_2}{\Sigma} = \frac{2}{\kappa}\ln\frac{\Sigma}{\sqrt{\epsilon_1\epsilon_2}} \equiv L \tag{16.287}$$

$$R = \text{fixed},\ \ \Delta y = i\frac{\Delta\theta}{\kappa} = i\frac{\pi}{\kappa} \equiv ih. \tag{16.288}$$

This is a finite strip of length L and width h. The accessible region A corresponds to fixed θ between $-\pi/2$ and 0 and thus corresponds to the upper side of the strip, whereas the inaccessible region B corresponds to fixed θ between $-\pi$ and $-\pi/2$ and thus to the lower side of the strip. The 'upper' and 'lower' are with respect to the width direction h.

The ground state wave functional $\Psi(\phi_0(x))$ can be defined via a path integral with an appropriate boundary condition specifying the field on the Cauchy surface $C = A \cup B$. Explicitly we have

$$\Psi(\phi_0(x)) = \int_{\tau=-\infty}^{\tau=0} \mathcal{D}\phi\,\exp(-S(\phi)),\ \ \phi_0(x) = \phi(0,\,x). \tag{16.289}$$

The field is also assumed to vanish in the limit $\tau \longrightarrow -\infty$. The complex conjugate wave functional $\bar{\Psi}(\phi_0'(x))$ is given similarly by

$$\bar{\Psi}(\phi_0'(x)) = \int_{\tau=0}^{\tau=+\infty} \mathcal{D}\phi\,\exp(-S(\phi)),\ \ \phi_0'(x) = \phi'(0,\,x). \tag{16.290}$$

We can write $\phi_0 = XY$ where $\phi_0 = X$ on A (upper side of the first copy of the strip) and $\phi_0 = Y$ on B (lower side of this strip) and similarly we can write $\phi_0' = X'Y'$ where $\phi_0' = X'$ on A (lower side of another copy of the strip) and $\phi_0' = Y'$ on B (upper side of the second copy of the strip).

The total density matrix is then given by

$$\rho_{\phi_0\phi_0'} = \Psi(\phi_0(x))\bar{\Psi}(\phi_0'(x)). \tag{16.291}$$

However, the reduced density matrix ρ_A, which describes the observations from the subsystem A, is obtained by tracing ρ over the inaccessible subsystem B. Thus, we need to integrate ϕ_0 on the region B with the condition $\phi_0(x) = \phi_0'(x)$ when $x \in B$. Then the reduced density matrix is given by

$$(\rho_A)_{XX'} = \int \mathcal{D}Y \Psi(X,\ Y)\bar{\Psi}(Y,\ X'). \tag{16.292}$$

This is generally a mixed density matrix as opposed to the total density matrix ρ which is a pure density matrix. This integral involves pasting together two copies of the strip along the inaccessible region B. Thus, it corresponds to a functional integral over a strip of height $2\pi/\kappa$ with boundary conditions given by $\phi_0 = X$ on the upper side of the strip and $\phi_0 = X'$ on the lower side of the strip given by

$$(\rho_A)_{XX'} = \frac{1}{Z_1}\int \mathcal{D}\phi \exp(-S(\phi)). \tag{16.293}$$

The path integral Z_1 is determined by the normalization condition $\mathrm{tr}\rho_A = 1$. This is clearly periodic in the height direction h since the fields are identified by the tracing. If we also impose a periodic boundary condition in the length direction L then Z_1 is nothing other than the partition function on a torus.

The entropy is actually going to be calculated using the so-called replica trick given by the relation

$$S_A = -\mathrm{tr}\rho_A \log \rho_A = -\frac{\partial}{\partial n}\mathrm{tr}_A\rho_A^n|_{n=1}. \tag{16.294}$$

The trace $\mathrm{tr}_A\rho_A^n$ is computed by pasting together n copies of the strip along the inaccessible region B. We start thus from n copies of the reduced density matrix, viz

$$(\rho_A)_{X_{1+}X_{1-}}(\rho_A)_{X_{2+}X_{2-}} \cdots (\rho_A)_{X_{n+}X_{n-}}. \tag{16.295}$$

The pasting or gluing is done by the conditions $X_{i-}(x) = X_{i+1+}(x)$, $\forall\, i = 1,\ \ldots,\ n-1$, and then integrating over X_{i+}. If we choose $X_{1+} = X$ and $X_{n-} = X'$ then we obtain the matrix element

$$(\rho_A^n)_{XX'} = \frac{1}{Z_1^n}\int \mathcal{D}\phi \exp(-S(\phi)). \tag{16.296}$$

The functional integral is over a strip of width $2\pi n/\kappa$ with boundary conditions given by $\phi_0 = X$ on the upper side of the strip and $\phi_0 = X'$ on the lower side of the strip. By setting $X = X'$ and integrating over X we obtain the desired trace as

$$\mathrm{tr}_A\rho_A^n = \frac{1}{Z_1^n}\int \mathcal{D}X \int \mathcal{D}\phi \exp(-S(\phi)) = \frac{Z_n}{Z_1^n}. \tag{16.297}$$

The Z_n is the partition function on the torus of lengths $2\pi n/\kappa$ and L around its two cycles. The entanglement entropy S_A takes finally the form

$$S_A = \left(\left[1 - n\frac{\partial}{\partial n}\right] \ln Z_n\right)|_{n=1}. \tag{16.298}$$

16.8.4 Ryu–Takayanagi formula

In this section we follow mainly [39].

The Bekenstein–Hawking formula states that the entropy of a black hole $\mathcal{S}_{BH}$ is proportional to the surface area of the event horizon A_H, viz

$$\mathcal{S}_{BH} = \frac{A_H}{4G_N}. \tag{16.299}$$

On the other hand, the entanglement entropy S_A for observers accessible to a subsystem A (outside event horizon) who cannot receive any signals from the subsystem B (inside the black hole) is given by

$$S_A = -\mathrm{tr}_A \rho_A \log \rho_A. \tag{16.300}$$

The entanglement entropy satisfies the following properties:

For three subsystems A, B and C, which do not intersect each other, we have the so-called strong subadditivity relations

$$\begin{aligned} &S_{A+B+C} + S_B \leqslant S_{A+B} + S_{B+C} \\ &S_A + S_C \leqslant S_{A+B} + S_{B+C}. \end{aligned} \tag{16.301}$$

By choosing B empty in the above relations we obtain

$$S_{A+B} \leqslant S_A + S_B. \tag{16.302}$$

The mutual information is defined by

$$I(A, B) = S_A + S_B - S_{A+B} \geqslant 0. \tag{16.303}$$

If we choose B to be the complement of A then

$$S_A = S_B \Rightarrow S_{A+B} \leqslant 2S_A. \tag{16.304}$$

Hence, the entanglement entropy is not an extensive quantity.

In a QFT on a $(d+1)$-dimensional manifold $\mathbf{R} \times \mathbf{N}$ where $d \geqslant 2$ and $\mathbf{N} = A \cup B$ it is found that the entanglement entropy (1) depends only on the geometry of A (this is why entanglement entropy is also called geometric entropy), (2) is UV divergent and hence the continuum theory should be regularized by a lattice a, and (3) it is proportional to the area of the boundary ∂A of A since the entanglement between A and B occurs strongly obviously on the boundary. We have explicitly [37, 41]

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{subleading terms}. \tag{16.305}$$

This entanglement entropy formula (includes UV divergences, proportional to the number of matter fields) is very similar to the Bekenstein–Hawking formula (does not include UV divergences, is not proportional to the number of matter fields). In fact, the quantum corrections to the Bekenstein–Hawking black hole entropy in the presence of matter fields is given by the entanglement entropy [42–45].

The Ryu–Takayanagi formula is a generalization of the Bekenstein–Hawking formula, based on the $\text{AdS}_{d+2}/\text{CFT}_{d+1}$ correspondence, in which we identify the entanglement entropy in $(d+1)$-dimensional QFT with a geometric quantity in $(d+2)$-dimensional gravity.

We consider the metric in AdS_{d+2} in a Poincaré patch given by

$$ds^2 = g^{\mu\nu} dx_\mu dx_\nu = \frac{R^2}{z^2}\left(-dt^2 + \sum_{i=1}^{d} dx_i^2 + dz^2\right). \tag{16.306}$$

The dual $(d+1)$-dimensional CFT lives on the boundary located at $z = 0$. The radial coordinate z as we have seen is a lattice spacing and the theory on the boundary should be properly understood as the continuum limit (in the sense of RG) of a regularize CFT, i.e. with a cutoff Λ. This regularized CFT lives on a surface $z = a$ where $a = 1/\Lambda$ and $a \longrightarrow 0$.

Our observers live on the boundary $z = a$. The accessible region A and the inaccessible region B are both on this boundary $z = a$. The entanglement entropy in the CFT_{d+1} which lives on this boundary can be computed from the gravity theory which lives in the bulk AdS_{d+2} as follows. We extend $\mathbf{N} = A \cup B$ to the time slice M of the bulk spacetime and we extend ∂A to a d-dimensional surface $\Gamma_A \in M$ such that $\partial\Gamma_A = \partial A$ (figure 16.4). The time slice M is the $(d+1)$-dimensional hyperbolic space $\mathbf{H}_{d+1}$ which is a Euclidean manifold, whereas Γ_A is a minimal area surface. The entanglement entropy in the CFT_{d+1} is then given by the formula [46, 47]

$$S_A = \frac{\text{Area}(\Gamma_A)}{4G_N^{(d+2)}}. \tag{16.307}$$

As an example, we consider the case of AdS_3. The dual field theory is a two-dimensional conformal field theory with central charge c. The corresponding partition function is given by the formula (16.269). This is the partition function of the conformal field theory of free bosons on the torus with periods 1 (real axis) and τ (imaginary axis). We have then

$$Z(\tau, \bar{\tau}) = (q\bar{q})^{-c/24} \text{tr} q^{L_0} \bar{q}^{\bar{L}_0}. \tag{16.308}$$

The variables q and $\bar{q}$ are given in terms of the modular parameter τ by the relations

$$q = \exp(2i\pi\tau), \quad \bar{q} = \exp(-2i\pi\bar{\tau}). \tag{16.309}$$

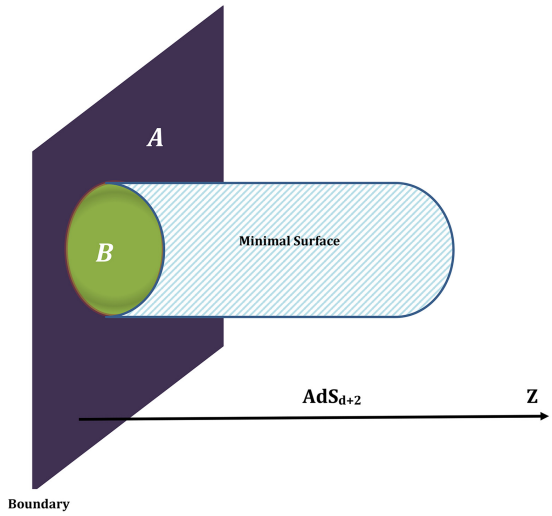

Figure 16.4. The accessible and inaccessible regions in AdS.

On the other hand, the entanglement entropy is given in terms of the partition function Z_n on the torus of lengths $2\pi n/\kappa$ and L around its two cycles by the relation

$$S_A = \left(\left[1 - n\frac{\partial}{\partial n}\right]\ln Z_n\right)|_{n=1}. \tag{16.310}$$

We can take the modular parameter τ to be $\tau = (i2\pi n/\kappa)/L = 2i\pi n/\kappa L$. But the partition function is invariant under the modular transformation $\tau \longrightarrow -1/\tau$ and hence we can take τ to be $\tau = i\kappa L/2\pi n$. In other words, $q = \bar{q} = \exp(-\kappa L/n)$. The entanglement entropy becomes

$$\begin{aligned} S_A &= \left(1 + \ln q\frac{\partial}{\partial \ln q}\right)\ln Z(\tau, \bar{\tau}) \\ &= \left(1 + \ln q\frac{\partial}{\partial \ln q}\right)\left(-\frac{c}{12}\ln q + \ln \mathrm{tr} q^{L_0+\bar{L}_0}\right) \\ &= -\frac{c}{6}\ln q + \left(1 + \ln q\frac{\partial}{\partial \ln q}\right)\ln \mathrm{tr} q^{L_0+\bar{L}_0} \\ &= -\frac{c}{6}(-\kappa L) \\ &= \frac{c}{3}\ln\frac{\Sigma}{\epsilon}. \end{aligned} \tag{16.311}$$

In going from the third to the fourth lines we have assumed that $\ln \mathrm{tr} q^{L_0+\bar{L}_0}$ is exponentially suppressed, whereas in the last line we have used the result (16.287).

This result can be re-derived from the gravity side as follows.

The metric in AdS_3 can be given by (Poincaré coordinates)

$$ds^2 = g^{\mu\nu}dx_\mu dx_\nu = \frac{R^2}{z^2}(-dt^2 + dx^2 + dz^2). \tag{16.312}$$

The boundary lies at $z = 0$. We will regularize by taking the restriction $z \geqslant a$. On the boundary $z = a$ we are interested in the line segment $x \in [-l/2, l/2]$. This segment is extended in the bulk (the plane xz) to a line of minimal length, i.e. a space-like geodesic, which can be found as follows. The length is written as (fixed time)

$$L = \int \frac{R}{z}\sqrt{\dot{z}^2 + \dot{x}^2}\, ds. \tag{16.313}$$

Then we write Euler–Lagrange equations for x and z as (with c a constant)

$$\begin{aligned} x &\longrightarrow \frac{\dot{x}}{z} = c\sqrt{\dot{z}^2 + \dot{x}^2} \\ z &\longrightarrow \frac{\dot{x}^2}{z^2} = -\frac{\sqrt{\dot{z}^2 + \dot{x}^2}}{z}\frac{d}{ds}\left(\frac{\dot{z}}{\sqrt{\dot{z}^2 + \dot{x}^2}}\right). \end{aligned} \tag{16.314}$$

The solution is given by the circle

$$z = A\sin s, \;\; x = A\cos s. \tag{16.315}$$

By imposing the boundary conditions that the circle starts at $(a, +l/2)$ and terminates at $(a, -l/2)$ we obtain

$$z = \frac{l}{2}\sin s, \;\; x = \frac{l}{2}\cos s, \;\; \epsilon \leqslant s \leqslant \pi - \epsilon, \;\; \epsilon = \frac{2a}{l} \longrightarrow 0. \tag{16.316}$$

We compute now the actual length of the minimal line in the bulk as

$$L = \int_\epsilon^{\pi-\epsilon} \frac{R}{\sin s} ds = 2R\ln\frac{l}{a}. \tag{16.317}$$

The central charge c of CFT_2 is related to the radius R of AdS_3 by the relation (16.166). The proportionality factor is given precisely by [48]

$$c = \frac{3R}{2G_N^{(3)}}. \tag{16.318}$$

The entanglement entropy becomes

$$S_A = \frac{L}{4G_N^{(3)}} = \frac{c}{3}\ln\frac{l}{a}. \tag{16.319}$$

16.9 Einstein's gravity from quantum entanglement

A sample of the original literature for this section is [4–7]. However, a very good, concise and pedagogical review of the formalism relating spacetime geometry to

quantum entanglement due to Van Raamsdonk and collaborators can be found in [8].

16.9.1 The CFT/black hole correspondence

The starting point is the statement that Einstein's theory of general relativity on anti-de Sitter spacetime AdS_{d+1} is dual to a conformal field theory CFT_d on its boundary M^d. Let $|\psi(0)\rangle$ be the vacuum state of the CFT_d. This state is dual to the Poincaré patch of the pure AdS spacetime given by the metric

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_\mu dx^\mu). \tag{16.320}$$

Let $|\psi(\zeta)\rangle$ be a one-parameter family of CFT_d excited states which are dual to the perturbed metrics

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \Gamma_{\mu\nu}(z, x)dx^\mu dx^\nu). \tag{16.321}$$

This corresponds to a spacetime M_ζ with boundary at $z \longrightarrow 0$ denoted by ∂M_ζ where the state $|\psi(\zeta)\rangle$ is living.Thus, $\Gamma_{\mu\nu} \longrightarrow \eta_{\mu\nu}$ and $M_\zeta \longrightarrow \text{AdS}_{d+1}$ when $\zeta \longrightarrow 0$. For small z (near the boundary) the metric behaves as

$$\begin{aligned}\Gamma_{\mu\nu}(z, x) &= \eta_{\mu\nu} + z^d \bar{h}_{\mu\nu}(z, x) \Rightarrow ds^2 \\ &= \frac{L^2}{z^2}(dz^2 + dx_\mu dx^\mu + z^d \bar{h}_{\mu\nu}(z, x)dx^\mu dx^\nu).\end{aligned} \tag{16.322}$$

This is called the Fefferman–Graham coordinates.

However, this metric can also be understood as corresponding to a spacetime M_ζ, which is a perturbation of pure AdS, dual to a small perturbation $|\psi(\zeta)\rangle_{\zeta \longrightarrow 0}$ of the CFT_d vacuum $|\psi(0)\rangle$. This is an asymptotically AdS spacetime.

For higher excited states $|\psi(\zeta)\rangle$ we cannot assume a classical supergravity solution since l_s is no longer much less than L and as a consequence stringy corrections of the order l_s^2 and higher become important. The geometry (and even the topology) of AdS_{d+1} becomes, therefore, very different.

An example of a non-trivial dual spacetime is the Schwarzschild–AdS black hole in $d + 1$ dimensions given by the metric

$$ds^2 = -f_M(r)dt^2 + \frac{dr^2}{f_M(r)} + r^2 d\Omega_{d-2}. \tag{16.323}$$

The function f_M is given by the difference of two pieces (the first being the usual Schwarzschild term)

$$f_M(r) = 1 - \frac{2\mu}{r^{d-3}} + \frac{r^2}{L^2},\ \mu = \frac{8\pi G_N M}{(d-1)\text{Vol}(\mathbf{S}^{d-1})},\ \mu_{d=4} = G_N M. \tag{16.324}$$

This is an eternal black hole in $d+1$ dimensions which is asymptotically a pure AdS in contrast to the eternal Schwarzschild black hole which is asymptotically a flat Minkowski spacetime.

Indeed, if we set $L=\infty$ we obtain Schwarzschild black hole. This is characterized by the Penrose diagram 16.5 which summarizes the causal structure of the maximally extended Schwarzschild spacetime in the Kruskal–Szekeres coordinates $-\infty < R < +\infty$ and $-\infty < T < +\infty$. The two light-like infinities $\mathcal{J}^{\pm}$ (where light rays begin and end) and the space-like infinity i^0 ($r=\infty$) are the same as those of Minkowski spacetime and hence the eternal Schwarzschild black hole is asymptotically a flat Minkowski spacetime. The time-like trajectories begin and end on the two time-like infinities i^- and $i+$, respectively, which are distinct surfaces from the singularity at $r=0$. The horizon is located at $r_h=(2\mu)^{1/d-3}$. Region II is the interior of the black hole, whereas region I is the exterior. Region III lies also outside the black hole but it is spatially separated and, therefore, causally disconnected from region I. Region IV is the interior of a white hole, i.e. we can never go there but things can emerge from that region towards region I.

The Penrose diagram of the Schwarzschild–AdS spacetime is shown in figure 16.6. The two light-like infinities $\mathcal{J}^{\pm}$ and the space-like infinity i^0 are replaced with the universal covering of global AdS spacetime in both regions I and III. These asymptotic regions are denoted A and B and they are the conformal boundary of AdS spacetime, i.e. $\mathbf{R}\times\mathbf{S}^{d-1}$. We have two possible situations:

- For $r_h \ll L$ we can neglect the term r^2/L^2 and we end up with an ordinary Schwarzschild spacetime. The Schwarzschild black hole at the center will then evaporate due to Hawking emission.
- For $r_h \gg L$ the Schwarzschild–AdS spacetime is in equilibrium with its Hawking radiation, i.e. emission and absorption rates are equal and the

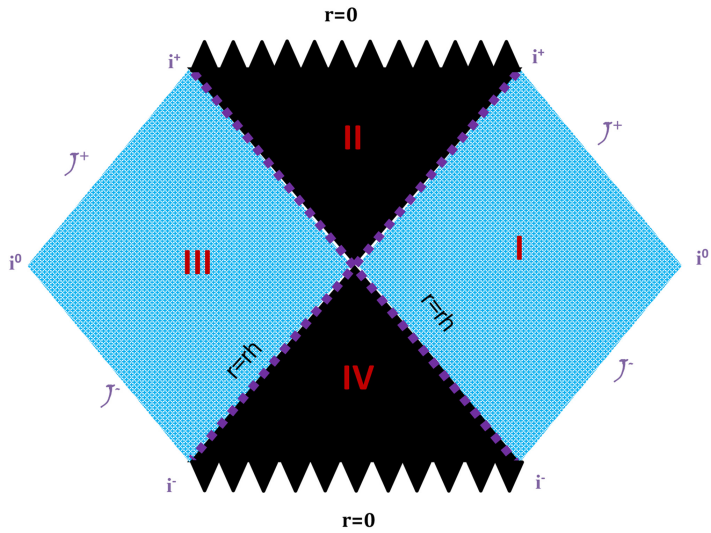

Figure 16.5. The Schwarzschild spacetime.

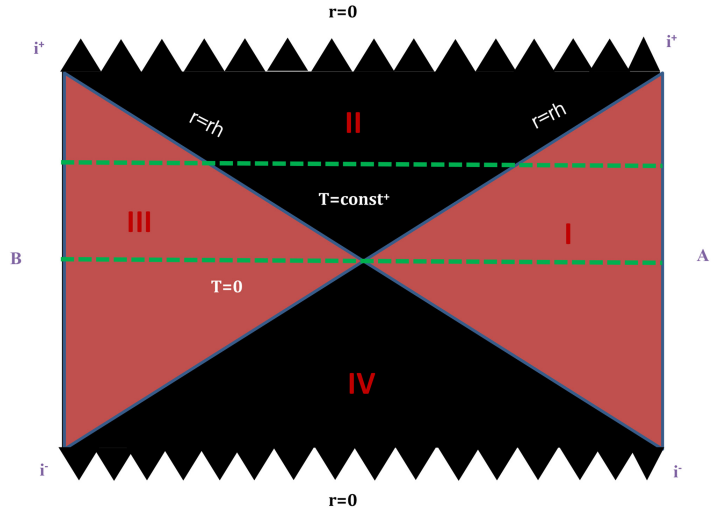

Figure 16.6. The Schwarzschild–AdS spacetime.

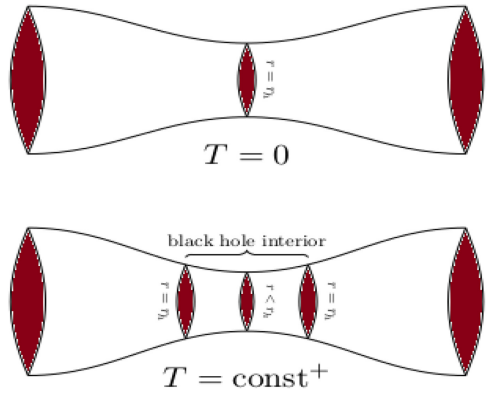

Figure 16.7. The two-sided Schwarzschild–AdS black hole as a wormhole.

black hole will not evaporate. This is due to the fact that radiation can reach the asymptotic AdS boundary and return in a finite time.

Regions I and III are causally disconnected but signals from region I can intersect with signals from region III behind the horizon. This is then a two-sided black hole, i.e. with two different exterior regions, which can also be viewed as a wormhole, as depicted in figure 16.7 for $T = 0$ and $T > 0$.

The Schwarzschild–AdS black hole in $d + 1$ dimensions is conjectured in [49] to be dual to the thermofield double state $|\Psi\rangle$ of two identical non-interacting copies of the conformal field theory CFT_d living on the cylinder $\mathbf{R} \times \mathbf{S}^{d-1}$ (the asymptotic boundaries A and B).

The CFT dual to the Schwarzschild–AdS black is defined, therefore, on $A \cup B$ and as a consequence the quantum system of interest is constituted of two subsystems Q_A and Q_B which contain the degrees of freedom of the local CFT living on A and B respectively. The two subsystems Q_A and Q_B can only interact via entanglement.

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be the Hilbert spaces associated with Q_A and Q_B, respectively, and let $\{|E_i^A\rangle\}$ and $\{|E_i^B\rangle\}$ be the corresponding bases. The thermofield double state $|\Psi\rangle$ dual to the Schwarzschild-AdS black hole is then given by

$$|\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_i \exp(-\beta E_i/2) \left| E_i^A \right\rangle \otimes \left| E_i^B \right\rangle. \tag{16.325}$$

The partition function at inverse temperature β is given by

$$Z(\beta) = \sum_i \exp(-\beta E_i). \tag{16.326}$$

The reduced density matrix for the subsystem Q_A is immediately given by integrating out the degrees of freedom associated with the subsystem Q_B, viz

$$\rho_A = \mathrm{tr}_B |\Psi\rangle\langle\Psi| = \frac{1}{\sqrt{Z(\beta)}} \sum_i \exp(-\beta E_i) \left| E_i^A \right\rangle \left\langle E_i^A \right|. \tag{16.327}$$

The corresponding entanglement entropy is given by

$$\begin{aligned} S_A &= -\,\mathrm{tr}_A \rho_A \log \rho_A \\ &= \frac{1}{\sum_i e^{-\beta E_i}} \sum_i e^{-\beta E_i} \beta E_i + \log \left(\sum_i e^{-\beta E_i} \right). \end{aligned} \tag{16.328}$$

This entanglement entropy is always non-zero except in the limit $\beta \longrightarrow \infty$. Indeed, in the limit of zero temperature the reduced density matrix ρ_A approaches $\rho_A = |E_0^A\rangle\langle E_0^A|$ where E_0 denotes the energy of the ground state and $E_0 = 0$. Hence, in this limit only the ground state is occupied and as a consequence the entanglement entropy vanishes. We have then in the limit of zero temperature the behavior

$$\begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{Z(\beta)}} \sum_i \exp(-\beta E_i/2) |E_i^A\rangle \otimes |E_i^B\rangle \longrightarrow |\Phi\rangle \\ &= |E_0^A\rangle \otimes |E_0^B\rangle. \end{aligned} \tag{16.329}$$

This is a product state with zero entanglement entropy describing two completely uncorrelated subsystems Q_A and Q_B.

The thermofield double state $|\Psi\rangle$ where Q_A and Q_B are entangled is dual to the Schwarzschild–AdS black hole, which is a connected spacetime in which light signals traveling from A and B can intersect. Similarly, the product state $|\Phi\rangle$ where Q_A and Q_B are now disentangled or uncorrelated is dual to a spacetime consisting of the

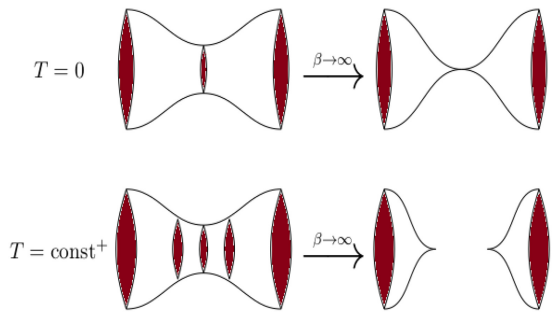

Figure 16.8. The limit $\beta \longrightarrow \infty$ in the Schwarzschild–AdS black hole.

product of two pieces corresponding to two disconnected regions A and B, i.e. light signals traveling from A and B cannot intersect. Thus, in the limit $\beta \longrightarrow \infty$ entanglement between Q_A and Q_B is removed and correspondingly connectivity between regions A and B is reduced until they become disconnected in the limit of zero temperature. This shows clearly that entanglement between Q_A and Q_B is a necessary condition for classical connectivity between A and B.

This picture can also be confirmed from the Ryu–Takayanagi formula $S_{\rm BH} = A_{\rm BH}/4G$. In the limit $\beta \longrightarrow \infty$ in the thermofield double state the entanglement between Q_A and Q_B is removed and thus $S_{\rm BH} \longrightarrow 0$ or equivalently $A_{\rm BH} \longrightarrow 0$. In other words, the limit $\beta \longrightarrow \infty$ in the thermofield double state decreases the connectivity in the dual black hole spacetime, as seen in figure 16.8. Indeed, the proper distance between any two points in A and B (encoded in the mutual information between S_A and S_B) goes to ∞ as the entanglement between Q_A and Q_B goes to zero in the limit $\beta \longrightarrow \infty$.

This relation between quantum entanglement and spacetime connectivity generalizes to any quantum state with a classical spacetime dual.

16.9.2 The first law of thermodynamics and CFT

As before, we will consider a one-parameter family of CFT states $|\psi(\zeta)\rangle$ living on the boundary of AdS spacetime AdS_{d+1}. The dual spacetime of the vacuum state $|\psi(0)\rangle$ of the CFT is assumed to be given by the Poincaré patch of the anti-de Sitter spacetime defined by the metric

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_\mu dx^\mu). \tag{16.330}$$

The dual spacetime of the perturbed state $|\psi(\zeta)\rangle$, i.e. with $\zeta \longrightarrow 0$, is denoted by $\mathcal{M}_\psi$ and is a small perturbation of anti-de Sitter spacetime with boundary $\partial\mathcal{M}_\psi$ which is itself a small perturbation of Minkowski spacetime.

Let B be some spatial region on the boundary $\partial\mathcal{M}_\psi$ and let $\bar{B}$ be its complement. The accessible region B is a ball shaped region of radius R on the Cauchy surface $\Sigma_{\partial\mathcal{M}_\psi} \in \partial\mathcal{M}_\psi$. The entanglement entropy of this spatial region in the CFT is equal to the von Neumann entropy of the reduced density matrix

$$\rho_B = \mathrm{tr}_{\bar{B}}|\psi(\zeta)\rangle\langle\psi(\zeta)|. \tag{16.331}$$

The entanglement entropy is thus given by

$$S_B = -\mathrm{tr}_B\rho_B \log\rho_B. \tag{16.332}$$

We can compute

$$\frac{d}{d\zeta}S_B = -\mathrm{tr}_B\frac{d}{d\zeta}\rho_B \cdot \log\rho_B. \tag{16.333}$$

We define the so-called modular Hamiltonian by the relation

$$H_B = -\log\rho_B(\zeta = 0). \tag{16.334}$$

Thus

$$\begin{aligned}\frac{d}{d\zeta}S_B &= \mathrm{tr}_B H_B\frac{d}{d\zeta}\rho_B + O(\zeta)\\ &= \frac{d}{d\zeta}\mathrm{tr}_B H_B\rho_B + O(\zeta)\\ &= \frac{d}{d\zeta}\langle H_B\rangle + O(\zeta).\end{aligned} \tag{16.335}$$

The expectation value of the modular Hamiltonian is what we call the hyperbolic energy, viz $\langle H_B\rangle = E_B^{\mathrm{Hyp}}$. We get then

$$\frac{d}{d\zeta}S_B = \frac{d}{d\zeta}E_B^{\mathrm{Hyp}}. \tag{16.336}$$

This is effectively the first law of thermodynamics $dE = dS$ which holds in the CFT for arbitrary perturbations ζ of the vacuum state $|\psi(0)\rangle$ and not only for thermal and equilibrium configurations as is usually the case for the first law of thermodynamics.

16.9.3 Holographic or AdS entanglement entropy

The variation in the entanglement entropy in the CFT is given by $dS_B/d\zeta$. In the AdS the variation in the entanglement entropy is given by the Ryu–Takayanagi formula, viz

$$\delta S_B = \frac{\delta A_B}{4G_N} \equiv \frac{d}{d\zeta}S_B. \tag{16.337}$$

A_B is the area of the extremal surface $\tilde{B}$ in the bulk such that $\partial\tilde{B} = \partial B$ where the accessible region B is a ball shaped region of radius R. This is the holographic interpretation of the entanglement entropy.

Thus, the variation in the entanglement entropy $dS_B/d\zeta$ of the CFT state $|\psi\rangle$ with a dual spacetime given by pure AdS corresponding to a ball shaped region B is

proportional to the variation δA_B in the area of the extremal surface $\tilde{B}$ due to the corresponding perturbation of the AdS space.

Let g^0_{ab} be the metric of pure AdS corresponding to the vacuum state $|\psi(0)\rangle$ and g_{ab} be the metric of the perturbed AdS corresponding to the excited state $|\psi(\zeta)\rangle$. We have

$$g_{ab} = g^0_{ab} + \delta g_{ab}. \tag{16.338}$$

The perturbation is given by Fefferman–Graham coordinates

$$\delta g_{ab} = L^2 z^{d-2} \bar{h}_{ab}(z, x) = z^{d-2} h_{ab}(z, x). \tag{16.339}$$

The surface $\tilde{B}$ is an extension of the spatial region B into the bulk and thus it is a co-dimension two surface characterized by some embedding functions $X^a(\sigma)$ with area A_B given by

$$A_B(g, X) = \int d^{d-1}\sigma \sqrt{\det \gamma_{ij}}. \tag{16.340}$$

The induced metric γ_{ij} on this surface $\tilde{B}$ is given by

$$\gamma_{ij} = g_{ab} \frac{\partial X^a}{\partial \sigma^i} \frac{\partial X^b}{\partial \sigma^j}. \tag{16.341}$$

In pure AdS this area is extremized by some embedding functions X^0_{ext}, whereas in perturbed AdS it is extremized by some other embedding functions X_{ext}. We have then the variation in the area of the extremal surface given by

$$\begin{aligned} \delta A_B(g, X_{\text{ext}}) &= \frac{\delta A_B(g, X^0_{\text{ext}})}{\delta g} \delta g + \frac{\delta A_B(g^0, X_{\text{ext}})}{\delta X^a} \delta X^a \\ &= \frac{\delta A_B(g, X^0_{\text{ext}})}{\delta g} \delta g + O(\delta g^2). \end{aligned} \tag{16.342}$$

In this equation we have used the fact that δX is of order δg and as a consequence the second term is subleading of order δg^2. The variation in the area of the extremal surface is thus obtained by holding the embedding functions fixed and varying the spacetime metric.

Hence, the variation in the induced metric, which in fact controls directly the variation in $A_B(g, X_{\text{ext}})$, should be obtained by holding the embedding functions fixed and varying the spacetime metric, viz

$$\delta \gamma_{ij} = \delta g_{ab} \frac{\partial X^a_{\text{ext}}}{\partial \sigma^i} \frac{\partial X^b_{\text{ext}}}{\partial \sigma^j}. \tag{16.343}$$

In this equation $X_{\text{ext}} = X^0_{\text{ext}}$ and obviously

$$\delta \gamma_{ij} = \frac{d}{d\zeta} \gamma_{ij}|_{\zeta=0}. \tag{16.344}$$

The variation in the area of the extremal surface is given explicitly in terms of the variation of the induced metric by

$$\delta A_B(g,\, X_{\text{ext}}) = \int d^{d-1}\sigma \sqrt{\det \gamma^0_{ij}} \left(\frac{1}{2} \gamma_0^{kl} \delta \gamma_{kl} \right). \tag{16.345}$$

The embedding functions X^a_{ext} are mappings from the spatial surface $\tilde{B}$ to the bulk whose boundary at $z = 0$ is the spatial surface B which is a ball shaped region of radius R. Both B and $\tilde{B}$ are co-dimension two surfaces.

For pure AdS the bulk surface $\tilde{B}$, which turns out to be extremal, is given by the sphere in d dimensions of radius R, i.e.

$$\vec{x}^2 + z^2 = R^2. \tag{16.346}$$

This has been shown explicitly for AdS_3 and the argument for higher-dimensional AdS spacetimes is similar.

We can now parametrize the extremal surface $\tilde{B}$ for pure AdS by $\sigma^i = x^i$ and we choose the embedding functions $X^a_{\text{ext}} : \mathbf{R}^{d-1} \longrightarrow \mathbf{R}^{d+1}$ as follows

$$X^0_{\text{ext}} = t_0,\;\; X^1_{\text{ext}} = x^1,\;\; \ldots\,,\; X^{d-1}_{\text{ext}} = x^{d-1},\;\; X^d_{\text{ext}} = z = \sqrt{R^2 - \vec{x}^2}\,. \tag{16.347}$$

We compute in the radial gauge (defined by $h_{zz} = h_{z\mu} = 0$) the variation in the induced metric given by

$$\delta \gamma_{ij} = z^{d-2} h_{ij}. \tag{16.348}$$

By using the above embedding functions we can easily compute the unperturbed induced metric

$$\begin{aligned} \gamma^0_{ij} &= g^0_{ab} \frac{\partial X^a}{\partial x^i} \frac{\partial X^b}{\partial x^j} \\ &= \frac{L^2}{z^2} \left(\delta_{ij} + \frac{x_i x_j}{z^2} \right). \end{aligned} \tag{16.349}$$

By using these two last results the variation in the area of the extremal surface can now be computed explicitly to obtain

$$\delta A_B = \frac{L^{d-3} R}{2} \int_{\tilde{B}} d^{d-1}x \left(\delta^{ij} - \frac{1}{R^2} (x - x_0)^i (x - x_0)^j \right) h_{ij}. \tag{16.350}$$

The point $\vec{x}_0$ is the center of the ball shaped region B. The variation in the entanglement entropy in the AdS is thus given by

$$\delta S_B = \frac{L^{d-3} R}{8 G_N} \int_{\tilde{B}} d^{d-1}x \left(\delta^{ij} - \frac{1}{R^2} (x - x_0)^i (x - x_0)^j \right) h_{ij}. \tag{16.351}$$

16.9.4 Modular Hamiltonian and hyperbolic energy

In this section we will follow closely [7]. First, we will define the so-called modular Hamiltonian in the CFT without holography then we will construct the AdS expression.

Modular Hamiltonian

We start by defining the causal development (also called domain of dependence) $\mathcal{D}$ of the region B by the set of all points p for which every inextensible causal (time-like or light-like) curve through p necessarily intersects B. Classically, the field values on $\mathcal{D}$ are completely determined in terms of the initial field values on B. Quantum mechanically, any operator in $\mathcal{D}$ is determined in terms of the fields in B alone and hence it can be calculated using the reduced density matrix ρ_B. The domain of dependence $\mathcal{D}$ is also the causal development of any other space-like surface whose boundary is the same as the boundary of B.

The reduced density matrix ρ_B is hermitian and positive semi-definite and thus it can be rewritten in terms of a hermitian operator H_B as

$$\rho_B = \exp(-H_B). \tag{16.352}$$

This operator H_B is called the modular Hamiltonian or entanglement Hamiltonian. Generically, this Hamiltonian is not local since it cannot be represented by a local expression in terms of fields defined on B. This Hamiltonian generates the modular group given by the transformations

$$\mathrm{tr}\rho\mathcal{O} = \mathrm{tr}\rho U(s)\mathcal{O}U(-s), \;\; U(s) \;=\; \rho^{is} = \exp(-isH). \tag{16.353}$$

The operator $U(s)$, although it looks like a time evolution transformation along s, does not generate a local flow on $\mathcal{D}$ since H_B is not local. Indeed, the operator defined by $\mathcal{O}(s) = U(s)\mathcal{O}U(-s)$ is not local, i.e. it is not defined at a point x, even if we start with a local operator $\mathcal{O}$ defined at a point x, e.g. $\mathcal{O} = \phi(x)$.

Rindler wedge

However, there are special circumstances where the modular Hamiltonian is local. The first example is Rindler space $\mathcal{R}$ [7]. The metric is given by ($T = \rho \sinh\omega$ and $Z = \rho\cosh\omega$)

$$\begin{aligned} ds^2 &= \rho^2 d\omega^2 - d\rho^2 - dX^2 - dY^2 \\ &= dT^2 - dZ^2 - dX^2 - dY^2. \end{aligned} \tag{16.354}$$

The Rindler wedge is region I which is the part of Minkowski spacetime accessible to a uniformly accelerating observer. Quadrants II and III have no causal relations with quadrant I. Quadrant IV provides initial data for the Rindler wedge. This is the Rindler decomposition of Minkowski spacetime. See figure 16.9.

A translation in the Rindler time $\omega \longrightarrow \omega + c$ corresponds to a Lorentz boost along the Z direction in Minkowski spacetime. The corresponding generator is the Rindler Hamiltonian H_ω, which is precisely the Hamiltonian in quadrant I, given by

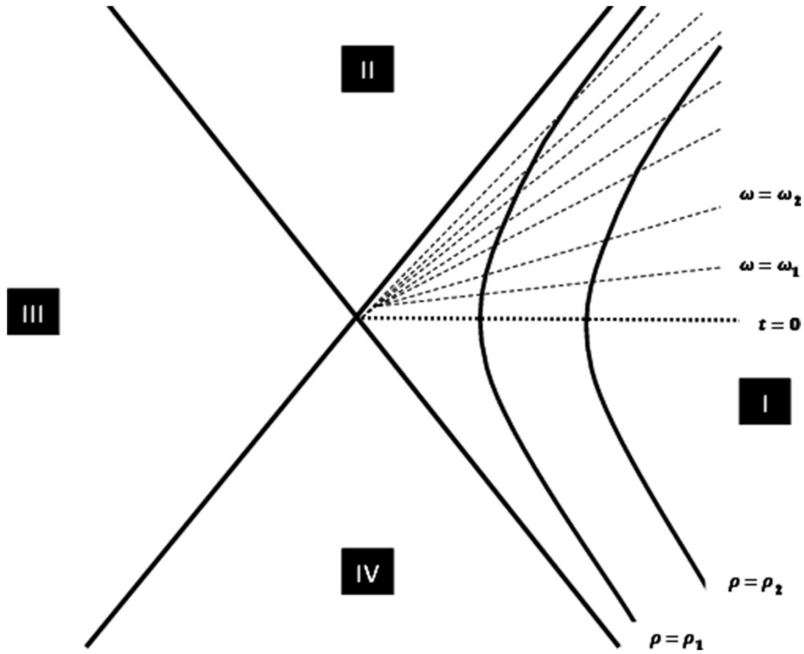

Figure 16.9. Rindler decomposition.

$$H_\omega = \int_{\rho=0}^{\rho=\infty} \rho d\rho dX dY T^{00}(\rho,\ X,\ Y). \tag{16.355}$$

T^{00} is the Hamiltonian density with respect to the Minkowski observer. Note also that in the T–Z plane the lines of constant Rindler time ω are straight lines through the origin. The proper time separation between these lines is $\delta\tau = \rho\delta\omega$. This is the origin of the ρ factor multiplying T^{00}.

The surface $T = 0$ is divided into two halves. The first half in region I and the second half in region III. The fields in region I ($Z > 0$) act in the Hilbert space $\mathcal{H}_R$ and those in region III ($Z < 0$) act in the Hilbert space $\mathcal{H}_L$. We have then

$$\phi(X,\ Y,\ Z) = \phi_R(X,\ Y,\ Z),\ \ Z > 0 \tag{16.356}$$

$$\phi(X,\ Y,\ Z) = \phi_L(X,\ Y,\ Z),\ \ Z < 0. \tag{16.357}$$

The general wave functional of interest is the pure state

$$\Psi = \Psi(\phi_L,\ \phi_R). \tag{16.358}$$

This is the ground state of the Minkowski Hamiltonian which can be computed using Euclidean path integrals. We get, after some calculation, the transition matrix element [50]

$$\Psi(\phi_L,\ \phi_R) = \langle\phi_R|\langle\phi_L|\Omega\rangle \propto \sum_i e^{-\pi E_i}\langle\phi_R|i_R\rangle\left\langle\phi_L|i_L^*\right\rangle. \tag{16.359}$$

In other words, we get the ground state

$$|\Omega\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\pi E_i} |i_R\rangle |i_L^*\rangle. \tag{16.360}$$

The entanglement between the left and right wedges is now fully manifest.

However, we want to compute the reduced density matrix ρ_A used by observers in the Rindler quadrant (quadrant I) to describe the Minkowski vacuum. This is the density matrix associated with the spatial half-space $Z > 0$, $X = Y = 0$ corresponding to the time slice $T = 0$ which is denoted by A. In other words, the Rindler wedge $\mathcal{R}$ is the causal development of the half-space A. We can define the reduced matrix ρ_A by the relation

$$\begin{aligned} \rho_A(\phi_R, \phi_R') &= \int \Psi^*(\phi_L, \phi_R)\Psi(\phi_L, \phi_R') d\phi_L \\ &= \frac{1}{Z} \sum_i e^{-2\pi E_i} \langle i_R | \phi_R \rangle \langle \phi_R' | i_R \rangle \\ &= \frac{1}{Z} \langle \phi_R' | e^{-2\pi H_\omega} | \phi_R \rangle. \end{aligned} \tag{16.361}$$

We get then the reduced density matrix

$$\rho_A = \frac{1}{Z} e^{-2\pi H_\omega}. \tag{16.362}$$

Thus, the fiducial observers in the Rindler wedge $\mathcal{R}$ see the vacuum as a thermal ensemble with a Maxwell–Boltzmann distribution at a temperature

$$T_A = \frac{1}{2\pi}. \tag{16.363}$$

This is the Unruh effect.

The modular flow on the Rindler wedge $\mathcal{R}$ corresponds, therefore, to the Rindler time translations $\omega \longrightarrow \omega + c$ and the modular Hamiltonian is given by

$$H_A = 2\pi H_\omega + \log Z. \tag{16.364}$$

The ball shaped region B

Another very important example in which the modular flow and the modular Hamiltonian can be shown to be local is a conformal field theory defined on the causal development $\mathcal{D}$ of a ball shaped region B of radius R. It was shown in [7] that this causal development $\mathcal{D}$ is conformally related to the Rindler wedge by a special conformal transformation.

Let us denote now the coordinates on the Rindler wedge by X^μ with metric $ds^2 = \eta_{\mu\nu} dX^\mu dX^\mu$. The Rindler Wedge is the causal development or domain of dependence of the half-space A corresponding to $X^0 = 0$, $X^1 > 0$ and $X^2 = X^3 = 0$.

This causal development can be given by $X^1 \geqslant |X^0|$ or equivalently in terms of the null coordinates $X^{\pm} = X^1 \pm X^0$ by $\{X^+ \geqslant 0\} \cap \{X^- \geqslant 0\}$.

The causal development $\mathcal{D}$ with coordinates x^μ and metric $ds^2 = \Omega^{-2}\eta_{\mu\nu}dx^\mu dx^\mu$ is obtained from the Rindler edge via the special conformal transformation [7]

$$\begin{aligned} x^\mu &= \frac{X^\mu - (X\cdot X)C^\mu}{\Omega} + 2R^2 C^\mu, \;\; C^\mu \\ &= \left(0, \frac{1}{2R}, 0, \; \ldots, 0\right), \;\; \Omega \\ &= 1 - 2(X\cdot C) + (X\cdot X)(C\cdot C). \end{aligned} \tag{16.365}$$

This maps the domain of dependence $\{X^+ \geqslant 0\} \cap \{X^- \geqslant 0\}$ to the domain of dependence $\{x^+ \leqslant R\} \cap \{x^- \leqslant R\}$ where the null coordinates $x^\pm$ are defined by $x^\pm = r \pm t$ where $r = \sqrt{(x^1)^2 + \cdots + (x^{d-1})^2}$.

Let us introduce the hyperbolic coordinates on the Rindler wedge by $X^1 = \rho\cosh\omega$ and $X^0 = \rho\sinh\omega$. We have then $X^\pm = \rho\exp(\pm\omega)$. The modular flow on the Rindler wedge is given by the time translations $\omega \longrightarrow \omega' = \omega + c$. We will write $c = 2\pi s$ since the period is given precisely by the inverse temperature $1/T = 2\pi$. Under this modular flow the coordinates transform as $X^\pm(0) = X^\pm \longrightarrow X^\pm(s) = X^\pm\exp(\pm 2\pi s)$. Under the special conformal transformation (16.365) this modular flow on the Rindler wedge transforms to the modular flow on $\mathcal{D}$ given explicitly by

$$x^\pm(s) = R\frac{R + x^\pm - e^{\mp 2\pi s}(R - x^\pm)}{R + x^\pm + e^{\mp 2\pi s}(R - x^\pm)}. \tag{16.366}$$

We compute for the Rindler wedge $\mathcal{R}$ the variation of the modular flow

$$\delta X^\pm(s) = \pm 2\pi\delta s X^\pm(s) \Leftrightarrow \delta X^1(s) = 2\pi\delta s X^0(s), \;\; \delta X^0(s) = 2\pi\delta s X^1(s). \tag{16.367}$$

We focus on the time slice $s = \omega = 0$. We get the equations

$$\delta X^1 = \delta\rho = 0, \;\; \delta X^0 = \rho\delta\omega = 2\pi\rho\delta s. \tag{16.368}$$

The corresponding generator H_A is then given by

$$H_A = 2\pi\int d\rho dX dY \rho T^{00}(\rho, X, Y) + \text{constant}. \tag{16.369}$$

We go through the same steps for the casual development $\mathcal{D}$. We compute for $\mathcal{D}$ the variation of the modular flow

$$\delta x^\pm(s) = \frac{\pm 4\pi R\delta s e^{\mp 2\pi s}(R^2 - (x^\pm)^2)}{\left(R + x^\pm + e^{\mp 2\pi s}(R - x^\pm)\right)^2}. \tag{16.370}$$

We focus on the ball B of radius R corresponding to the time slice $s = t = 0$ and hence $x^\pm = r$. The variation of the modular flow becomes

$$\delta x^{\pm} = \pm \frac{\pi \delta s (R^2 - r^2)}{R}. \tag{16.371}$$

This gives the shift

$$\delta r = 0, \;\; \delta t = \frac{2\pi \delta s (R^2 - r^2)}{2R}. \tag{16.372}$$

The corresponding generator H_B is then given by

$$H_B = 2\pi \int_B d^{d-1}x \frac{R^2 - r^2}{2R} T^{00}(x) + \text{constant}. \tag{16.373}$$

A careful derivation of H_B is given in [7]. The special conformal transformation (16.365), which is a symmetry of Minkowski spacetime, is associated with a unitary operator U_0 in the CFT which leaves the vacuum invariant, viz $U_0|0\rangle = |0\rangle$. This unitary operator acts on primary operators as

$$\phi(x) \;=\; \Omega^{\Delta} U_0 \phi(X) U_0^{-1}. \tag{16.374}$$

This unitary operator must then act on the density matrices as follows

$$\begin{aligned} \rho_B &= U_0 \rho_A U_0^{-1} \\ &= \frac{1}{Z} e^{-2\pi U_0 H_\omega U_0^{-1}} \\ &= \frac{1}{Z} e^{-H_B}. \end{aligned} \tag{16.375}$$

Next, if $U_{\mathcal{R}}$ is the quantum operator which generates the modular flow on the Rindler wedge $\mathcal{R}$ then the quantum operator which generates the modular flow on $\mathcal{D}$ must be given by

$$U_{\mathcal{D}}(s) \;=\; U_0 U_{\mathcal{R}}(s) U_0^{-1}. \tag{16.376}$$

We can then act on primary operators as follows

$$\begin{aligned} U_{\mathcal{D}}(s)\phi(x[s_0])U_{\mathcal{D}}(-s) &= U_0 U_{\mathcal{R}}(s) U_0^{-1} \phi(x[s_0]) U_0 U_{\mathcal{R}}(-s) U_0^{-1} \\ &= \Omega^{\Delta}(x[s_0]) U_0 U_{\mathcal{R}}(s) \phi(X[s_0]) U_{\mathcal{R}}(-s) U_0^{-1} \\ &= \Omega^{\Delta}(x[s_0]) U_0 \phi(X[s_0 + s]) U_0^{-1} \\ &= \Omega^{\Delta}(x[s_0]) \Omega^{-\Delta}(x[s_0 + s]) \phi(x[s_0 + s]). \end{aligned} \tag{16.377}$$

The pre-factor $\Omega^{\Delta}(x[s_0])\Omega^{-\Delta}(x[s_0 + s])$ evaluated on the surface $t = 0$ is equal to 1 at the order δs and hence it does not contribute to the infinitesimal shift (16.372). The modular Hamiltonian is, therefore, given indeed by equation (16.373).

Hyperbolic energy

The hyperbolic energy is the expectation value of the modular Hamiltonian. We are interested in the first-order variation in the ζ of the hyperbolic energy.

If we assume that the ball shaped region B of radius R is centered at $\vec{x}_0$ then the hyperbolic energy must be given by

$$E_B^{\text{Hyp}} = 2\pi \int_B d^{d-1}x \frac{R^2 - (\vec{x} - \vec{x}_0)^2}{2R} \langle T^{00}(x) \rangle. \tag{16.378}$$

For an infinitesimal ball B we can safely replace $\langle T^{00}(x) \rangle$ with $\langle T^{00}(x_0) \rangle$. We can then easily compute

$$\begin{aligned} \delta E_B &= \frac{d}{d\zeta} \Big|_{\zeta=0} E_{B_{\text{inf}}}^{\text{Hyp}} \\ &= 2\pi \int_B d^{d-1}x \frac{R^2 - (\vec{x} - \vec{x}_0)^2}{2R} \frac{d}{d\zeta} \Big|_{\zeta=0} \langle T^{00}(x) \rangle \\ &= 2\pi \frac{d}{d\zeta} \Big|_{\zeta=0} \langle T^{00}(x_0) \rangle \int_B d^{d-1}x \frac{R^2 - (\vec{x} - \vec{x}_0)^2}{2R} \\ &= \frac{2\pi R^d S_{d-2}}{d^2 - 1} \frac{d}{d\zeta} \Big|_{\zeta=0} \langle T^{00}(x_0) \rangle, \end{aligned} \tag{16.379}$$

where $S_{d-2} = \int d\Omega_{d-2}$ is the area of the unit sphere $\mathbf{S}^{d-2}$.

However, by using the first law of thermodynamics, the holographic interpretation of the entanglement entropy and the Ryu–Takayanagi formula we obtain

$$\frac{d}{d\zeta} \Big|_{\zeta=0} E_{B_{\text{inf}}}^{\text{Hyp}} = \frac{d}{d\zeta} \Big|_{\zeta=0} S_{B_{\text{inf}}} = \lim_{R \to 0} \delta S_B = \lim_{R \to 0} \frac{\delta A_B}{4G_N}. \tag{16.380}$$

We employ now the holographic result (16.350) to obtain

$$\frac{d}{d\zeta} \Big|_{\zeta=0} \langle T^{00}(x_0) \rangle = \frac{L^{d-3} d}{16\pi G_N} \delta^{ij} h_{ij}(x_0, z = 0). \tag{16.381}$$

We have thus another crucial holographic result given by

$$\frac{d}{d\zeta} \Big|_{\zeta=0} \langle T^{00}(x) \rangle = \frac{L^{d-3} d}{16\pi G_N} \delta^{ij} h_{ij}(x, z = 0). \tag{16.382}$$

By substituting back into the variation of the hyperbolic energy for a ball shaped region B of radius R we obtain the holographic interpretation of the second side of the first law of thermodynamics, viz

$$\delta E_B = \frac{L^{d-3} d}{16 G_N} \int_B d^{d-1}x \frac{R^2 - (\vec{x} - \vec{x}_0)^2}{R} \delta^{ij} h_{ij}(x, z = 0). \tag{16.383}$$

16.9.5 Einstein's equations

In the remainder we will consider a ball shaped region B of radius R centered around the origin, i.e. $\vec{x}_0 = 0$. This is a spatial surface at the time slice $t = 0$.

The first law of thermodynamics in the CFT is given by

$$\frac{d}{d\zeta} S_B = \frac{d}{d\zeta} E_B^{\text{Hyp}}. \tag{16.384}$$

The holographic interpretation of the two sides of this equation is given by the two equations (16.351) and (16.383), viz

$$\frac{d}{d\zeta} S_B \equiv \delta S_B = \frac{L^{d-3} R}{8 G_N} \int_{\tilde{B}} d^{d-1}x \left(\delta^{ij} - \frac{1}{R^2} x^i x^j \right) h_{ij}(x, z), \tag{16.385}$$

where $\tilde{B}$ is the extremal surface in the bulk such that $\partial \tilde{B} = \partial B$ and z should be understood as $z = \sqrt{R^2 - \vec{x}^2}$. And

$$\frac{d}{d\zeta} E_B^{\text{Hyp}} \equiv \delta E_B = \frac{L^{d-3} d}{16 G_N} \int_B d^{d-1}x \frac{R^2 - r^2}{R} \delta^{ij} h_{ij}(x, z = 0). \tag{16.386}$$

The first law of thermodynamics in the AdS is then given by the non-local constraints

$$\int_{\tilde{B}} d^{d-1}x (R^2 \delta^{ij} - x^i x^j) h_{ij}(x, z) = \frac{d}{2} \int_B d^{d-1}x (R^2 - r^2) \delta^{ij} h_{ij}(x, z = 0). \tag{16.387}$$

Note that this equation relates the metric perturbation on the boundary $z = 0$ to the metric perturbation in the bulk $z \neq 0$. The central claim is that these non-local equations are precisely the Einstein's equations linearized around AdS [4, 5].

Also we have seen that the modular Hamiltonian of a ball shaped region B of radius R evaluated at time $t = 0$ is given by the following expression

$$H_B = 2\pi \int_B d^{d-1}x \frac{R^2 - r^2}{2R} T^{00}(x) + \text{constant}. \tag{16.388}$$

The constant is chosen such that the trace of the density matrix is equal to 1. The corresponding modular flow (16.366) is a symmetry which is generated by the conformal time-like Killing vector of Minkowski spacetime given by

$$\xi = \frac{\pi}{R} \left((R^2 - t^2 - r^2) \partial_t - 2 t x^i \partial_i \right). \tag{16.389}$$

We associate to this a bulk Killing vector ξ_B which obviously approaches ξ as we approach the AdS boundary $z = 0$. This is given explicitly by

$$\xi_B = \frac{\pi}{R} ((R^2 - z^2 - t^2 - r^2) \partial_t - 2t(z \partial_z + x^i \partial_i)). \tag{16.390}$$

This is indeed a symmetry which preserves the metric, i.e. an isometry, since the Lie derivative of the metric with respect to ξ_B is zero, viz $\mathcal{L}_{\xi_B} g_{\text{AdS}} = 0$. The corresponding

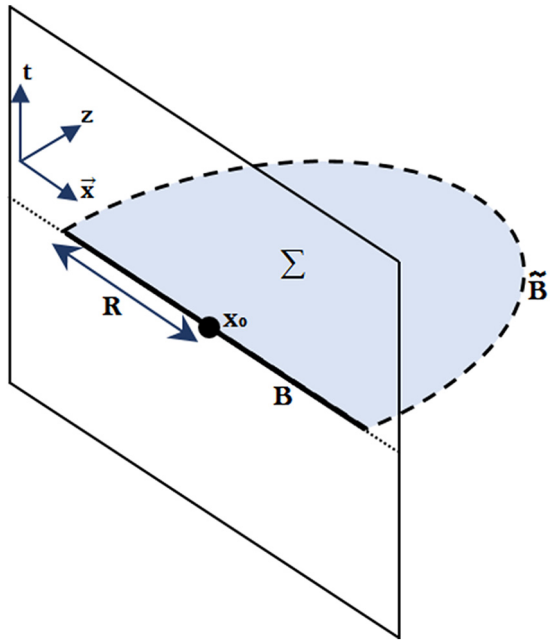

Figure 16.10. The volume Σ bounded by $B \cup \tilde{B} = \partial\Sigma$.

conserved quantity is the so-called canonical energy associated with the Killing vector ξ_B. This Killing vector ξ_B also vanishes on the Rindler horizon, i.e. on the extremal surface $\tilde{B}$ in the unperturbed AdS given by $\vec{x}^2 + z^2 = R^2$.

Clearly the function ξ_B is defined in the spatial region Σ between B and $\tilde{B}$, i.e. Σ is the volume bounded by $B \cup \tilde{B} = \partial\Sigma$. In other words, Σ is a co-dimension d-dimensional surface in the bulk. See figure 16.10. The volume form on this region Σ is given by

$$\epsilon_{a_1} = \frac{1}{d!}\epsilon_{a_1 a_2 \dots a_{d+1}} dx^{a_2} \wedge \cdots \wedge dx^{a_{d+1}}, \tag{16.391}$$

where $\epsilon_{a_1 \dots a_{d+1}}$ is the anti-symmetric tensor in $d+1$ dimensions normalized such that $\epsilon_{ztx^1 \dots x^{d-1}} = \sqrt{-g}$. Since the region Σ is a spatial surface all components but ϵ^t vanish and as a consequence the volume form is simply given by

$$\text{vol}_\Sigma = \epsilon^t. \tag{16.392}$$

The approach to convert the above non-local constraints (16.387) to local equations is similar to the way we convert the integral form of Maxwell's equations to differential forms where we mainly use Stokes' theorem. Indeed, the non-local constraints (16.387) can be converted into the local Einstein's equations if for every choice of B there exists a differential form χ_B such that

$$\delta S_B = \int_{\tilde{B}} \chi_B \tag{16.393}$$

$$\delta E_B = \int_B \chi_B . \tag{16.394}$$

The non-local constraints $\delta E_B = \delta S_B$ become, therefore,

$$\int_B \chi_B = \int_{\tilde{B}} \chi_B \Rightarrow \int_B \chi_B - \int_{\tilde{B}} \chi_B = 0 \Rightarrow \int_{\partial\Sigma} \chi_B = 0 \Rightarrow \int_\Sigma d\chi_B = 0. \tag{16.395}$$

The form χ_B is obtained by means of the Iyer–Wald formalism [51]. Explicitly the form χ_B can be shown to satisfy

$$d\chi_B = 2\xi^0_B \delta E_{00} \mathrm{vol}_\Sigma. \tag{16.396}$$

In this equation δE_{00} is time–time component of the Einstein equations linearized about AdS spacetime given explicitly by

$$\delta E_{00} = \frac{z^d}{2L^2}\left(\partial_z^2 h_i^{\ i} + \frac{d+1}{z}\partial_z h_i^{\ i} + \partial_j \partial^j h_i^{\ i} - \partial^i \partial^j h_{ij}\right). \tag{16.397}$$

Hence, the first law of thermodynamics in the CFT, in which the energy is understood as hyperbolic energy and the entropy is understood as entanglement entropy, yields under holographic extension, for ball shaped regions in slices of constant time, a set of non-local constraints in the bulk which are precisely the time–time component of the Einstein equations linearized about AdS spacetime, i.e.

$$\delta E_{00} = 0. \tag{16.398}$$

To obtain the other components of the Einstein equations we repeat the same steps for ball shaped regions in general frames of reference. We obtain then

$$\delta E_{\mu\nu} = 0. \tag{16.399}$$

The remaining equations $\delta E_{zz} = \delta E_{z\mu} = 0$ are in fact constraints in the dual geometry which are equivalent to the conservation and tracelessness of the energy–momentum tensor $T_{\mu\nu}$ in the CFT at the boundary. This can be verified explicitly by considering infinitesimal balls.

In summary, we have thus shown that the first law of thermodynamics for ball shaped regions in the CFT is exactly equivalent to linearized Einstein equations in the gravity dual theory. This is a very concrete mechanism for the emergence of spacetime geometry from quantum entanglement.

16.10 Exercises

Exercise 1:

Show that in anti-de Sitter the proper time to go from the center $r = 0$ to the boundary $r = 1$ and back is finite equal to $\tau = \pi$. What do you conclude?

Exercise 2:

Show that the boundary $z = 0$ of AdS_{d+1} in the Poincaré patch becomes $\mathbf{R}^d$ in the Euclidean rotation, whereas the horizon $z = \infty$ shrinks to a point. What do you get by adding the point $z = \infty$ to the boundary $\mathbf{R}^d$ and what is the resulting compactified Euclidean AdS_{d+1}?

Exercise 3:

1. Show that the generators of $SO(2, 1)$ in the global coordinates τ, r are given by the differential operators

$$L_2^{\ 0} = \partial_\tau \tag{16.400}$$

$$L_1^{\ 0} = \sin\tau \sin r\partial_\tau - \cos\tau \cos r\partial_r \tag{16.401}$$

$$L_2^{\ 1} = -\cos\tau \sin r\partial_\tau - \sin\tau \cos r\partial_r. \tag{16.402}$$

2. Verify that they satisfy the algebra

$$\left[L_2^{\ 0}, L_1^{\ 0}\right] = -L_2^{\ 1}, \ \left[L_2^{\ 0}, L_2^{\ 1}\right] = L_1^{\ 0}, \ \left[L_1^{\ 0}, L_2^{\ 1}\right] = L_2^{\ 0}. \tag{16.403}$$

3. Show that the lowering and raising operators (special conformal and momentum generators) K and P take in the global coordinates τ, r the form

$$K = \exp(-i\tau)(i \sin r\partial_\tau - \cos r\partial_r) \tag{16.404}$$

$$P = -\exp(i\tau)(i \sin r\partial_\tau + \cos r\partial_r). \tag{16.405}$$

4. Show that the Casimir operator for the conformal group $SO(2, 1)$ which is precisely the Klein–Gordon Laplacian is given by

$$\begin{aligned} \nabla &= D^2 + \frac{1}{2}(PK + KP) \\ &= \cos^2 r(-\partial_\tau^2 + \partial_r^2). \end{aligned} \tag{16.406}$$

Exercise 4:

Show that the diffeomorphisms

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \longrightarrow (a\tilde{\sigma}^1 + b\tilde{\sigma}^2, c\tilde{\sigma}^1 + d\tilde{\sigma}^2). \tag{16.407}$$

are compatible with the equivalence relation

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \sim (\tilde{\sigma}^1 + n, \tilde{\sigma}^2 + m), \;\; n, m \in \mathbf{Z}. \tag{16.408}$$

Determine the domain of the coefficients a, b, c and d. By requiring invertibility and one-to-one determine the group structure corresponding to the above transformations.

Exercise 5:

- Show that

$$\int 1 = \int 2i dz \wedge d\bar{z} = \int 4\tau_2 d\sigma^1 \wedge d\sigma^2 = 4\tau_2. \tag{16.409}$$

- Show that the Fourier transform of the function

$$f(n'r) = \exp\left(-2\pi \frac{1}{\tau_2}(n'r - \tau_1 nr)^2\right) \tag{16.410}$$

is

$$\tilde{f}(p) = \sqrt{\frac{\tau_2}{2}} \exp\left(2\pi i \tau_1 nrp - \frac{1}{2}\pi\tau_2 p^2\right). \tag{16.411}$$

- Show that the partition function of the conformal field theory of free bosons on the torus is given by

$$\int \mathcal{D}\Phi \exp(-S[\Phi]) = (q\bar{q})^{-c/24} \mathrm{tr} q^{L_0} \bar{q}^{\bar{L}_0}. \tag{16.412}$$

Verify its modular invariance under the modular transformation $\tau \longrightarrow -1/\tau$.

Exercise 6:

Show that the dilatation operator, the translation and Lorentz generators, and the special conformal generators are represented on scalar fields with scaling dimension Δ by the differential operators

$$D\Phi = -i(x^\mu \partial_\mu + \Delta)\Phi \tag{16.413}$$

$$P_\mu \Phi = -i\partial_\mu \Phi \tag{16.414}$$

$$M_{\mu\nu}\Phi = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi + S_{\mu\nu}\Phi \tag{16.415}$$

$$K_\mu \Phi = (-2i\Delta x_\mu - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu)\Phi. \tag{16.416}$$

Work for simplicity in three dimensions. Show that the transformation law of the scalar field Φ under conformal transformations is given by

$$\Phi(x) \longrightarrow \Phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \Phi(x). \tag{16.417}$$

Exercise 7:

Show that under special conformal transformation we have the behavior

$$\begin{aligned} x^\mu \longrightarrow x'^\mu &= \frac{x^\mu + b^\mu x^2}{1 + 2bx + b^2x^2} \Rightarrow r'^2_{12} \\ &= \frac{r_{12}^2}{\left(1 + 2bx_1 + b^2x_1^2\right)\left(1 + 2bx_2 + b^2x_2^2\right)} \end{aligned} \tag{16.418}$$

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(1 + 2bx + b^2x^2)^d}. \tag{16.419}$$

Exercise 8:

Show that the requirement of invariance under translations, rotations, scalings and special conformal transformations constrains the three-point function of a conformal field theory such that

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{13}^{\Delta_1+\Delta_3-\Delta_2} r_{23}^{\Delta_2+\Delta_3-\Delta_1}}. \tag{16.420}$$

Exercise 9:

Show that for a scalar field in Euclidean AdS_{d+1} the canonical momentum with respect to z is given by

$$\Pi = \eta \sqrt{g}\phi g^{zz}\partial_z\phi. \tag{16.421}$$

Exercise 10:

Show that the required counter term to renormalize the action (16.218) is given by

$$\begin{aligned} S_{\mathrm{ct}} &= \frac{\eta}{2}\eta_1 \int \sqrt{\gamma}\, d^dx\phi^2 \\ &= \frac{\eta}{2}\eta_1 L^d \int \frac{d^dk}{(2\pi)^d}(A(k)A(-k)\epsilon^{-2\nu} + 2A(k)B(-k)). \end{aligned} \tag{16.422}$$

Determine the value of η_1.

Exercise 11:

Show that the two-point function (16.225) takes in position space the form

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{2\nu L^{d-1}\eta}{\pi^{d/2}} \frac{\Gamma\left(\frac{d}{2} + \nu\right)}{\Gamma(-\nu)} \frac{1}{|x|^{2\Delta}}. \tag{16.423}$$

Use the identity

$$\int \frac{d^d k}{(2\pi)^d} \exp(ikx) k^n = \frac{2^n}{\pi^{d/2}} \frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)} \frac{1}{|x|^{d+n}}. \tag{16.424}$$

Exercise 12:

Show that $\mathrm{ltr} q^{L_0+\bar{L}_0}$ admits an expansion in positive powers of q due to the requirement of positive dimensions of all fields (locality). Show then that the contributions from $\mathrm{ltr} q^{L_0+L_0}$ to the entanglement entropy are exponentially suppressed. Recall that $q = \exp(-\kappa L)$.

Exercise 13:

The thermofield double state $|\psi\rangle$ dual to the Schwarzschild–AdS black hole is given by

$$|\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_i \exp(-\beta E_i/2) |E_i^A\rangle \otimes \left| E_i^B \right\rangle. \tag{16.425}$$

Compute the reduced density matrix for the subsystem Q_A and its entanglement entropy. Show that in the limit $\beta \longrightarrow \infty$ we have the behavior

$$|\Psi\rangle = \rangle \longrightarrow |\Phi\rangle = |E_0^A\rangle \otimes |E_0^B\rangle. \tag{16.426}$$

Exercise 14:

The entanglement entropy of a spatial region B in the CFT is equal to the von Neumann entropy of the reduced density matrix

$$\rho_B = \mathrm{tr}_{\bar{B}} |\psi(\zeta)\rangle\langle\psi(\zeta)|. \tag{16.427}$$

Show that

$$\frac{d}{d\zeta} S_B = -\mathrm{tr}_B \frac{d}{d\zeta} \rho_B \cdot \log \rho_B. \tag{16.428}$$

Exercise 15:

1. Show that

$$\frac{\delta A_B(g^0, X_{\mathrm{ext}})}{\delta X^a} \delta X^a \tag{16.429}$$

   is of order δg^2.
2. Show that the variation in the area of the extremal surface is given in terms of the variation of the induced metric by

$$\delta A_B(g,\, X_{\text{ext}}) = \int d^{d-1}\sigma \sqrt{\det \gamma^0_{\mu\nu}} \left(\frac{1}{2} \gamma_0^{\rho\lambda} \delta\gamma_{\rho\lambda} \right). \tag{16.430}$$

3. Show that in the radial gauge the variation in the induced metric given by

$$\delta\gamma_{ij} = z^{d-2} h_{ij}. \tag{16.431}$$

4. Verify that the unperturbed induced metric is given by

$$\gamma^0_{ij} = \frac{L^2}{z^2} \left(\delta_{ij} + \frac{x_i x_j}{z^2} \right). \tag{16.432}$$

5. Show that the variation in the area of the extremal surface is given by

$$\delta A_B = \frac{L^{d-3} R}{2} \int_{\tilde{B}} d^{d-1}x \left(\delta^{ij} - \frac{1}{R^2} (x - x_0)^i (x - x_0)^j \right) h_{ij}. \tag{16.433}$$

Exercise 16:

1. Verify that under the special conformal transformation (16.365) the metric transforms as

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\mu = \Omega^{-2} \eta_{\mu\nu} dx^\mu dx^\mu. \tag{16.434}$$

2. Show that the domain of dependence $\{X^+ \geqslant 0\} \cap \{X^- \geqslant 0\}$ of the Rindler wedge maps under the special conformal transformation (16.365) to the domain of dependence $\{x^+ \leqslant R\} \cap \{x^- \leqslant R\}$ of $\mathcal{D}$ where the null coordinates $x^\pm$ are defined by $x^\pm = r \pm t$ where $r = \sqrt{(x^1)^2 + \cdots + (x^{d-1})^2}$.
3. Show that the modular flow on the Rindler wedge given by $X^\pm(s) = X^\pm \exp(\pm 2\pi s)$ transforms under the special conformal transformation (16.365) to the modular flow on $\mathcal{D}$ given explicitly by

$$x^\pm(s) = R \frac{R + x^\pm - e^{\mp 2\pi s}(R - x^\pm)}{R + x^\pm + e^{\mp 2\pi s}(R - x^\pm)}. \tag{16.435}$$

Exercise 17:

By employing the holographic result (16.350) verify that

$$\frac{d}{d\zeta} \bigg|_{\zeta=0} \langle T^{00}(x) \rangle = \frac{L^{d-3} d}{16\pi G_N} \delta^{ij} h_{ij}(x,\, z = 0). \tag{16.436}$$

Exercise 18:

1. Show that the time–time component of the Einstein equations linearized about AdS spacetime is given by

$$\delta E_{00} = \frac{z^d}{2L^2}\left(\partial_z^2 h_i^{\ i} + \frac{d+1}{z}\partial_z h_i^{\ i} + \partial_j \partial^j h_i^{\ i} - \partial^i \partial^j h_{ij}\right). \tag{16.437}$$

Use the Fefferman–Graham metric.

2. By employing the Iyer–Wald formalism show that the form χ_B satisfies

$$d\chi_B = 2\xi_B^0 \delta E_{00} \mathrm{vol}_\Sigma. \tag{16.438}$$

3. Show that the form χ_B is given explicitly by

$$\begin{aligned}\chi_B|_\Sigma = \frac{z^d}{16\pi G_N}\Bigg\{&\epsilon_z^{\ t}\left(\frac{2\pi z}{R} + \frac{d}{z}\xi^t + \xi^t\partial_z\right)h_i^{\ i} \\ &+ \epsilon_i^{\ t}\left[\left(\frac{2\pi x^i}{R} + \xi^t\partial^i\right)h_j^{\ j} - \left(\frac{2\pi x^j}{R} + \xi^t\partial^j\right)h_j^{\ i}\right]\Bigg\}.\end{aligned} \tag{16.439}$$

4. Verify explicitly that

$$\delta S_B = \int_{\tilde{B}} \chi_B \tag{16.440}$$

$$\delta E_B = \int_B \chi_B. \tag{16.441}$$

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Appendix D

Lie algebra representation theory: a primer

In this primer I follow the excellent pedagogical presentation of Zuber [1].

D.1 The Cartan subalgebra

We consider a semi-simple[1] Lie algebra $\mathbf{g}$ of finite dimension over the complex numbers $\mathbf{C}$. The Cartan subalgebra is the maximal abelian subalgebra $\mathbf{h}$ of $\mathbf{g}$ such that all its elements are diagonalizable in the adjoint representation. The elements of $\mathbf{g}$ can be chosen to be hermitian. The Cartan subalgebra is not unique and different choices are related by an automorphism of the algebra, i.e. as $h \longrightarrow ghg^{-1}$. The dimension l of $\mathbf{h}$ is called the rank of $\mathbf{g}$. For example, for $su(n)$ the Cartan subalgebra can be generated by the following $l = n - 1$ diagonal traceless matrices

$$H_1 = (1, -1, \dots, 0),\ H_2 = (0, 1, -1, \dots, 0), \dots,\ H_{n-1} = (0, \dots, 1, -1). \quad \text{(D.1)}$$

Let H_i, $i = 1, \dots, l$ be the basis elements of $\mathbf{h}$ which are chosen to be hermitian. By definition we have

$$[H_i, H_j] = 0 \Leftrightarrow [\text{ad}\ H_i, \text{ad}\ H_j] = 0. \quad \text{(D.2)}$$

Thus, from $\text{ad}\ H_i H_j = [H_i, H_j] = 0$ we see that H_j are eigenvectors of $\text{ad}\ H_i$ with eigenvalues 0. The other linearly independent eigenvectors with eigenvalues α_i, not all vanishing, will be denoted by E_α, viz

$$\text{ad}\ H_i E_\alpha = [H_i, E_\alpha] = \alpha_i E_\alpha. \quad \text{(D.3)}$$

[1] A semi-simple algebra has no *abelian ideal* but {0}, whereas a simple algebra has no *ideal* but {0}. In other words, the semi-simple is more general since it can have ideals. Thus, any semi-simple algebra can be decomposed into a direct sum of simple algebras.

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These eigenvalues are real since ad H_i are hermitian after multiplication by i. We compute for $H = \sum_i h^i H_i$ that

$$\text{ad } HE_\alpha = \alpha(H)E_\alpha, \;\; \alpha(H) = \sum_i h^i \alpha_i. \tag{D.4}$$

Obviously, $\alpha(H) = \sum_i h^i \alpha_i$ is a linear form on $\mathbf{h}$. Linear forms on a vector space E form the dual vector space E^*. Thus the set of all $\alpha(H)$ form the dual space $\mathbf{h}^*$ of $\mathbf{h}$. This dual space is called the root space and $\alpha(H)$ is called a root. We note that

$$\alpha(H_i) = \alpha_i. \tag{D.5}$$

The total number of the eigenvectors H_j and E_α of H_i is equal to the dimension d of the adjoint representation of the Lie algebra $\mathbf{g}$. Since the number of the H_j is equal to the rank l and the roots are not degenerate we conclude that the number of the roots is $d - l$. This number is even since if α is a root $-\alpha$ is also a root.

The Killing form in the basis (H_i, E_α) of $\mathbf{g}$ is given by

$$(H_i, E_\alpha) = 0, \;\; (E_\alpha, E_\beta) = 0 \text{ unless } \alpha + \beta = 0. \tag{D.6}$$

The Killing form is defined by the trace in the adjoint representation in an obvious way, viz

$$(H_i, E_\alpha) = \text{tr ad } H_i \text{ ad } E_\alpha. \tag{D.7}$$

The Killing form on $\mathbf{h}$ is non-degenerate, i.e. there exists an isomorphism between $\mathbf{h}$ and $\mathbf{h}^*$ which allows us to associate to every element $\alpha \in \mathbf{h}^*$ a unique element $H_\alpha = \sum_i h_\alpha^i H_i \in \mathbf{H}$ as follows

$$\begin{aligned}(H_\alpha, H) &= \sum_i h_\alpha^i \sum_j h^j (H_i, H_j) = \sum_i h_\alpha^i \sum_j h^j g_{ij} \\ &= \sum_j h^j \alpha_j = \alpha(H) \equiv \alpha.\end{aligned} \tag{D.8}$$

In the above equation, we have used the fact that the metric $g_{ij} = (H_i, H_j)$ is invertible since the Killing form was assumed to be non-degenerate, and $\alpha_j = \sum_i h_\alpha^i g_{ij}$. The bilinear form on the space of roots $\mathbf{H}^*$ can then be defined by

$$\langle \alpha, \beta \rangle = (H_\alpha, H_\beta). \tag{D.9}$$

In order to compute the commutation relations of the E_α, we compute using Jacobi identity the following

$$\begin{aligned}\text{ad } H_i[E_\alpha, E_\beta] &= [H_i, [E_\alpha, E_\beta]] \\ &= -[E_\beta, [H_i, E_\alpha]] - [E_\alpha, [E_\beta, H_i]] \\ &= (\alpha + \beta)_i [E_\alpha, E_\beta].\end{aligned} \tag{D.10}$$

We have three possibilities. If $\alpha + \beta$ is a root then $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$. If $\alpha + \beta \neq 0$ is not a root then $[E_\alpha, E_\beta] = 0$. If $\alpha + \beta = 0$ then $[E_\alpha, E_{-\alpha}] = H \in \mathbf{h}$. We compute

$$\begin{aligned}(H_i, [E_\alpha, E_{-\alpha}]) &= \text{tr ad } H_i[\text{ad } E_\alpha, \text{ad } E_{-\alpha}] \\ &= \alpha_i \text{ tr ad } E_\alpha \text{ ad } E_{-\alpha} \\ &= \alpha_i(E_\alpha, E_{-\alpha}) \\ &= (H_i, H_\alpha)(E_\alpha, E_{-\alpha}) \Rightarrow [E_\alpha, E_{-\alpha}] \\ &= (E_\alpha, E_{-\alpha})H_\alpha.\end{aligned} \tag{D.11}$$

In summary, we have obtained the commutation relations

$$[H_i, H_j] = 0, \ [H_i, E_\alpha] = \alpha_i E_\alpha \tag{D.12}$$

$$\begin{aligned}[E_\alpha, E_\beta] &= N_{\alpha\beta}E_{\alpha+\beta}, \ \alpha + \beta = \text{root} \\ [E_\alpha, E_{-\alpha}] &= (E_\alpha, E_{-\alpha})H_\alpha, \ \alpha + \beta = 0 \\ [E_\alpha, E_{-\alpha}] &= 0, \ \text{otherwise.}\end{aligned} \tag{D.13}$$

Since the restriction of the Killing form to $\mathbf{h}$ is positive definite we can choose the normalization

$$(H_i, H_j) = \delta_{ij}, \ (E_\alpha, E_\beta) = \delta_{\alpha+\beta,0} \tag{D.14}$$

Thus $g_{ij} = \delta_{ij}$, $\alpha_i = h_\alpha^i$ and hence

$$H_\alpha = \sum_i \alpha_i H_i, \ \langle \alpha, \alpha \rangle = (H_\alpha, H_\alpha) = \sum_i \alpha_i^2. \tag{D.15}$$

We get finally the commutation relations

$$[H_\alpha, E_{\pm\alpha}] = \pm\langle \alpha, \alpha \rangle E_{\pm\alpha}, \ [E_\alpha, E_{-\alpha}] = H_\alpha. \tag{D.16}$$

Thus we get an $su(2)$ Lie algebra for every root.

D.2 Roots, Cartan matrix and Dynkin diagrams

We have established that we have $d - l$ roots. Their inner product is inherited from the Killing form, viz $\langle \alpha, \beta \rangle = (H_\alpha, H_\beta) = \sum_i \alpha_i \beta_i$. It is obvious that $H_\alpha / \langle \alpha, \alpha \rangle$ plays the role of J_z in $su(2)$, whereas the E_α and $E_{-\alpha}$ play the role of the raising and lowering operators J_+ and J_-, respectively, in $su(2)$. We also compute

$$\text{ad } H_\alpha E_\beta = [H_\alpha, E_\beta] = \langle \alpha, \beta \rangle E_\beta. \tag{D.17}$$

Let $p \leqslant 0$ be the smallest integer such that $(\text{ad } E_{-\alpha})^{|p|} E_\beta$ is non-zero, i.e. $(\text{ad } E_{-\alpha})^{|p|+1} E_\beta = 0$. This means that $E_{-k\alpha+\beta}$, $k = 1, \ldots, |p|$, correspond to non-zero roots $-k\alpha + \beta$. Let $q \geqslant 0$ be the largest integer such that $(\text{ad } E_\alpha)^q E_\beta$ is non-zero, i.e. $(\text{ad } E_\alpha)^{q+1} E_\beta = 0$. This means that $E_{k\alpha+\beta}$, $k = 1, \ldots, q$, correspond to non-zero roots $k\alpha + \beta$. The set $\{E_{\beta'}; \ \beta' = \beta + q\alpha, \ldots, \beta, \ldots, \beta + p\alpha\}$ is called the α-chain through β. Similarly to $su(2)$, the $E_{\beta'}$ form a representation of the $su(2)$ algebra generated by $E_{\pm\alpha}$ and H_α. Furthermore, the eigenvalues corresponding to the highest state $E_{\beta+q\alpha}$ and the lowest state $E_{\beta+p\alpha}$ given, respectively, by

$$\begin{aligned}[H_\alpha, E_{\beta+q\alpha}] &= \langle \alpha, \beta + q\alpha\rangle E_{\beta+q\alpha},\\ [H_\alpha, E_{\beta+p\alpha}] &= \langle \alpha, \beta + p\alpha\rangle E_{\beta+p\alpha},\end{aligned} \tag{D.18}$$

are opposite to each other, i.e.

$$\begin{aligned}\langle \alpha, \beta + q\alpha\rangle = -\langle \alpha, \beta + p\alpha\rangle \Rightarrow 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\\ = -q - p = m \in \mathbf{Z}.\end{aligned} \tag{D.19}$$

The set of all roots, which are not linearly independent in $\mathbf{h}^*$, is denoted by Δ and it is decomposed into positive and negative roots. Obviously, the opposite of a positive root is a negative root and vice versa. A basis in Δ can be given by the so-called simple roots denoted by α_i, $i = 1, \ldots, l$, such that the positive (negative) roots are linear combinations of simple roots with positive (negative) integer coefficients. A simple root cannot be rewritten as the sum of two positive roots. The choice of simple roots is not unique and different choices are related by the Weyl group, which leaves the set of roots globally invariant.

Let α and β be two simple roots. Then obviously $\alpha - \beta$ cannot be a root and as a consequence $p = 0$ and $m = -q \leqslant 0$. In other words, $\langle \alpha, \beta\rangle \leqslant 0$. The Cartan matrix is defined in terms of the simple roots by

$$C_{ij} = 2\frac{\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle}. \tag{D.20}$$

This is not a symmetric matrix. We compute for $i \neq j$, using $\langle \alpha, \beta\rangle = |\alpha||\beta| \cos\theta$, that

$$C_{ij} = 2\frac{\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} = 2\frac{|\alpha_i|}{|\alpha_j|}\cos\theta_{ij} = m_i \leqslant 0 \tag{D.21}$$

$$C_{ji} = 2\frac{\langle \alpha_j, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} = 2\frac{|\alpha_j|}{|\alpha_i|}\cos\theta_{ij} = m_j \leqslant 0. \tag{D.22}$$

By multiplying the last two equations and using the Schwarz inequality $\langle \alpha, \beta\rangle^2 < \langle \alpha, \alpha\rangle\langle \beta, \beta\rangle$ we obtain

$$m_i m_j < 4. \tag{D.23}$$

The value $m_i m_j = 4$ is forbidden because $\alpha_i \neq -\alpha_j$. We also compute by dividing the above two equations the following

$$\frac{|\alpha_i|}{|\alpha_j|} = \sqrt{\frac{m_i}{m_j}}, \quad \cos\theta_{ij} = -\frac{1}{2}\sqrt{m_i m_j}. \tag{D.24}$$

The only allowed angles are, therefore, $\pi/2$, $2\pi/3$, $3\pi/4$ and $5\pi/6$ corresponding to the cosine equals $0, -1/2, -\sqrt{2}/2, -\sqrt{3}/2$. The allowed ratios of lengths of roots are, therefore, 0, 1, $\sqrt{2}$ and $\sqrt{3}$.

If the set of roots decomposes into two mutually orthogonal subsets then the corresponding semi-simple algebra **g** decomposes into the direct sum of two simple algebras corresponding to the orthogonal sets. In the following we will only consider simple algebras following [1].

For rank 1 there is one complex simple Lie algebra which is $A_1 = sl(2, \mathbf{C})$. For rank 2 there are two complex simple Lie algebras which are $A_2 = sl(3, \mathbf{C})$ and $B_2 = so(5, \mathbf{C})$ and one complex semi-simple Lie algebra which is $D_2 = so(4, \mathbf{C})$. There is also another rank 2 complex simple algebra G_2 which is an exceptional algebra of dimension 14. For higher ranks, the Cartan classification of complex simple algebra is given by

$$\begin{aligned} &A_l = sl(l+1, \mathbf{C}), \;\; B_l = so(2l+1, \mathbf{C}), \\ &C_l = sp(2l, \mathbf{C}), \;\; D_l = so(2l, \mathbf{C}). \end{aligned} \tag{D.25}$$

These four infinite families are the so-called classical Lie algebra. The unique real compact forms of these algebras are given, respectively, by $A_l = su(l+1)$, $B_l = so(2l+1)$, $C_l = usp(2l)$, $D_l = so(2l)$.

Furthermore, the full Cartan classification of the exceptional complex simple algebra is given by the following five cases

$$E_6(78), \;\; E_7(133), \;\; E_8(248), \;\; F_4(52), \;\; G_2(14). \tag{D.26}$$

The number in brackets is the dimension.

For higher ranks we use the Dynkin diagram which encodes the Cartan matrix to visualize the root system. The Dynkin diagram is constructed as follows

- Each simple root is represented by a vertex.
- Any two roots such that $\langle \alpha_i, \alpha_j \rangle \neq 0$ are linked by a line.
- The line is simple if

$$C_{ij} = C_{ji} = -1, \;\; \left(\theta_{ij} = \frac{2\pi}{3}, \;\; \frac{|\alpha_i|}{|\alpha_j|} = 1 \right). \tag{D.27}$$

- The line is double if

$$C_{ij} = -2, = C_{ji} = -1, \;\; \left(\theta_{ij} = \frac{3\pi}{4}, \;\; \frac{|\alpha_i|}{|\alpha_j|} = \sqrt{2} \right). \tag{D.28}$$

- The line is triple if

$$C_{ij} = -3, = C_{ji} = -1, \;\; \left(\theta_{ij} = \frac{5\pi}{6}, \;\; \frac{|\alpha_i|}{|\alpha_j|} = \sqrt{3} \right). \tag{D.29}$$

- The line carries an arrow from i to j if $|\alpha_i| > |\alpha_j|$.

D.3 Weights, Dynkin labels and representations

The roots studied so far provide one particular finite dimensional irreducible unitary representation of the algebra **g** known as the adjoint representation. A general finite

dimensional irreducible unitary representation of the algebra **g** is given by the so-called weights. We start with the Cartan subalgebra generated by the commuting elements H_i. These can be obviously diagonalized simultaneously. Let $|\lambda_a\rangle$ be the eigenvectors of the Cartan elements H_i with eigenvalues λ_i, viz

$$H_i|\lambda_a\rangle = \lambda_i|\lambda_a\rangle. \tag{D.30}$$

The vector of eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_l)$ is called a weight and is associated with the vectors $|\lambda_a\rangle$ where the index a denotes possible degeneracy of the eigenvalue λ. Obviously, for $H = \sum_i h^i H_i$ we have

$$H|\lambda_a\rangle = \lambda(H)|\lambda_a\rangle, \;\; \lambda(H) = \sum_i h^i \lambda_i. \tag{D.31}$$

Thus, λ is a linear form on **h** and thus an element in $\mathbf{h}^*$ which is the space of roots. Since H is hermitian the weights are real. The set of weights in a given representation forms what we call a weight diagram.

Let us call the representation space E. The dimension of the representation space E is equal to the total number of λ_a including their multiplicities. Since there is an $su(2)$ Lie algebra for every root α, the space E contains subspaces corresponding to the $d-l$ $su(2)$ Lie algebras $\{H_\alpha, E_\alpha, E_{-\alpha}\}$ satisfying $[H_\alpha, E_{\pm\alpha}] = \pm\langle\alpha, \alpha\rangle E_{\pm\alpha}$, $[E_\alpha, E_{-\alpha}] = H_\alpha$.

We compute

$$H_i E_{\pm\alpha}|\lambda_a\rangle = (\lambda \pm \alpha)_i E_{\pm\alpha}|\lambda_a\rangle. \tag{D.32}$$

Thus, the vector $E_\alpha|\lambda_a\rangle$ is an eigenvector associated with the eigenvalue $\lambda + \alpha$. In other words, all vectors in the representation space E are obtained from each other by the action of the $E_{\pm\alpha}$. We also conclude that any two weights of the same representation can only differ by a linear combination of roots with integer coefficients.

The weight λ will be associated with the operator E_λ. Let $p' \leqslant 0$ be the smallest integer such that $(E_{-\alpha})^{|p|}|\lambda_a\rangle$ is non-zero, i.e. $(E_{-\alpha})^{|p'|+1}|\lambda_a\rangle = 0$. This means that $(E_{-\alpha})^k|\lambda_a\rangle$, $k = 1, \ldots, |p'|$, correspond to non-zero weights $-k\alpha + \lambda$. Let $q' \geqslant 0$ be the largest integer such that $(E_\alpha)^{q'}|\lambda_a\rangle$ is non-zero, i.e. $(E_\alpha)^{q'+1}|\lambda_a\rangle = 0$. This means that $E_{k\alpha+\lambda}|\lambda_a\rangle$, $k = 1, \ldots, q'$, correspond to non-zero weights $k\alpha + \lambda$. The weights $\{\beta' = \lambda - |p'|\alpha, \ldots, \lambda, \ldots, \lambda + q'\alpha\}$ will be associated with the operator $E_{\beta'}$. Similar to before, the operators $\{E_{\beta'}\}$ form a representation of the $su(2)$ algebra generated by $E_{\pm\alpha}$ and H_α. Furthermore, the eigenvalues corresponding to the highest state $E_{\lambda+q'\alpha}$ and the lowest state $E_{\lambda+p'\alpha}$ given, respectively, by

$$\begin{aligned} [H_\alpha, E_{\lambda+q'\alpha}] &= \langle\alpha, \lambda + q'\alpha\rangle E_{\lambda+q'\alpha}, \\ [H_\alpha, E_{\lambda+p'\alpha}] &= \langle\alpha, \lambda + p'\alpha\rangle E_{\lambda+p'\alpha}, \end{aligned} \tag{D.33}$$

are opposite to each other, i.e.

$$\begin{aligned}\langle\alpha, \lambda + q'\alpha\rangle = -\langle\alpha, \lambda + p'\alpha\rangle \Rightarrow 2\frac{\langle\alpha, \lambda\rangle}{\langle\alpha, \alpha\rangle} \\ = -q' - p' = m' \in \mathbf{Z}.\end{aligned} \tag{D.34}$$

Since we are dealing with an $su(2)$ algebra, the number of the states $\{E_{\beta'}\}$ is exactly given by $q' - p' + 1 = 2j + 1$, i.e. $q' - p' = 2j$.

In the above space we can clearly introduce an ordering given by $\lambda' > \lambda$ if $\lambda' - \lambda = \sum_i n_i\alpha_i$ where n_i are positive integers. There exists, therefore, a highest weight state denoted by $|\Lambda\rangle$ such that if α is any positive root we have

$$E_\alpha|\Lambda\rangle = 0. \tag{D.35}$$

In this case $q' = 0$ and $p' = -2j$ and hence

$$2\langle\Lambda, \alpha\rangle = j\langle\alpha, \alpha\rangle > 0. \tag{D.36}$$

We define the Dynkin labels of a weight λ by

$$\lambda_i = 2\frac{\langle\lambda, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} \in \mathbf{Z}. \tag{D.37}$$

The α are simple roots and, therefore, there are l Dynkin labels. If we choose the weight λ to be the highest weight Λ we obtain

$$\lambda_i = 2\frac{\langle\Lambda, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} = j \in \mathbf{N}. \tag{D.38}$$

We define the fundamental weights Λ_i by the formula

$$\delta_{ij} = 2\frac{\langle\Lambda_i, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}. \tag{D.39}$$

Clearly, there are l of them and they provide a basis in $\mathbf{h}^*$. Also, it is obvious that any highest weight state can be rewritten as a linear combination of the fundamental weights with coefficients given by the Dynkin labels, viz

$$\Lambda = \sum_{i=1}^{l} \lambda_i\Lambda_i, \;\; \lambda_i \in \mathbf{N}. \tag{D.40}$$

Any irreducible unitary representation is completely characterized by its highest weight state. The fundamental weights define the so-called fundamental representations. Hence, we have l fundamental representations characterized by Λ_i.

The dimension and the Casimir operators of a given irreducible representation characterized by a highest weight state Λ are given in terms of the Weyl vector defined by the half sum of the positive roots or by the sum of the fundamental roots as follows

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_i \Lambda_i. \tag{D.41}$$

The dimension and the quadratic Casimir of the irreducible representation Λ are given respectively by the Weyl formulas

$$\dim(\Lambda) = \prod_{\alpha>0}\frac{\langle\Lambda+\rho,\,\alpha\rangle}{\langle\rho,\,\alpha\rangle} \tag{D.42}$$

$$C_2(\Lambda) = \frac{1}{2}\langle\Lambda,\,\Lambda+2\rho\rangle. \tag{D.43}$$

D.4 Explicit construction of Lie algebra representations

D.4.1 $A_l = su(l+1)$

In this case the dimension of $\mathbf{h}^*$ is $l = n-1$, i.e. $\mathbf{h}^* = \mathbf{R}^{n-1}$. Let $\hat{e}_i$, $i = 1, \ldots, n$, be an orthonormal basis in $\mathbf{R}^n$ satisfying

$$\langle\hat{e}_i,\,\hat{e}_j\rangle = \delta_{ij}. \tag{D.44}$$

We define any hyperplane in $\mathbf{R}^n$ by the vector normal to it. We consider the hyperplane given by the normal vector

$$\hat{\rho} = \sum_{i=1}^{n}\hat{e}_i. \tag{D.45}$$

We project the $\hat{e}_i$ on this hyperplane to obtain vectors e_i given by

$$e_i = \hat{e}_i - \frac{1}{n}\hat{\rho}. \tag{D.46}$$

Indeed, we check that $\langle\hat{\rho},\,e_i\rangle = 0$. Furthermore, we check that

$$\langle e_i,\,e_j\rangle = \delta_{ij} - \frac{1}{n}. \tag{D.47}$$

We also check that $\sum_i e_i = 0$. In other words, the e_i are not linearly independent and they form a basis in $\mathbf{R}^{n-1}$. We define the $l = n-1$ simple roots and the $(d-l)/2 = n(n-1)/2$ positive roots, respectively, by

$$\alpha_i = \alpha_{ii+1} = e_i - e_{i+1},\ \ i = 1, \ldots, l \tag{D.48}$$

$$\alpha_{ij} = e_i - e_j,\ \ i < j = 1, \ldots, n. \tag{D.49}$$

The simple roots satisfy

$$\langle\alpha_i,\,\alpha_j\rangle = 2\delta_{ij} - \delta_{ij+1} - \delta_{i+1j}. \tag{D.50}$$

From this equation, we can infer directly the value of the Cartan matrix. We have

$$C_{ij} = \langle \alpha_i, \alpha_j \rangle = 2\delta_{ij} - \delta_{ij+1} - \delta_{i+1j}. \tag{D.51}$$

In order to compute the Weyl vector we divide the positive roots as $\{\alpha_{1i} = e_1 - e_i,\ i = 2, \ldots, n\}$ $(n-1$ roots$)$, $\{\alpha_{2i} = e_2 - e_i, i = 3, \ldots, n\}$ $(n-2$ roots$)$, $\{\alpha_{3i} = e_3 - e_i, i = 4, \ldots, n\}$ $(n-3$ roots$)$, $\ldots$, $\{\alpha_{n-2i} = e_{n-2} - e_i, i = n-1, n\}$ (2 roots), $\{\alpha_{n-1i} = e_{n-1} - e_i, i = n\}$ (1 root). We have then

$$\begin{aligned} 2\rho = \sum_{\alpha>0} \alpha = \sum_{i<j} \alpha_{ij} \\ = (n-1)e_1 + (n-3)e_2 + (n-5)e_3 + \cdots \\ + (n-2i+1)e_i + \cdots - (n-1)e_n. \end{aligned} \tag{D.52}$$

The index i in the above formula goes from $i = 1$ to n. Now, if we re-express the e_i in terms of the simple roots α_i we find that the coefficient of each α_i is the sum of the coefficients of e_j with $j \leqslant i$, viz

$$\sum_{j=1}^{i} (n - 2j + 1) = i(n-i),\ \ i = 1, \ldots, n-1 \tag{D.53}$$

We obtain then

$$\begin{aligned} 2\rho = (n-1)\alpha_1 + 2(n-2)\alpha_2 + 3(n-3)\alpha_3 + \cdots \\ + i(n-i)\alpha_i + \cdots + (n-1)\alpha_{n-1}. \end{aligned} \tag{D.54}$$

By using $\sum_i e_i = 0$ we can also rewrite the Weyl vector as

$$\rho = \sum_{i=1}^{l} (n-i)e_i. \tag{D.55}$$

From equation (D.50) we have $\langle \alpha_i, \alpha_i \rangle = 2$. Thus, the fundamental weights must be defined by the condition $\langle \Lambda_j, \alpha_i \rangle = \delta_{ij}$. A solution is given by

$$\Lambda_i = \sum_{j=1}^{i} e_j. \tag{D.56}$$

We verify that $\langle \Lambda_j, \alpha_i \rangle = \sum_{k=1}^{j} \delta_{ki} - \sum_{k=1}^{j-1} \delta_{ki}$. For $i = j$ we get $1 - 0$, for $i > j$ we get $0 - 0$, whereas for $i < j$ we get $1 - 1$. Hence

$$2\frac{\langle \Lambda_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij}. \tag{D.57}$$

Explicitly, we have

$$\begin{aligned} e_1 &= \Lambda_1 \\ e_i &= \Lambda_i - \Lambda_{i-1},\ \ i = 2, \dots, n-1 \\ e_n &= -\Lambda_{n-1}. \end{aligned} \tag{D.58}$$

Furthermore, we compute

$$\langle \Lambda_j, \alpha_i \rangle = \frac{i(n-j)}{n},\ \ i \leqslant j. \tag{D.59}$$

We need now to compute the dimension of the irreducible representation with highest weight state $\Lambda = \sum_{i=1}^{l} \lambda_i \Lambda_i$. We have

$$\begin{aligned} \langle \Lambda, \alpha_{ij} \rangle &= \langle \Lambda, e_i \rangle - \langle \Lambda, e_j \rangle \\ &= \sum_{k=1}^{l} \lambda_k \sum_{k'=1}^{k} \left(\delta_{k'i} - \frac{1}{n} \right) - (i \longrightarrow j) \\ &= \sum_{p=1}^{l} f_p \left(\delta_{pi} - \frac{1}{n} \right) - (i \longrightarrow j) \\ &= f_i - f_j . \end{aligned} \tag{D.60}$$

The f_i is defined by the equation

$$f_i = \sum_{q=i}^{l} \lambda_q,\ \ i = 1, \dots, l;\ f_n = 0. \tag{D.61}$$

Further, we compute in the same way

$$\begin{aligned} \langle \rho, \alpha_{ij} \rangle &= \langle \rho, e_i \rangle - \langle \rho, e_j \rangle \\ &= \frac{1}{2} \sum_{k=1}^{l} (n - 2k + 1) \langle e_k, e_i \rangle - (i \longrightarrow j) \\ &= j - i. \end{aligned} \tag{D.62}$$

Hence, we find the dimension

$$\dim(\Lambda) = \prod_{i<j} \frac{f_i - f_j + j - i}{j - i}. \tag{D.63}$$

The Casimir operator is given by

$$C_2(\Lambda) = \frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{l} \lambda_i (\lambda_j + 2) \langle \Lambda_i, \Lambda_j \rangle. \tag{D.64}$$

We compute

$$\langle \Lambda_i, \Lambda_j \rangle = \frac{i(n-j)}{n}, \;\; i \leqslant j \tag{D.65}$$

$$\langle \Lambda_i, \Lambda_j \rangle = \frac{j(n-i)}{n}, \;\; i > j. \tag{D.66}$$

Example: $su(4)$ In this case $n = 4$, $d = n^2 - 1 = 15$, $l = 3$ and thus $\mathbf{h}^* = \mathbf{R}^3$. We have $(d - l)/2 = 6$ positive roots. The simple roots are

$$\begin{aligned} &\alpha_1 = \alpha_{12} = e_1 - e_2, \;\; \alpha_2 = \alpha_{23} = e_2 - e_3, \\ &\alpha_3 = \alpha_{34} = e_3 - e_4. \end{aligned} \tag{D.67}$$

The other positive roots are

$$\alpha_{13} = e_1 - e_3, \;\; \alpha_{14} = e_1 - e_4, \;\; \alpha_{24} = e_2 - e_4. \tag{D.68}$$

The Weyl vector is (using also $\sum_i e_i = 0$)

$$\begin{aligned} 2\rho &= 3e_1 + e_2 - e_3 - 3e_4 = 6e_1 + 4e_2 + 2e_3 \\ &= 3\alpha_1 + 4\alpha_2 + 3\alpha_3. \end{aligned} \tag{D.69}$$

The weights are

$$\Lambda_1 = e_1, \;\; \Lambda_2 = e_1 + e_2, \;\; \Lambda_3 = e_1 + e_2 + e_3. \tag{D.70}$$

The Weyl vector can also be rewritten as

$$\rho = \Lambda_1 + \Lambda_2 + \Lambda_3. \tag{D.71}$$

The Casimir operator and the dimension are given by

$$\begin{aligned} C_2(\Lambda) = &\frac{1}{8}\lambda_1(3\lambda_1 + 2\lambda_2 + \lambda_3 + 12) + \frac{1}{4}\lambda_2(\lambda_1 + 2\lambda_2 + \lambda_3 + 8) \\ &+ \frac{1}{8}\lambda_3(\lambda_1 + 2\lambda_2 + 3\lambda_3 + 12) \end{aligned} \tag{D.72}$$

$$\begin{aligned} \dim(\Lambda) = &\frac{1}{12}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_3 + 1)(\lambda_1 + \lambda_2 + 2) \\ &\times (\lambda_2 + \lambda_3 + 2)(\lambda_1 + \lambda_2 + \lambda_3 + 3). \end{aligned} \tag{D.73}$$

For example $C_2(n, 0, n) = n(n + 3)$ and $\dim(n, 0, n) = (n + 1)^2(n + 2)^2(2n + 3)/12$.

D.4.2 $B_l = so(2l + 1)$

In this case the dimension is $d = l(2l + 1)$ and the rank is l, i.e. $\mathbf{h}^* = \mathbf{R}^l$. We choose a basis in $\mathbf{R}^l$ given by e_i, $i = 1, \ldots, l$, such that $\langle e_i, e_j \rangle = \delta_{ij}$. The l simple roots are

$$\alpha_i = e_i - e_{i+1}, \;\; i = 1, \ldots, l - 1; \;\; \alpha_l = e_l. \tag{D.74}$$

There are $(d-l)/2 = l^2$ positive roots. Here they are

$$e_i = \sum_{k=i}^{l} \alpha_k, \;\; i = 1, \ldots, l \tag{D.75}$$

$$e_i - e_j = \sum_{k=i}^{j-1} \alpha_k, \;\; 1 \leqslant i < j \leqslant l \tag{D.76}$$

$$e_i + e_j = \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{l} \alpha_k, \;\; 1 \leqslant i < j \leqslant l. \tag{D.77}$$

The Weyl vector is the half sum of the positive roots. We get

$$\begin{aligned} 2\rho &= \sum_{i=1}^{l} e_i + \sum_{i<j}(e_i - e_j) + \sum_{i<j}(e_i + e_j) \\ &= \sum_{i=1}^{l} e_i(2l - 2i + 1) \\ &= \sum_{i=1}^{l}\sum_{k=i}^{l} \alpha_k(2l - 2i + 1) \\ &= \sum_{i=1}^{l}\sum_{k=1}^{i} \alpha_i(2l - 2k + 1) \\ &= \sum_{i=1}^{l} \alpha_i \cdot i(2l - i). \end{aligned} \tag{D.78}$$

The non-zero elements of the Cartan matrix are given by

$$\langle \alpha_i, \alpha_i \rangle = 2, \;\; i = 1, \ldots, l-1 \tag{D.79}$$

$$\langle \alpha_l, \alpha_l \rangle = 1 \tag{D.80}$$

$$\langle \alpha_i, \alpha_j \rangle = -1, \;\; 1 \leqslant i = j+1, \;\; j = i+1 \leqslant l. \tag{D.81}$$

The fundamental weights are defined by

$$\Lambda_i = \sum_{j=1}^{i} e_j, \;\; i = 1, \ldots, l-1; \;\; \Lambda_l = \frac{1}{2}\sum_{j=1}^{l} e_j. \tag{D.82}$$

We can re-express the basis vectors in terms of the fundamental weights as

$$e_1 = \Lambda_1; \;\; e_i = \Lambda_i - \Lambda_{i-1}, \;\; 2 \leqslant i \leqslant l-1; \;\; e_l = 2\Lambda_l - \Lambda_{l-1}. \tag{D.83}$$

The Dynkin labels of the roots are given by

$$\alpha_1 = 2\Lambda_1 - \Lambda_2 = (2, -1, \ldots) \tag{D.84}$$

$$\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1} = (\ldots, -1, 2, -1, \ldots), \quad i = 2, \ldots, l-2 \tag{D.85}$$

$$\alpha_{l-1} = -\Lambda_{l-2} + 2\Lambda_{l-1} - 2\Lambda_l = (\ldots, -1, 2, -2) \tag{D.86}$$

$$\alpha_l = -\Lambda_{l-1} + 2\Lambda_l = (\ldots, -1, 2). \tag{D.87}$$

In general, irreducible representations of $so(2l+1)$ are characterized by the highest weight vectors Λ which can be rewritten as $\Lambda = \sum_i n_i e_i = (n_1, \ldots, n_l)$ with $n_1 \geqslant \cdots \geqslant n_{l-1} \geqslant n_l \geqslant 0$ with dimensions (see [2] page 407)

$$\begin{aligned} \dim(n_1, \ldots, n_l) &= \prod_{i<j} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} \prod_i \frac{l_i}{m_i} \\ l_i &= n_i + l - i + \frac{1}{2}, \; m_i = l - i + \frac{1}{2}. \end{aligned} \tag{D.88}$$

Example: $so(5)$ In this case $l = 2$ and thus $\mathbf{h}^* = \mathbf{R}^2$. The simple roots are

$$\alpha_1 = e_1 - e_2, \;\; \alpha_2 = e_2. \tag{D.89}$$

There are four positive roots. These are the two simple roots plus

$$e_1 = \alpha_1 + \alpha_2, \;\; e_1 + e_2 = \alpha_1 + 2\alpha_2. \tag{D.90}$$

The Weyl vector is

$$\rho = \frac{3}{2}e_1 + \frac{1}{2}e_2 = \frac{3}{2}\alpha_1 + 2\alpha_2. \tag{D.91}$$

The fundamental weights are

$$\begin{aligned} \Lambda_1 &= e_1 = \alpha_1 + \alpha_2 \\ \Lambda_2 &= \frac{1}{2}(e_1 + e_2) = \frac{1}{2}\alpha_1 + \alpha_2. \end{aligned} \tag{D.92}$$

A straightforward calculation gives the dimension and the Casimir

$$\dim(\Lambda) = \frac{1}{6}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2)(2\lambda_1 + \lambda_2 + 3) \tag{D.93}$$

$$C_2(\Lambda) = \frac{1}{4}\lambda_1(2\lambda_1 + \lambda_2 + 6) + \frac{1}{4}\lambda_2(\lambda_1 + \lambda_2 + 4). \tag{D.94}$$

The highest weight state $\Lambda = \sum_i \lambda_i \Lambda_i$ is usually expressed as $\Lambda = \sum_i n_i e_i$ where the spin quantum numbers n_i are defined in terms of the Dynkin labels λ_i by $\lambda_1 = n_1 - n_2$

and $\lambda_2 = 2n_2$. The dimension and the Casimir operator become in terms of n_i given by

$$\dim(\Lambda) = \frac{1}{6}(2n_1 + 3)(2n_2 + 1)(n_1 - n_2 + 1)(n_1 + n_2 + 2) \tag{D.95}$$

$$C_2(\Lambda) = \frac{1}{2}n_1(n_1 + 3) + \frac{1}{2}n_2(n_2 + 1). \tag{D.96}$$

D.4.3 $D_l = so(2l)$

In this case the dimension is $d = l(2l - 1)$ and the rank is l, i.e. $\mathbf{h}^* = \mathbf{R}^l$. We choose a basis in $\mathbf{R}^l$ given by e_i, $i = 1, \ldots, l$, such that $\langle e_i, e_j \rangle = \delta_{ij}$. The l simple roots are

$$\alpha_i = e_i - e_{i+1}, \;\; i = 1, \ldots, l - 1; \;\; \alpha_l = e_{l-1} + e_l. \tag{D.97}$$

There are $(d - l)/2 = l(l - 1)$ positive roots. Here they are

$$e_i - e_j = \sum_{k=i}^{j-1} \alpha_k, \;\; 1 \leqslant i < j \leqslant l \tag{D.98}$$

$$e_i + e_j = \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{l-2} \alpha_k + \alpha_{l-1} + \alpha_l, \;\; 1 \leqslant i < j \leqslant l - 1 \tag{D.99}$$

$$e_i + e_l = \sum_{k=i}^{l-2} \alpha_k + \alpha_l, \;\; 1 \leqslant i \leqslant l - 1. \tag{D.100}$$

The Weyl vector is the half sum of the positive roots. We get

$$\begin{aligned} 2\rho &= \sum_{i=1}^{l-1} 2(l - i)e_i \\ &= \sum_{i=1}^{l-2} \alpha_i \cdot i(2l - i - 1) + \frac{l(l - 1)}{2}(\alpha_{l-1} + \alpha_l). \end{aligned} \tag{D.101}$$

The non-zero elements of the Cartan matrix are given by

$$\langle \alpha_i, \alpha_i \rangle = 2, \;\; i = 1, \ldots, l \tag{D.102}$$

$$\langle \alpha_i, \alpha_j \rangle = -1, \;\; i = l, \;\; j = l - 2; \;\; i = l - 2, \;\; j = l \tag{D.103}$$

$$\langle \alpha_i, \alpha_j \rangle = -1, \;\; 1 \leqslant i = j + 1, \;\; j = i + 1 \leqslant l - 2. \tag{D.104}$$

The fundamental weights are defined by

$$\begin{aligned}\Lambda_i &= \sum_{j=1}^{i} e_j \\ &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \cdots + \alpha_{l-2}) \\ &\quad + \frac{i}{2}(\alpha_{l-1} + \alpha_l),\ \ i = 1, \ldots, l-2\end{aligned} \tag{D.105}$$

$$\begin{aligned}\Lambda_{l-1} &= \frac{1}{2}(e_1 + \cdots + e_{l-1} - e_l) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2}) + \frac{l}{4}\alpha_{l-1} + \frac{l-2}{4}\alpha_l\end{aligned} \tag{D.106}$$

$$\begin{aligned}\Lambda_l &= \frac{1}{2}(e_1 + \cdots + e_{l-1} + e_l) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2}) + \frac{l-2}{4}\alpha_{l-1} + \frac{l}{4}\alpha_l.\end{aligned} \tag{D.107}$$

In general, irreducible representations of $so(2l)$ are characterized by the highest weight vectors Λ which can be rewritten as $\Lambda = \sum_i n_i e_i = (n_1, \ldots, n_l)$ with $n_1 \geqslant \cdots \geqslant n_{l-1} \geqslant |n_l| \geqslant 0$ with dimensions (see [2] page 409)

$$\begin{aligned}\dim(n_1, \ldots, n_l) &= \prod_{i<j} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} \\ l_i &= n_i + l - i,\ m_i = l - i.\end{aligned} \tag{D.108}$$

Example: $so(6)$ In this case $l = 3$ and thus $\mathbf{h}^* = \mathbf{R}^3$. The simple roots are

$$\alpha_1 = e_1 - e_2,\ \ \alpha_2 = e_2 - e_3,\ \ \alpha_3 = e_2 + e_3. \tag{D.109}$$

There are six positive roots. These are the three simple roots plus

$$\begin{aligned}&e_1 - e_3 = \alpha_1 + \alpha_2,\ \ e_1 + e_3 = \alpha_1 + \alpha_3, \\ &e_1 + e_2 = \alpha_1 + \alpha_2 + \alpha_3.\end{aligned} \tag{D.110}$$

The Weyl vector is

$$\rho = 2e_1 + e_2 = 2\alpha_1 + \frac{3}{2}\alpha_2 + \frac{3}{2}\alpha_3. \tag{D.111}$$

The Cartan matrix is

$$C = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \tag{D.112}$$

The fundamental weights are

$$\begin{aligned}\Lambda_1 &= e_1 = \alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 \\ \Lambda_2 &= \frac{1}{2}(e_1 + e_2 - e_3) = \frac{1}{2}\alpha_1 + \frac{3}{4}\alpha_2 + \frac{1}{4}\alpha_3 \\ \Lambda_3 &= \frac{1}{2}(e_1 + e_2 + e_3) = \frac{1}{2}\alpha_1 + \frac{1}{4}\alpha_2 + \frac{3}{4}\alpha_3.\end{aligned} \tag{D.113}$$

A straightforward calculation gives the dimension and the Casimir

$$\begin{aligned}\dim(\Lambda) = \frac{1}{12}&(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_3 + 1)(\lambda_1 + \lambda_2 + 2) \\ &\times (\lambda_1 + \lambda_3 + 2)(\lambda_1 + \lambda_2 + \lambda_3 + 3)\end{aligned} \tag{D.114}$$

$$\begin{aligned}C_2(\Lambda) = &\frac{1}{4}\lambda_1(2\lambda_1 + \lambda_2 + \lambda_3 + 8) + \frac{1}{8}\lambda_2(2\lambda_1 + 3\lambda_2 + \lambda_3 + 12) \\ &+ \frac{1}{8}\lambda_3(2\lambda_1 + \lambda_2 + 3\lambda_3 + 12).\end{aligned} \tag{D.115}$$

In terms of the spin quantum numbers n_i defined by $\lambda_1 = n_1 - n_2$, $\lambda_2 = n_2 - n_3$ and $\lambda_3 = n_2 + n_3$ we have

$$\begin{aligned}\dim(\Lambda) = \frac{1}{12}&((n_1 + 2)^2 - n_3^2)((n_1 + 2)^2 \\ &- (n_2 + 1)^2)((n_2 + 1)^2 - n_3^2)\end{aligned} \tag{D.116}$$

$$C_2(\Lambda) = \frac{1}{2}n_1(n_1 + 4) + \frac{1}{2}n_2(n_2 + 2) + \frac{1}{2}n_3^2. \tag{D.117}$$

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Appendix E

On homotopy theory

- **Compactification:** In one dimension we can compactify $\mathbf{R}$ by adding one point (at) ∞ to obtain the one-sphere S^1. Conversely, by removing one point from the circle we get essentially $\mathbf{R}$. This is called Alexandroff one-point compactification. Generalization to higher dimensions is obvious.
- **Topological spaces and manifolds:** A topological space is a collection of open subsets (they do not contain any of their boundary points) with certain properties which allow the introduction of the concept of continuity (smoothness). Manifolds have the added property of differentiability.
- **Homeomorphic:** Two spaces are said to be homeomorphic if they can be mapped continuously and bijectively onto each other. Two homeomorphic spaces are topologically identical and posses the same connectedness properties (i.e. they are homotopically equivalent).
- **Homotopic equivalence:** Two spaces X and Y are homotopically equivalent if there exist continuous mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $g \circ f = \mathbf{1}_X$, $f \circ g = \mathbf{1}_Y$. A very important example is the sphere S^n and the punctured $\mathbf{R}^{n+1}$ (i.e. $\mathbf{R}^{n+1}/\{0\}$)[1].
- **Homotopy:** A homotopy (or a deformation) of a smooth map f between two smooth manifolds X and Y is a smooth map $F : X \times I \longrightarrow Y$, $I = [0, 1]$ with the property $F(x, 0) = f(x)$. The maps $f_t(x) = F(x, t)$ are said to be homotopic.
- **Homotopy classes:** The relation homotopy divides the set of smooth maps between two smooth manifolds X and Y into equivalence classes called homotopy classes.

[1] This can be shown using the stereographic projection.

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- **Connected spaces:** An arcwise connected space is one in which every two points are connected by some path.
- **Loops:** A loop is a closed path. A loop through a point $x_0 \in M$ is a map $\alpha : [0, 1] \longrightarrow M$ such that $\alpha(0) = \alpha(1) = x_0$.

  A product of two loops α and β is a loop $\gamma = \alpha * \beta$ which corresponds to traversing the original loops consecutively, viz $\gamma(t) = \alpha(2t)$, $0 \leqslant t \leqslant 1/2$ and $\gamma(t) = \beta(2t - 1)$, $1/2 \leqslant t \leqslant 1$. The inverse loop α^{-1} corresponds to traversing the loop α in the opposite direction. The constant loop is obviously given by $c(t) = x_0$ for all t.

  Two loops are said to be homotopic, and we write $\alpha \sim \beta$, if they can be continuously deformed into each other. Thus, there must exist a mapping $H : [0, 1] \times [0, 1] \longrightarrow M$ which satisfy $H(s, 0) = \alpha(s)$, $H(s, 1) = \beta(s)$ and $H(0, t) = H(1, t) = x_0$.
- **The fundamental group:** The fundamental group of M, denoted $\pi_1(M, x_0)$, consists of all the equivalence (homotopy) classes of loops through $x_0 \in M$. The product of homotopy classes is given by the product of their representatives and thus $\pi_1(M, x_0)$ is a group where the neutral element is given by the constant loop. For arcwise connected space the fundamental group is independent of the base point x_0.
- **Homotopic equivalence:** Homotopically equivalent spaces have the same fundamental group.
- **Simply connected:** In a simply connected space every loop can be contracted to a point. The fundamental group in this case is trivial, viz $\pi_1 = 0$.
- **The circle:** Let us consider $M = S^1$. The maps $\theta : S^1 \longrightarrow S^1$ are phases. We can divide the first S^1 by 2π to get the interval $[0, 1]$. An arbitrary phase on S^1 always satisfies $\theta(0) = 0$, $\theta(2\pi) = 2\pi m$. This can be continuously deformed to the linear function $m\phi$. To see this consider the map

$$H(\phi, t) = (1 - t)\theta(\phi) + t\phi\frac{\theta(2\pi)}{2\pi}. \tag{E.1}$$

  This satisfies $H(0, t) = \theta(0) = 0$, $H(2\pi, t) = \theta(2\pi)$. In other words, $H(\phi, t)$ is a homotopy and as a consequence $\theta(\phi)$ is in the same equivalence class as $m\phi$. The set of homotopy (equivalence) classes is therefore Z. The fundamental group of the circle is

$$\pi_1(S^1) = Z. \tag{E.2}$$

- **Higher spheres:** All higher spheres are simply connected and as a consequence

$$\pi_1(S^n) = 0, \quad n \geqslant 2. \tag{E.3}$$

  The reason is very simple. Any loop in $\mathbf{R}^{n+1}/\{0\}$ can always avoid the point defect at the origin and be shrunk to a point.
- **Torus:** The fundamental group of a product of spaces X and Y is $\pi_1(X \otimes Y) = \pi_1(X) \otimes \pi_1(Y)$. Thus for a two-dimensional torus we have

$$\pi_1(T) = Z \otimes Z. \tag{E.4}$$

- **Higher homotopy groups:** The fundamental group uses loops and their behavior under deformations to characterize topological properties. Furthermore, the fundamental group (using loops) cannot detect point defects in dimensions higher than two. The higher homotopy groups uses generalizations of the one-dimensional loop to detect point defects in higher dimensions.
- **The *n*-cubes and *n*-loops:** An n-cube is defined by

$$I^n = \{(s_1, \ldots, s_n) | 0 \leqslant s_i \leqslant 1 \text{ all } s_i\}. \tag{E.5}$$

The boundary is defined by

$$\partial I^n = \{(s_1, \ldots, s_n) \in I^n | s_i = 0 \text{ or } s_i = 1\}. \tag{E.6}$$

An n-loop is a continuous map from the n-cube to the topological space X, viz

$$\alpha : I^n \longrightarrow X, \tag{E.7}$$

which satisfies

$$\alpha(s) = x_0, \ \ s \in \partial I^n. \tag{E.8}$$

In other words, all the points on the boundary are mapped to a single point $x_0 \in X$. Thus, n-loops are topologically equivalent to n-spheres.

As before a homotopy is a continuous deformation of the above n-loop. We define

$$F : I^n \times I \longrightarrow X. \tag{E.9}$$

We demand

$$\begin{aligned} F(s_1, s_2, \ldots, 0) &= \alpha(s_1, s_2, \ldots, s_n), \\ F(s_1, s_2, \ldots, 0) &= \beta(s_1, s_2, \ldots, s_n) \end{aligned} \tag{E.10}$$

$$F(s_1, s_2, \ldots, t) = x_0, \ \ (s_1, s_2, \ldots, s_n) \in \partial I^n. \tag{E.11}$$

The two n-loops α, β are therefore homotopic, viz $\alpha \sim \beta$.

- **Higher homotopy groups:** Again the homotopy relation defines an equivalence relation and as a consequence the space of n-loops is turned into a set of equivalence classes. For arcwise connected spaces (i.e. the base point x_0 is irrelevant) the set of equivalence classes is denoted by $\pi_n(X)$ and it is a group. The higher homotopy groups for $n > 1$ are all abelian as opposed to the fundamental group which can be non-abelian. Some of the most important examples are

$$\pi_n(S^n) = Z \tag{E.12}$$

$$\pi_m(S^n) = 0, \ \ m < n. \tag{E.13}$$